Interval Type-2 Fuzzy Logic Systems *Made Simple* by Using Type-1 Mathematics

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Outline

- Motivation
- Type-2 Fuzzy Sets
- Interval Type-2 Fuzzy Sets
- Interval Type-2 FLSs
- Conclusions

Motivation

- Type-2 fuzzy sets (T2 FSs), originally introduced by Zadeh (1975), provide additional design degrees of freedom in Mamdani and TSK fuzzy logic systems (FLS)
- This can be very useful when such systems are used in situations where lots of uncertainties are present
- The resulting type-2 fuzzy logic systems (T2 FLS) have the potential to provide better performance than a type-1 (T1) FLS

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This section is self-explanatory. Just read on.

Motivation (Cont'd.)

- To-date, because of the computational complexity of using a general T2 FS, most people only use *interval* T2 FSs in a T2 FLS, the result being an *interval T2 FLS* (IT2 FLS)
- The computations associated with interval T2 FSs are very manageable, which makes an IT2 FLS quite practical

Motivation (Cont'd.)

- There is a heavy ***educational burden*** **even to using an IT2 FLS**

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Motivation (Cont'd.)

- There is a heavy ***educational burden*** even to using an IT2 FLS
- One must first become proficient about (top-down):
  - T1 FLSs
  - General T2 FSs
  - Operations performed upon them (***T2 FS mathematics***)
  - T2 fuzzy relations
  - T2 FLSs
  - Interval T2 FSs, their associated operations and relations, and IT2 FLSs—all as examples of the more general results

Motivation (Cont'd.)

- To obtain such a level of proficiency, one has to make a very significant investment of time, something that many practicing engineers do not have
- Courses about FL also do not have enough time to do this
- Requiring a person to use T2 FS mathematics represents a *barrier* to the use of IT2 FSs and FLSs

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Motivation (Cont'd.)

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Motivation (Cont'd.)

- **This talk demonstrates that it is unnecessary to take the above route, from general T2 FS to IT2 FS**
- **Our approach will be "bottom-up"**
- ***All of the results that are needed to implement an IT2 FLS can be obtained using T1 FS mathematics***
- **As such, this talk will make IT2 FLSs much more accessible to all**

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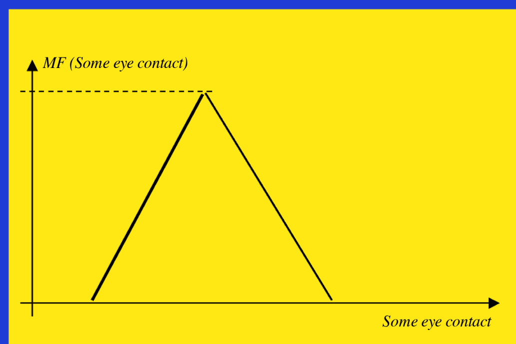
What is a T2 FS and How is it Different From a T1 FS?

- T1 FS has a grade of membership that is crisp



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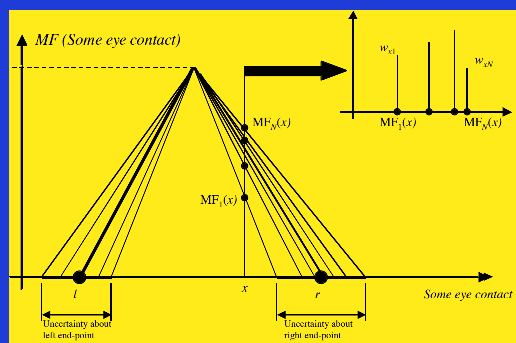
In this section I will cover some aspects of general type-2 fuzzy sets (T2 FSs).

For any value of the primary variable (*some eye contact*) when you project upwards you will intersect the membership function (MF) at a single point. In this case, "the T1 FS has a grade of membership that is *crisp*."

What is a T2 FS and How is it Different From a T1 FS?

- **T2 FS has a grade of membership that is fuzzy (a fuzzy-fuzzy set ☺)**



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I have assumed that there is some uncertainty about both the left-end and right-end vertices of the T1 MF. This is just one example, e.g. uncertainty could have also been assigned to the apex of the triangle.

I have drawn some additional triangle MFs that begin at some point in the interval of uncertainty for the left-end vertex, then pass through the apex and finally terminate at some point in the interval of uncertainty for the right-end vertex.

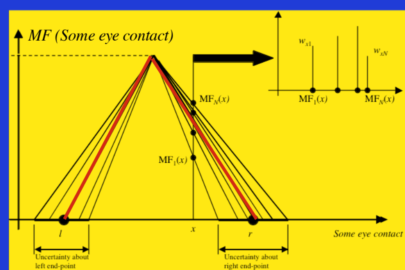
If all of the possible triangle MFs are shown (of course, we can't do that because there are an uncountable number of them), we would get a region that is called the *footprint of uncertainty (FOU)*—more about the FOU shortly.

When *some eye contact* has value x' then projecting vertically from x' a continuum of triangle MF values will be intersected. Note, however, that this continuum of MF values is bounded by both lower and upper values (bounds).

The insert shows that each point in the interval of MF values (the *primary memberships*) can be weighted differently. The weights—*secondary grades*—come out of the page, and provide the MF of a general T2 FS with a new *third dimension*.

Does a T2 FS reduce to a T1 FS?

- **Yes, when all uncertainties disappear**
- **Analogous to randomness reducing to determinism when random uncertainty disappears**



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So, a T2 FS reduces to a T1 FS when the FOU collapses to a T1 FS. This T1 FS is called a *primary MF*.

Consequently, T2 FS theory contains T1 FS theory as a special case.

What's the Main Difference Between T2 and T1 FSs?

- MF of a T1 FS is 2D

Nothing to add here.

What's the Main Difference Between T2 and T1 FSs?

- **MF of a T1 FS is 2D**
- **MF of a T2 FS is 3D**
  - **Too difficult to draw 3D functions**
  - **Convention is to focus on the 2D domain of the 3D MF**
  - **That domain is called the footprint of uncertainty**

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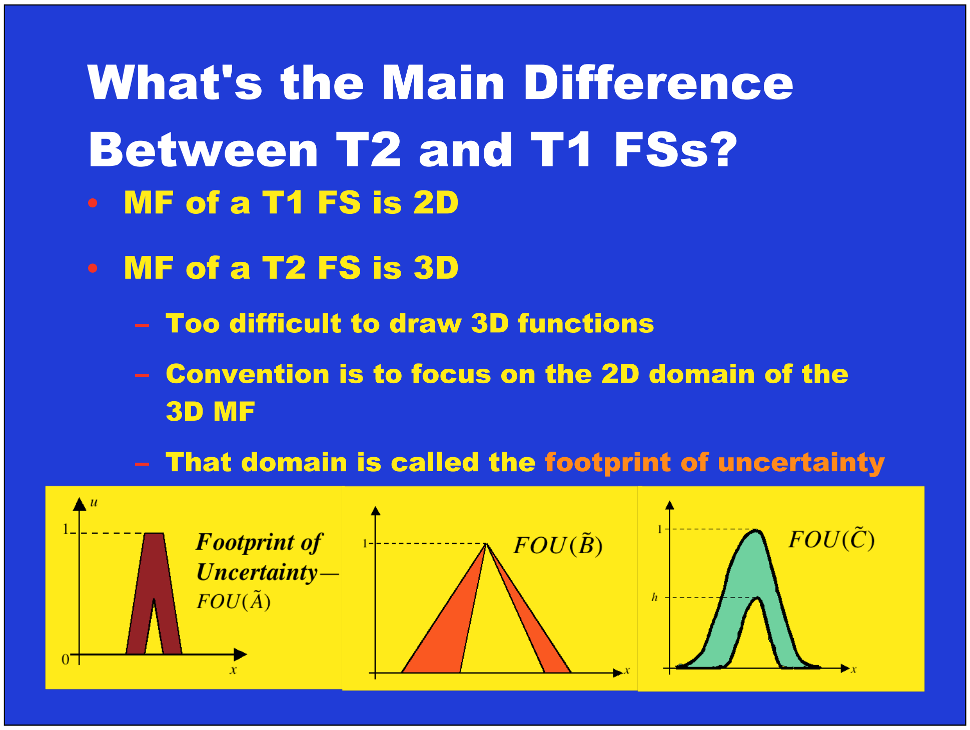
Recall that the MF of a general T2 FS is 3D.

So the FOU is the domain of the T2 FS MF. The secondary grades sit atop the FOU.



The left-hand FOU was obtained by beginning with a triangle primary MF and assuming uncertainty only about its apex. Assume the uncertainty is given by the interval [m - a, m + a]. Now beginning at m slide the triangle a units to the left and a units to the right. Doing this you will obtain the filled-in FOU.

The middle FOU demonstrates that uncertainty does not have to be symmetrical, i.e. the FOU to the left of the apex is "larger" (e.g., contains more area) than the FOU to the right of the apex. So, a FOU does not have to be symmetrical.

The right-hand FOU demonstrates that its bounds do not have to be piecewise linear. In this case, they are both Gaussian.

Observe that a FOU is described by more parameters than is the original primary MF. It is these additional parameters—degrees of freedom—that provide T2 FSs the potential to outperform T1 FSs.

Are There Limitations to Using T2 FSs?

- Today, using general T2 FSs is computationally infeasible
  - Efficient algorithms do not yet exist for computing operations
  - Finding such algorithms is a very hot and active research area

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Yes, T2 FSs require more computation than do T1 FSs. However, do we complain about using probability because it is also more complicated that deterministic analysis? We don't, because it provides us with a very useful model of unpredictability. Well, T2 FSs provide us with a very useful model of linguistic uncertainty.

So What Do We Do?

- We use Interval T2 FSs (IT2 FS)
  - They only use the FOU

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In an interval T2 FS (IT2 FS) all secondary grades have the same value, namely 1. Consequently, they convey no useful information, and can be discarded.

So What Do We Do?

- We use Interval T2 FSs (IT2 FS)
  - They only use the FOU
- Rationale
  - There is no best choice for even the MF of a type-1 FS (Triangle? Trapezoid? Gaussian?)
  - Let's uniformly weight the entire FOU ➟ IT2 FS

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T1 FS applications seem to be very robust to the specific choice made for the shape of the MF. The shape is usually fixed and then the associated parameters of that shape are optimized (tuned).

So What Do We Do?

- **We use Interval T2 FSs (IT2 FS)**
  - **They only use the FOU**
- **Rationale**
  - **There is no best choice for even the MF of a type-1 FS (Triangle? Trapezoid? Gaussian?)**
  - **Let's uniformly weight the entire FOU → IT2 FS**
- **Advantage**
  - **Formulas for computing all operations are simple—only have to keep tabs of the two end-points of an interval**

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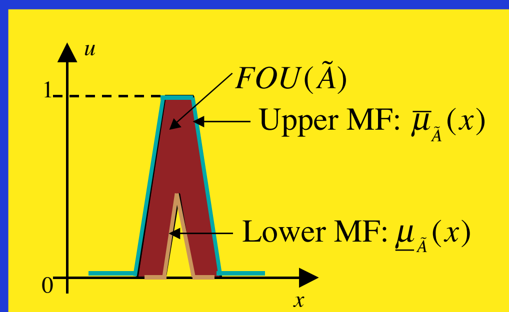
An interval is completely defined by its left-end and right-end points. You will see later in this talk that when operations are performed on IT2 FSs, intervals get mapped into other intervals. Simple formulas for the mapped intervals will be derived later in this talk.

Interval Type-2 Fuzzy Sets: Terminology-1



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Just as you learned new terminology in a course on probability, you have to learn some new terminology for IT2 FSs.

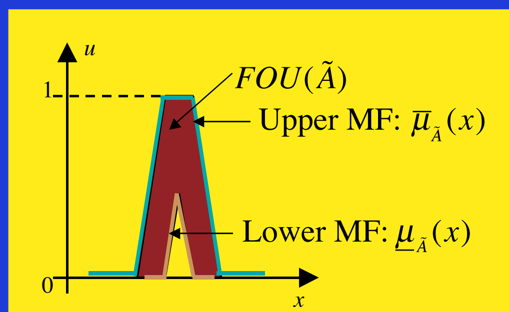
By the way, this terminology also applies to general T2 FSs.

First of all, it is now common to distinguish between a T2 FS and a T1 FS by using a tilde over the former symbol.

The upper MF is abbreviated to *UMF*, and lower MF is abbreviated to *LMF*. An over-bar on the T2 MF denotes the former, and an under-bar on the T2 MF denotes the latter.

The LMF and UMF play very important roles in all calculations involving IT2 FSs.

Interval Type-2 Fuzzy Sets: Terminology-1



- **Because all secondary grades of an IT2 FS equal 1, they convey no information—2D**
- **An IT2 FS is completely described by its Footprint of Uncertainty (FOU)**

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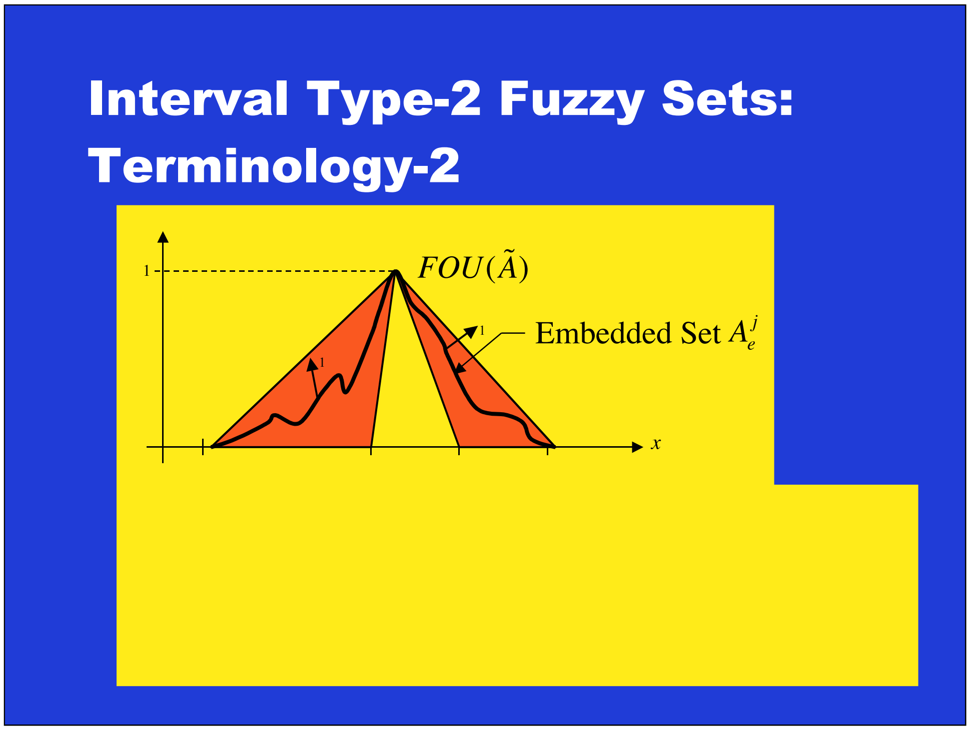
Nothing to add here.



An *embedded set* (also called an *embedded T1 FS*) is a function that lies within or on the FOU.

Two other examples of embedded sets are the LMF and UMF.

A short arrow labeled "1" is shown along the embedded T1 FS. When it is included with the embedded T1 FS, the result is an *embedded T2 FS*.

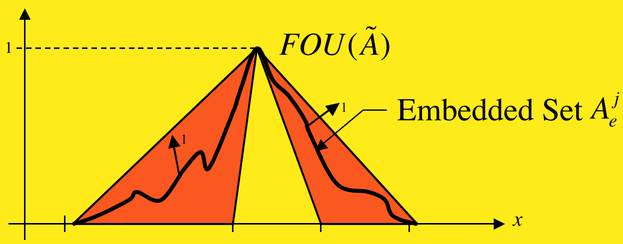
Because the third dimension of a general T2 FS is irrelevant for an IT2 FS, it is unnecessary to carry along the equal unit secondary grades. The FOU says it all for an IT2 FS.

For a continuous FOU (i.e., a completely filled-in FOU) there are an uncountable number of embedded sets. Don't' worry, though, because such sets will only be used for theoretical derivations, and never for computation.

If both the primary and secondary variable axes are discretized, then there will be a countable number of embedded sets, but there could still be an astronomical number of them. Again, don't worry because such sets will only be used for theoretical derivations, and never for computation

Observe that an embedded set looks like a *wavy slice* that cuts through the FOU.

Interval Type-2 Fuzzy Sets: Terminology-2



<u>Representation Theorem</u> (Wavy Slices)

$$\tilde{A} = 1/FOU(\tilde{A}) = 1\Big/\bigcup_{j=1}^{n_A} A_e^j = \begin{cases} 1\Big/\left\{\underline{\mu}_{\tilde{A}}(x), \ldots, \bar{\mu}_{\tilde{A}}(x)\right\} & \forall x \in X_d \\ 1\Big/\left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x)\right] & \forall x \in X \end{cases}$$

The *Representation Theorem* (also known as the *Mendel-John Representation Theorem*) is to-date the most important theoretical result in T2 FS theory.

It is covering theorem. In words, it states the obvious, namely the FOU can be covered by the union of all of its embedded sets.

The 1/FOU(·) notation is a shorthand notation. It means the secondary grades equal 1 at all points in the FOU.

The upper line in this equation is for a discrete universe of discourse, for which there will be a finite number of embedded sets. The second line in this equation is for a continuous universe of discourse, for which there will be an infinite and uncountable number (a continuum) of embedded sets.

So why is the Representation Theorem so important? Because it lets us represent the FOU (the IT2 FS) in terms of T1 FSs!

Although there are a lot of these T1 FSs, the Representation Theorem will only be used to derive formulas, and, because it only involves T1 FSs, this means derivations will only use T1 FS mathematics.

You will soon see that the resulting formulas are very easy to compute, and do NOT involve an astronomical number of T1 FS calculations.

Set Theoretic Operations

- We need formulas for union, intersection and complement
- They can be derived using the Representation Theorem
- We will show the details for the union of two interval T2 FSs

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Just as in T1 FS theory, where the union, intersection and complement of T1 FSs are frequently computed, these same quantities will also have to be computed for IT2 FSs.

By the way, all of the nitty-gritty details of derivations are in Reference 1 that is given at the end of this talk. I will only high-light the steps of derivations.

Details for the intersection and complement are in Ref. 1. Conceptually, these derivations are no different than the derivation of the union.

Union-Derivation: 1

- $\tilde{A} \cup \tilde{B} = 1 / FOU(\tilde{A} \cup \tilde{B})$

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This equation states that the union of two IT2 FSs is another IT2 FS, one that is completely described by its FOU.

Union-Derivation: 1

- $\tilde{A} \cup \tilde{B} = 1/FOU(\tilde{A} \cup \tilde{B})$

- $FOU(\tilde{A} \cup \tilde{B}) = \sum_{j=1}^{n_A} A_e^j \cup \sum_{i=1}^{n_B} B_e^i = \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i$

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In the second equation, the Representation Theorem has been used two times, once for IT2 FS *A* (sorry, I can't do tildes in these notes) and once for IT2 FS *B*.

In line 2, the sigma signs represent unions of elements *within* each of the two IT2 FSs, whereas the union sign represents the union *across* the two IT2 FSs.

To paraphrase the famous author Gertrude Stein (who said "A rose is a rose is a rose"), "a union is a union is a union." Consequently, the second part of line 2 reorganizes all of the unions.

IMPORTANT OBSERVATION: The union of the two embedded sets is the union of two T1 FSs, and this can be computed using T1 FS mathematics.

Union-Derivation: 1

- $\tilde{A} \cup \tilde{B} = 1 / FOU(\tilde{A} \cup \tilde{B})$

- $FOU(\tilde{A} \cup \tilde{B}) = \sum_{j=1}^{n_A} A_e^j \cup \sum_{i=1}^{n_B} B_e^i = \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i$

- $A_e^j \cup B_e^i = \max\left\{\mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k)\right\} \quad k = 1, 2, \ldots, N$

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In this derivation, the maximum operation is used for the disjunction operation.

In a more general derivation, the disjunction could be replaced by a t-conorm operator, of which the maximum is but one such operator.

The maximum is taken at all N points in the discretized domains of the two T1 embedded sets.

It is important to recall that the union of two T1 FSs is another T1 FS, i.e. it is a function (in this case it is defined at N points).

$$FOU(\tilde{A} \cup \tilde{B}) = \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i$$

Union-Derivation: 2

- $$\sup_{\forall j,i} \max \left\{ \mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k) \right\} = \max \left\{ \bar{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{B}}(x) \right\} \quad \forall x \in X$$
$$= \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x) \quad \forall x \in X$$

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The collection of all unions will have lower and upper bounding functions.

Consequently, the second critical step in the derivation is to return to the second line on the previous slide, and to find the largest and the smallest of the just-calculated unions.

The largest of these unions is found by determining the supremum of all of the unions.

Because the UMFs are legitimate T1 embedded sets, at each sampled value of the primary variable the supremum must occur at the UMFs of both IT2 FSs. The result of doing this for all values of the primary variable is the first line of the top equation.

The second line is just another way of writing the right-hand part of the first line.

$$FOU(\tilde{A} \cup \tilde{B}) = \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i$$

Union-Derivation: 2

- $$\sup_{\forall j,i} \max \left\{ \mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k) \right\} = \max \left\{ \bar{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{B}}(x) \right\} \quad \forall x \in X$$
  $$= \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x) \quad \forall x \in X$$

- $$\inf_{\forall j,i} \max \left\{ \mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k) \right\} = \max \left\{ \underline{\mu}_{\tilde{A}}(x), \underline{\mu}_{\tilde{B}}(x) \right\} \quad \forall x \in X$$
  $$= \underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x) \quad \forall x \in X$$

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The smallest of all the unions is found by determining the infimum of all of the unions.

Because the LMFs are legitimate T1 embedded sets, at each sampled value of the primary variable the infimum must occur at the LMFs of both IT2 FSs. The result of doing this for all values of the primary variable is the top line of the second equation.

The second line of that equation is just another way of writing the right-hand part of its first line.

$$FOU(\tilde{A} \cup \tilde{B}) = \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i$$

Union-Derivation: 2

- $$\sup_{\forall j,i} \max\left\{\mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k)\right\} = \max\left\{\bar{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{B}}(x)\right\} \quad \forall x \in X$$
  $$= \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x) \quad \forall x \in X$$
- $$\inf_{\forall j,i} \max\left\{\mu_{A_e^j}(x_k), \mu_{B_e^i}(x_k)\right\} = \max\left\{\underline{\mu}_{\tilde{A}}(x), \underline{\mu}_{\tilde{B}}(x)\right\} \quad \forall x \in X$$
  $$= \underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x) \quad \forall x \in X$$
- $$\tilde{A} \cup \tilde{B} = 1 \Big/ \sum_{j=1}^{n_A} \sum_{i=1}^{n_B} A_e^j \cup B_e^i = 1/[\underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x), \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x)]$$

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The last equation on this slide collects the two calculations in one place.

Observe that the union of two IT2 FSs is also an IT2 FS, one whose LMF is obtained from the LMFs of the original IT2 FSs, and whose UMF is obtained from the UMFs of the original IT2 FSs.

Observe, also, that although the Representation Theorem was used in the first steps of the derivation, additional analyses were needed to reach the final results. These analyses led to a simple *computational algorithm* for the union.

As promised, the Representation Theorem is a means to an end, and is not used as the final computational algorithm.

Summary Formulas

- $\tilde{A} \cup \tilde{B} = 1/[\underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x), \overline{\mu}_{\tilde{A}}(x) \vee \overline{\mu}_{\tilde{B}}(x)] \quad \forall x \in X$

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On this slide and the next two, computational formulas are stated for union, intersection and complement of two IT2 FSs.

It is straightforward to extend these formulas from two to a finite number of IT2 FSs. Just repeat the process adding in one new IT2 FS at a time.

Summary Formulas

- $\tilde{A} \cup \tilde{B} = 1/[\underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x), \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x)] \quad \forall x \in X$

- $\tilde{A} \cap \tilde{B} = 1/[\underline{\mu}_{\tilde{A}}(x) \wedge \underline{\mu}_{\tilde{B}}(x), \bar{\mu}_{\tilde{A}}(x) \wedge \bar{\mu}_{\tilde{B}}(x)] \quad \forall x \in X$

Summary Formulas

- $\tilde{A} \cup \tilde{B} = 1/[\underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{B}}(x), \bar{\mu}_{\tilde{A}}(x) \vee \bar{\mu}_{\tilde{B}}(x)] \quad \forall x \in X$

- $\tilde{A} \cap \tilde{B} = 1/[\underline{\mu}_{\tilde{A}}(x) \wedge \underline{\mu}_{\tilde{B}}(x), \bar{\mu}_{\tilde{A}}(x) \wedge \bar{\mu}_{\tilde{B}}(x)] \quad \forall x \in X$

- $\bar{\tilde{A}} = 1/[1-\bar{\mu}_{\tilde{A}}(x), 1-\underline{\mu}_{\tilde{A}}(x)] \quad \forall x \in X$

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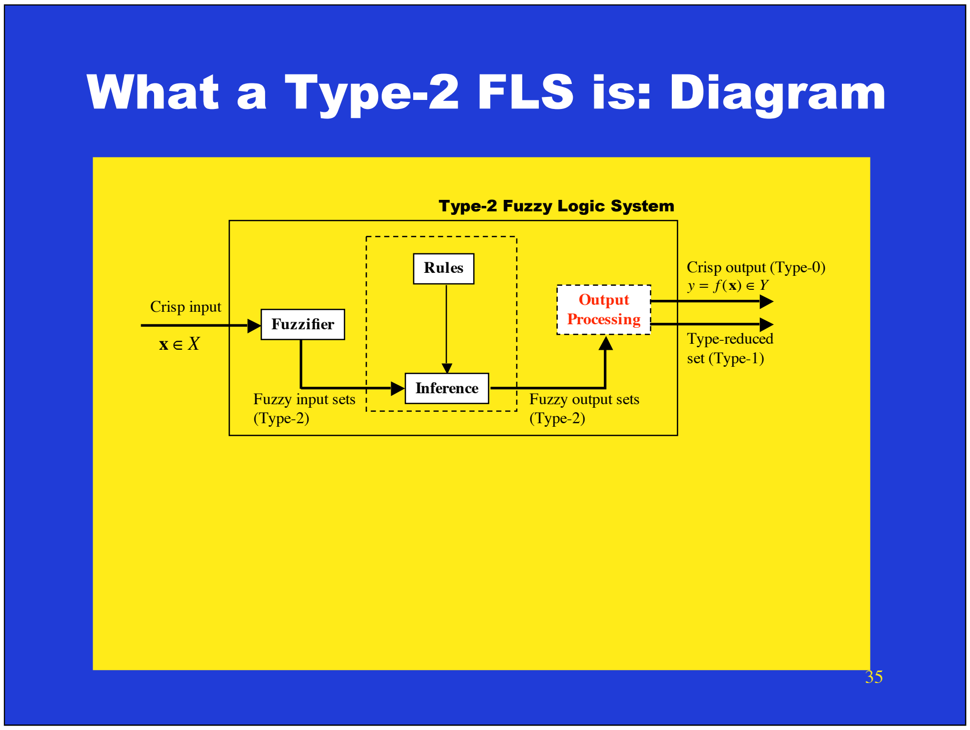
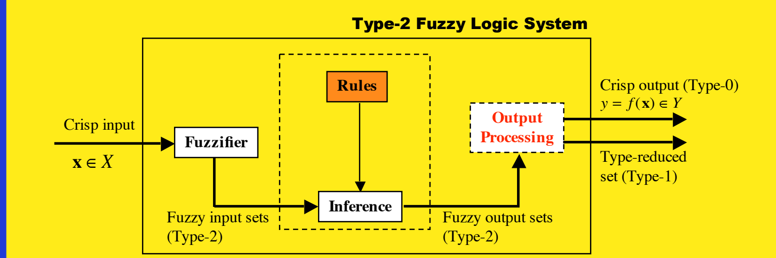


A T2 FLS has almost the same structure as a T1 FLS, the main difference being the box labeled "Output Processing."

Note that T2 and IT2 will be used interchangeably from here on in. All of our results are only for the latter.

Crisp inputs are fuzzified by the *Fuzzifier*. Unlike a T1 FLS, in which only two kinds of fuzzification are possible—singleton and T1—in an IT2 FLS three kinds of fuzzification are possible—singleton, T1 and IT2. The latter offers a possibility that is not available at all in a T1 FLS.

When either T1 or IT2 fuzzification is used, it is common to state that *non-singleton fuzzification* is used.



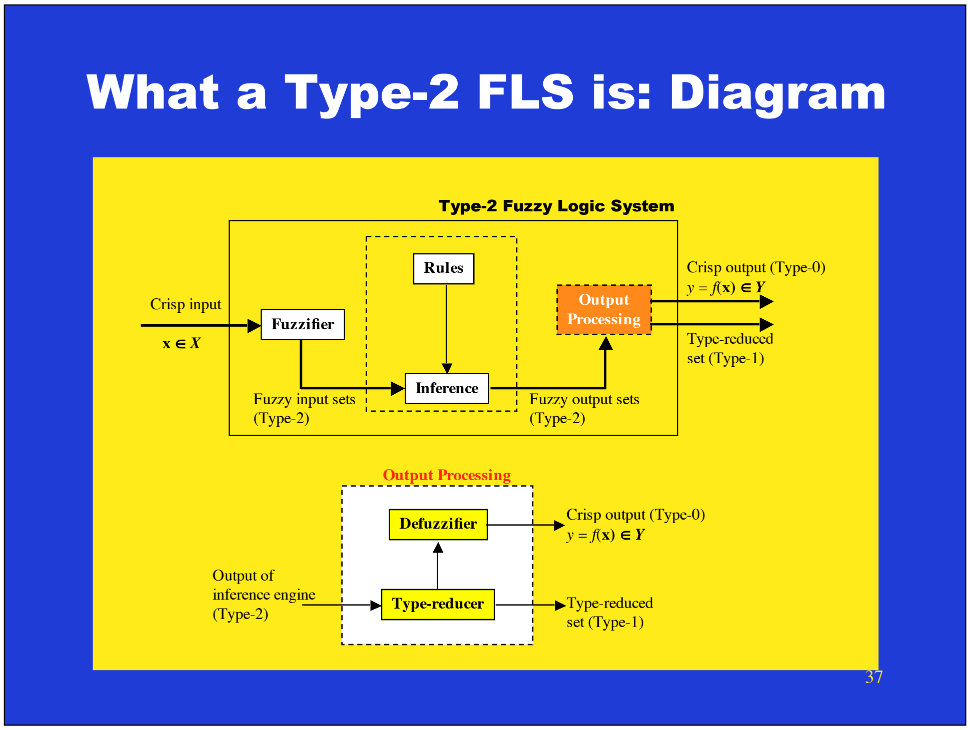
$$R^l:\ \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ and } \ldots \text{ and } x_p \text{ is } \tilde{F}_p^l,$$
$$\text{THEN } y \text{ is } \tilde{G}^l \qquad l = 1,\ldots,M$$

Rules don't change. What does change are the models used for antecedents and consequents in the rules.

Again, paraphrasing Gertrude Stein, "A rule is a rule is a rule."

As long as even one antecedent or consequent is modeled as T2, the entire FLS is T2. In fact, if all antecedents and consequents are modeled as T1 FSs, but at least one input is fuzzified as an IT2 FS, then the FLS is T2.

The Inference engine maps T2 FSs into fired-rule T2 FSs. Closed-form formulas that do this are derived below.



Usually, the desired output of a FLS is a number. *Output Processing* maps the fired rule IT2 FSs into a number, but it does so in two steps.

Type-reduction (discussed in detail below, and abbreviated TR) maps IT2 FSs into an interval set.

Defuzzification maps the TR set into a number.

Next, let us see how the Representation Theorem can be used to obtain closed-form formulas for inferencing.

Interval T2 FLS Formulas: SF and One Antecedent

IF $x =$ is $\tilde{F}$,THEN y is $\tilde{G}$

A bottom-up approach is taken, one that begins with the simplest situation, a single rule with one antecedent and singleton fuzzification (SF).

Once the details of the derivation of the fired-rule set are understood for this case, it will be very straightforward to extend the results to more general cases.

Interval T2 FLS Formulas: SF and One Antecedent

IF $x =$ is $\tilde{F}$,THEN y is $\tilde{G}$

- $\tilde{F} = 1 / FOU(\tilde{F}) = 1 \Big/ \sum_{i=1}^{n_F} F_e^i$
- $\tilde{G} = 1 / FOU(\tilde{G}) = 1 \Big/ \sum_{j=1}^{n_G} G_e^j$

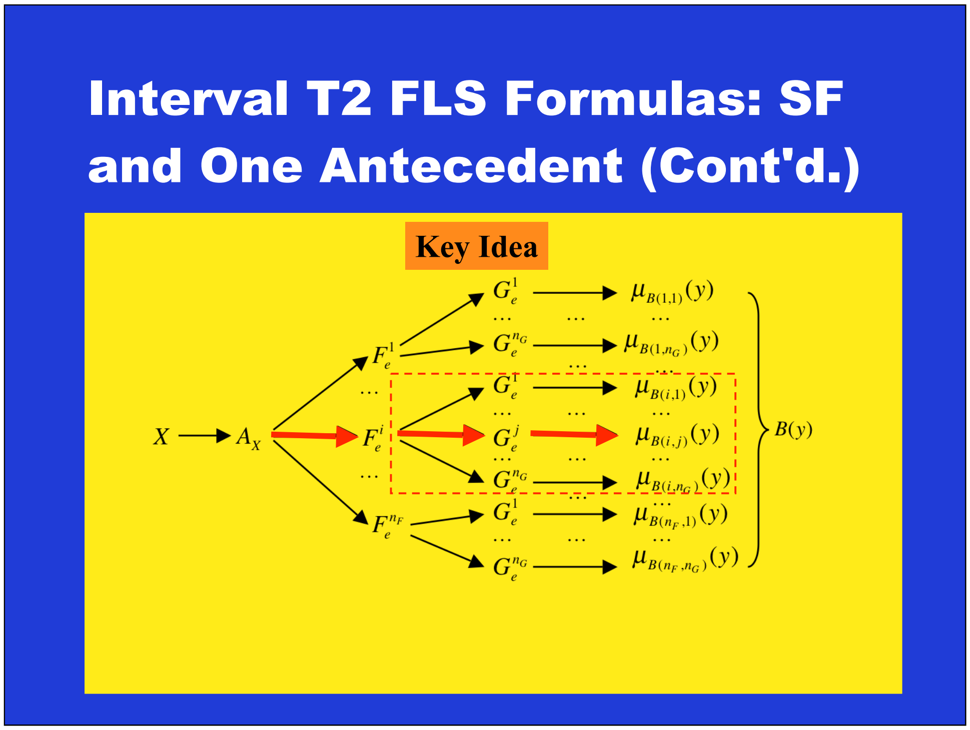
To begin, the Representation Theorem is used for both the antecedent and consequent IT2 FSs.

Interval T2 FLS Formulas: SF and One Antecedent

IF $x =$ is $\tilde{F}$,THEN y is $\tilde{G}$

- $\tilde{F} = 1 / FOU(\tilde{F}) = 1 \Big/ \sum_{i=1}^{n_F} F_e^i$
- $\tilde{G} = 1 / FOU(\tilde{G}) = 1 \Big/ \sum_{j=1}^{n_G} G_e^j$
- $n_F \times n_G$ possible combinations of embedded T1 antecedent and consequent FSs

The approach taken next is to consider all possible $n_F \times n_G$ type-1 rules.



This is a *directed graph* showing (from left-to-right) the singleton input (fuzzified into A_X) fanning out into *each* of the n_F embedded T1 FSs of the antecedent. Each of these then fan out into *each* of the n_G embedded T1 FSs of the consequent. In total, there are $n_F \times n_G$ paths. A group of n_G such paths is enclosed by the red dashed rectangle, and the path shown with red arrows is one such path.

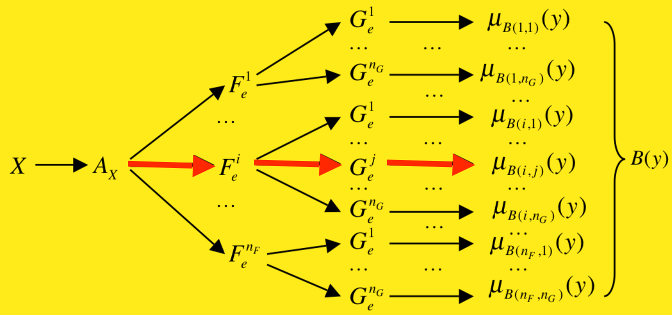
For convenience, this derivation is done for a discrete universe of discourse so that it is possible to use directed graphs.

So, the "Key Idea" is to use a directed graph to show all of the $n_F \times n_G$ paths, and each path is associated with a T1 calculation.

Observe the indexing of the T1 fired rule consequent set. The first index in $B(i,j)$ is associated with the superscript of the antecedent embedded T1 FS, and the second index in $B(i,j)$ is associated with the superscript of the consequent embedded T1 FS.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

Key Idea



$$B(y) \triangleq \sum_{i=1}^{n_F} \sum_{j=1}^{n_G} \mu_{B(i,j)}(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} \quad \forall y \in Y_d$$

All of the $n_F \times n_G$ T1 FS s are unioned in order to obtain the MF of the fired rule consequent set of the rule.

This union has a smallest and a largest member, and these are shown with the usual under and over bars.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) \triangleq \sum_{i=1}^{n_F} \sum_{j=1}^{n_G} \mu_{B(i,j)}(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} \quad \forall y \in Y_d$

Our task is to compute these two MFs.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) \triangleq \sum_{i=1}^{n_F} \sum_{j=1}^{n_G} \mu_{B(i,j)}(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} \quad \forall y \in Y_d$

- $\underline{\mu}_{\tilde{B}}(y) = \inf_{\forall i,j} (\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$
  $\bar{\mu}_{\tilde{B}}(y) = \sup_{\forall i,j} (\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$

The second and third equations are just formal statements of what needs to be computed for the LMF and UMF of the IT2 FS *B*.

These calculations can be done using T1 FS mathematics.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) \triangleq \sum_{i=1}^{n_F}\sum_{j=1}^{n_G} \mu_{B(i,j)}(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} \quad \forall y \in Y_d$
- $\underline{\mu}_{\tilde{B}}(y) = \inf_{\forall i,j}(\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$
  $\bar{\mu}_{\tilde{B}}(y) = \sup_{\forall i,j}(\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$
- $\mu_{B(i,j)}(y) = \mu_{F_e^i}(x') \bigstar \mu_{G_e^j}(y) \quad \forall y \in Y_d$ T1 formula

In the fourth equation, *Mamdani implication* has been assumed (I am assuming that you are familiar with a T1 Mamdani FLS), and so the formula for the MF of the T1 consequent set $B(i, j)$ is very simple.

The t-norm is either minimum or product.

Observe x' in the antecedent MF. It is due to singleton fuzzification for which $x = x'$.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) \triangleq \sum_{i=1}^{n_F} \sum_{j=1}^{n_G} \mu_{B(i,j)}(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} \quad \forall y \in Y_d$
- $\underline{\mu}_{\tilde{B}}(y) = \inf_{\forall i,j}(\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$
  $\bar{\mu}_{\tilde{B}}(y) = \sup_{\forall i,j}(\mu_{B(i,j)}(y)) \quad \forall y \in Y_d$
- $\mu_{B(i,j)}(y) = \mu_{F_e^i}(x') \bigstar \mu_{G_e^j}(y) \quad \forall y \in Y_d$ T1 formula
- $\underline{\mu}_{\tilde{B}}(y) = \underline{\mu}_F(x') \bigstar \underline{\mu}_G(y), \quad \forall y \in Y_d$
  $\bar{\mu}_{\tilde{B}}(y) = \bar{\mu}_F(x') \bigstar \bar{\mu}_G(y), \quad \forall y \in Y_d$

Analyzing the fourth equation, it is easy to see that its lower (upper) bound is achieved when both the antecedent and consequent MFs are their respective LMFs (UMFs).

This analysis leads to the last two equations on this slide.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y)=\left\{\underline{\mu}_{\tilde{B}}(y),\ldots,\ \bar{\mu}_{\tilde{B}}(y)\right\}=FOU(\tilde{B}),\quad \forall y\in Y$

In this first equation, the IT2 fired rule consequent set is equated to its FOU.

In effect, the Representation Theorem is now being used in reverse, i.e. the totality of T1 FSs contained within the bracketed set is the union of all MFs that describe the FOU of the IT2 fired rule consequent set.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} = FOU(\tilde{B}), \quad \forall y \in Y$
- $\tilde{B} = 1 / FOU(\tilde{B})$

The second equation is just the formal description of the IT2 fired rule consequent set given on the bottom of Slide 24.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y)=\left\{\underline{\mu}_{\tilde{B}}(y),\ldots,\ \bar{\mu}_{\tilde{B}}(y)\right\}=FOU(\tilde{B}),\quad \forall y\in Y$
- $\tilde{B}=1\big/FOU(\tilde{B})$
- $FOU(\tilde{B})=\left\{\underline{\mu}_F(x')\bigstar\underline{\mu}_G(y),\ldots,\ \bar{\mu}_F(x')\bigstar\bar{\mu}_G(y)\right\}$

The third equation was obtained by substituting the last two equations on Slide 46 into the first equation on the present slide.

Next, this third equation will be written in a more informative way.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} = FOU(\tilde{B}), \quad \forall y \in Y$
- $\tilde{B} = 1 / FOU(\tilde{B})$
- $$\begin{aligned} FOU(\tilde{B}) &= \left\{ \underline{\mu}_F(x') \star \underline{\mu}_G(y), \ldots, \bar{\mu}_F(x') \star \bar{\mu}_G(y) \right\} \\ &= \left\{ \underline{\mu}_F(x'), \ldots, \bar{\mu}_F(x') \right\} \star \left\{ \underline{\mu}_G(y), \ldots, \bar{\mu}_G(y) \right\} \end{aligned}$$

Convince yourselves that the second line of the third equation is the same as the first line of that equation.

Take all of the t-norms between the antecedent and consequent MFs, and note that this collection will have a smallest and a largest function, and those functions are the ones given on the second line.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y)=\left\{\underline{\mu}_{\tilde{B}}(y),\ldots,\ \bar{\mu}_{\tilde{B}}(y)\right\}=FOU(\tilde{B}),\quad \forall y\in Y$
- $\tilde{B}=1\big/FOU(\tilde{B})$
- $$\begin{aligned} FOU(\tilde{B}) &= \left\{\underline{\mu}_F(x')\bigstar\underline{\mu}_G(y),\ldots,\ \bar{\mu}_F(x')\bigstar\bar{\mu}_G(y)\right\} \\ &= \left\{\underline{\mu}_F(x'),\ldots,\ \bar{\mu}_F(x')\right\}\bigstar\left\{\underline{\mu}_G(y),\ldots,\bar{\mu}_G(y)\right\} \\ &\triangleq \underbrace{\left\{\underline{f}(x'),\ldots,\ \bar{f}(x')\right\}}_{\text{Firing Interval}}\bigstar\underbrace{\left\{\underline{\mu}_G(y),\ldots,\bar{\mu}_G(y)\right\}}_{\text{Consequent FS}}\quad \forall y\in Y \end{aligned}$$

It is common practice to label the elements of the excited antecedent MFs using the symbol "*f*" to denote "*firing*."

The elements in the first bracketed set are collectively called the "*firing interval*."

The elements in the second bracketed set constitute the entire IT2 consequent set.

The uncertainties about the antecedent as it is excited by the singleton input $x = x'$ are carried through the computations by means of the firing interval.

When the antecedent collapses to a T1 FS, then the firing interval collapses to a point value, the *firing level*.

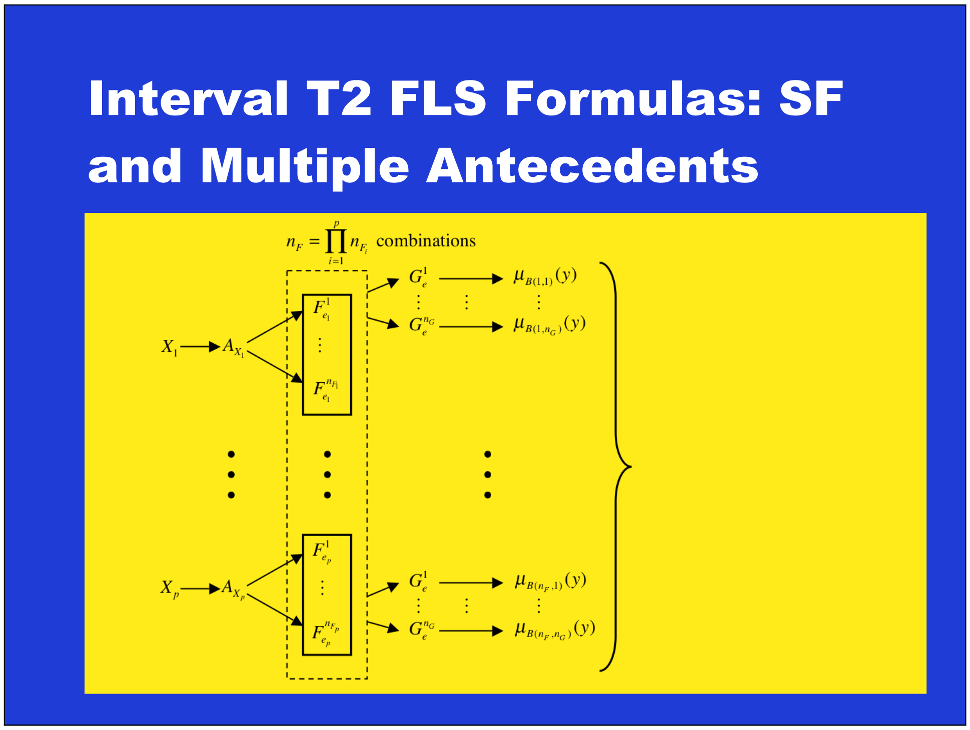
And, when the consequent is also a T1 FLS, then this is how an IT2 FLS reduces to a T1 FLS.

Interval T2 FLS Formulas: SF and One Antecedent (Cont'd.)

- $B(y) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\} = FOU(\tilde{B}), \quad \forall y \in Y$
- $\tilde{B} = 1 / FOU(\tilde{B})$
- $$\begin{aligned} FOU(\tilde{B}) &= \left\{ \underline{\mu}_F(x') \bigstar \underline{\mu}_G(y), \ldots, \bar{\mu}_F(x') \bigstar \bar{\mu}_G(y) \right\} \\ &= \left\{ \underline{\mu}_F(x'), \ldots, \bar{\mu}_F(x') \right\} \bigstar \left\{ \underline{\mu}_G(y), \ldots, \bar{\mu}_G(y) \right\} \\ &\triangleq \underbrace{\left\{ \underline{f}(x'), \ldots, \bar{f}(x') \right\}}_{\text{Firing Interval}} \bigstar \underbrace{\left\{ \underline{\mu}_G(y), \ldots, \bar{\mu}_G(y) \right\}}_{\text{Consequent FS}} \quad \forall y \in Y \end{aligned}$$

All using T1 FS math!!

All of the mathematics that has been used to derive the preceding results was T1 FS math., as promised!



Next, consider a single rule with p antecedents, and singleton fuzzification.

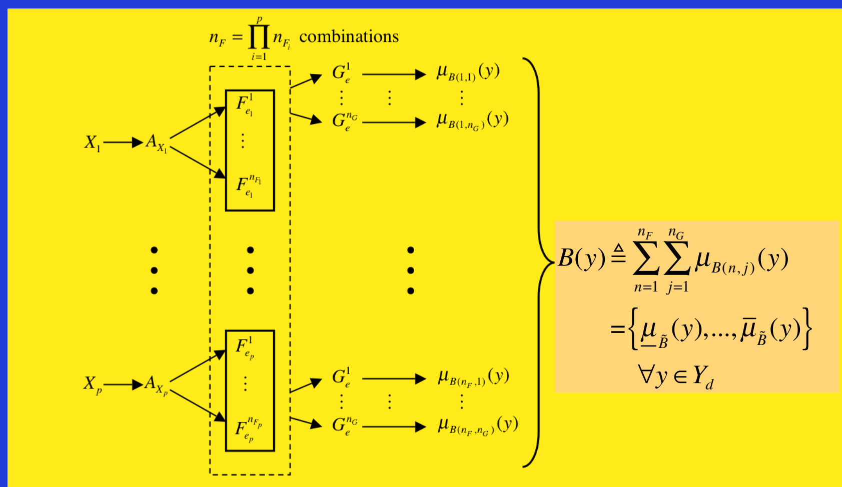
I had some difficulty coming up with a simple directed graph for this important case. My graph is above.

Each input is singleton-fuzzified and then fans out into its respective embedded T1 antecedent sets. Each of those in turn fans out into the n_G embedded T1 consequent sets.

The dashed block around all of the embedded antecedent sets denotes the fact that all possible n_F combinations of them must be formed, where the formula for n_F is given at the top of the dashed block.

Of course, in this case there are many more paths than in the previous case, but conceptually there is nothing new.

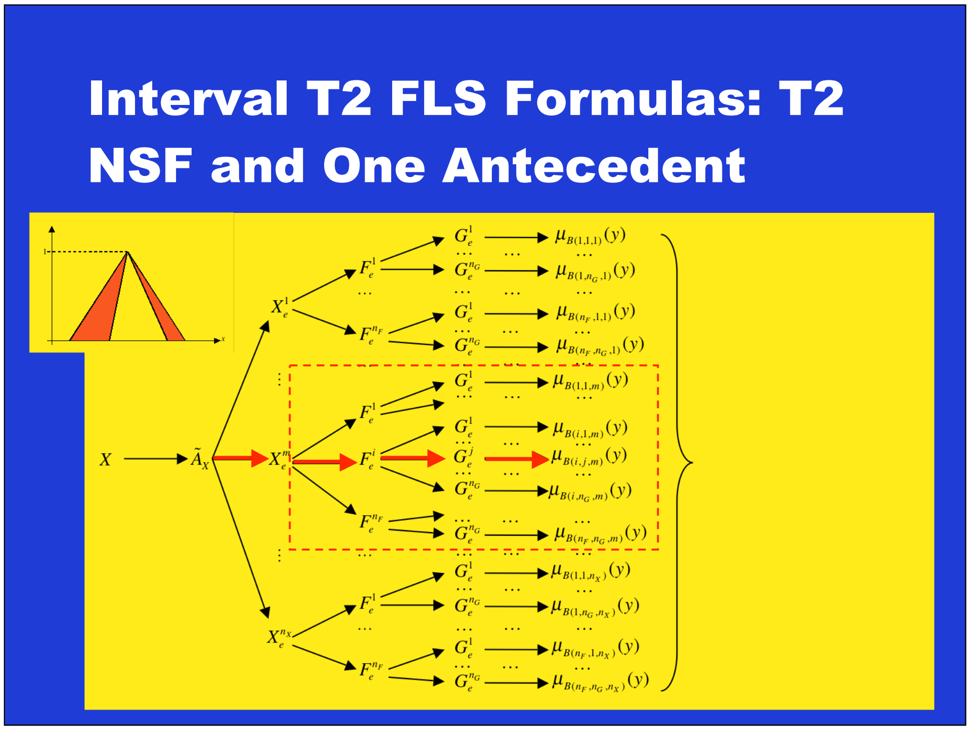
Interval T2 FLS Formulas: SF and Multiple Antecedents



You should compare the formula that is given on this slide for the IT2 fired rule consequent set with the one on Slide 42. They look exactly the same.

The difference in the two formulas is in the formulas for n_F.

The rest of the derivations proceed exactly as in the previous case and are not included here. See Ref. 1 for more details.

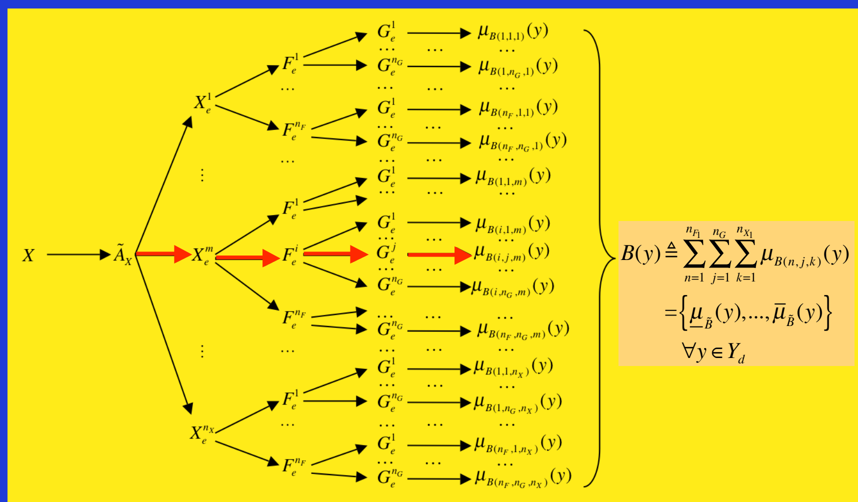


Returning to the case of a single antecedent rule, this time the single input is modeled as an IT2 FS. A representative FOU for the input is shown in the upper left-hand insert.

Now the Representation is used three times, once each for the input, antecedent and consequent.

Observe that each of the input's n_X embedded sets fans out into a directed graph (in the dashed red rectangle) that looks exactly like the one on Slide 41, and of course there are many more embedded sets in the present case than in the first case.

Interval T2 FLS Formulas: T2 NSF and One Antecedent



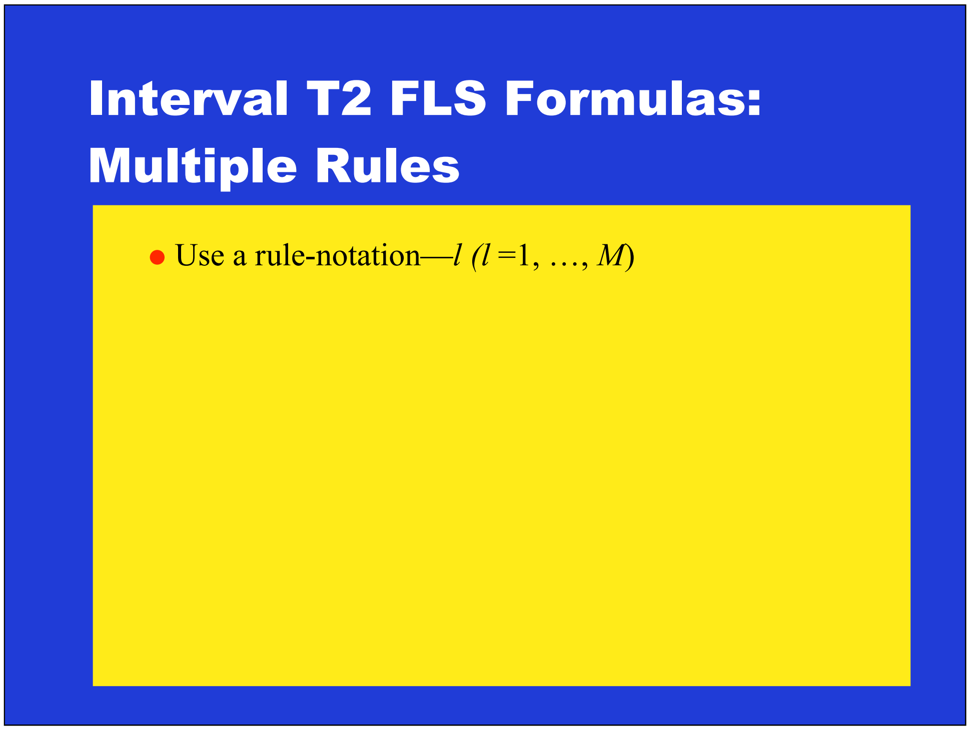
Now the formula for the IT2 fired rule consequent set has three unions, where the third one is for the input IT2 FS.

I have purposely made the new union the right-most one, and have also made the third index on the MF of the fired rule consequent set the third one. In this way it is east to connect the formula that is shown on this slide to the one that is shown on Slide 42.

For the rest of the derivation, see Ref. 1. It makes use of the sup-star composition that is used in a T1 FLS when the input is modeled as a T1 FS, the so-called *non-singleton T1 case*.

The final results can again be expressed as the t-norm between a firing interval and the entire IT2 consequent set, as on Slide 52, but the calculation of the firing interval is more complicated in the present case. It does not involve x', but instead it involves a value of x that derives from the sup-star composition calculation.

Once again, all of the mathematics that is needed to obtain formulas for the LMF and UMF of the IT2 fired-rule consequent set is T1 FS mathematics.



Our last generalization is from one rule to multiple rules. To that end, the rule-notation shown on this slide is introduced.

M rules are assumed.

Interval T2 FLS Formulas: Multiple Rules

- Use a rule-notation—l (l =1, …, M)
- If l fired rules are combined using the union operation, leading to a composite IT2 FS, $\tilde{B}$, then:

We don't necessarily advocate combining the fired rule consequent sets by the union. Union combining is used here merely as an illustration.

Recall, even in a T1 FLS there can be different ways of combining the fired rule consequent sets, and many times it is done as part of defuzzification.

The same is true for an IT2 FLS. Fired rule consequent sets can be combined as a part of the type-reduction process.

Interval T2 FLS Formulas: Multiple Rules

- Use a rule-notation—l (l =1, …, M)
- If l fired rules are combined using the union operation, leading to a composite IT2 FS, $\tilde{B}$, then:
- $\tilde{B} = 1/FOU(\tilde{B})$
- $FOU(\tilde{B}) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\}, \quad \forall y \in Y_d$

The symbolic structure of the two equations shown on this slide do not change as a result of multiple rules. That's because when fired rule consequent sets are combined by the union operation there is then only *one* combined fired rule consequent set.

Interval T2 FLS Formulas: Multiple Rules

- Use a rule-notation—l (l =1, …, M)
- If l fired rules are combined using the union operation, leading to a composite IT2 FS, $\tilde{B}$, then:
- $\tilde{B} = 1 / FOU(\tilde{B})$
- $FOU(\tilde{B}) = \left\{ \underline{\mu}_{\tilde{B}}(y), \ldots, \bar{\mu}_{\tilde{B}}(y) \right\}, \quad \forall y \in Y_d$
- $\underline{\mu}_{\tilde{B}}(y) = \underline{\mu}_{\tilde{B}^1}(y) \vee \underline{\mu}_{\tilde{B}^2}(y) \vee \ldots \vee \underline{\mu}_{\tilde{B}^M}(y)$
- $\bar{\mu}_{\tilde{B}}(y) = \bar{\mu}_{\tilde{B}^1}(y) \vee \bar{\mu}_{\tilde{B}^2}(y) \vee \ldots \vee \bar{\mu}_{\tilde{B}^M}(y)$

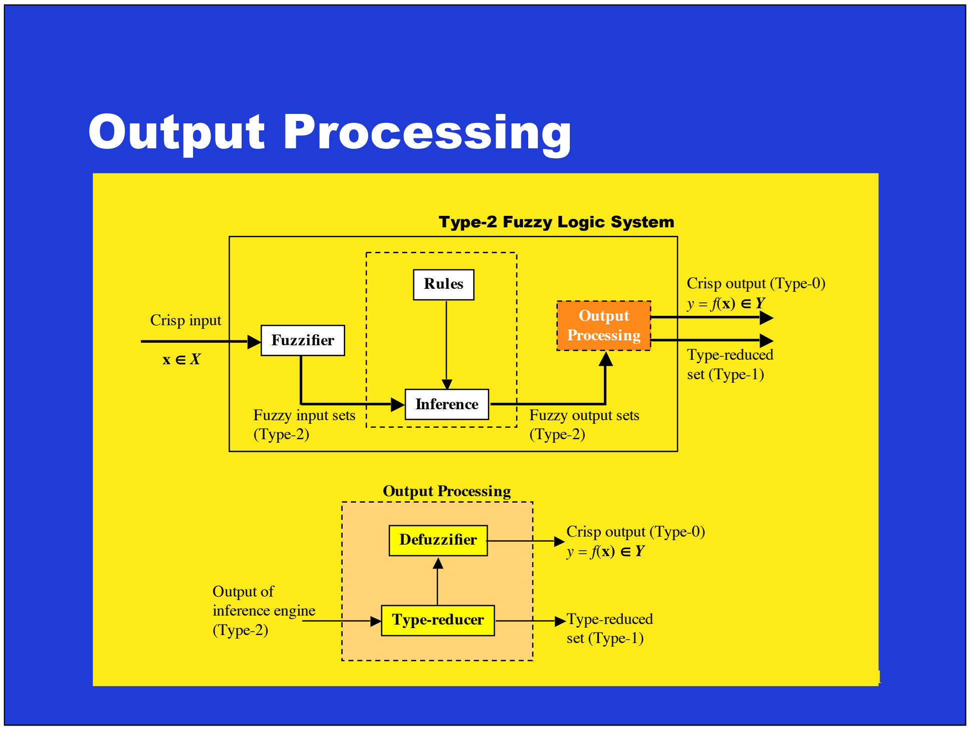
It is the formulas for the LMF and UMF that change.

In the last two equations on this slide, our earlier formula for the union of IT2 FSs, given on Slide 31, has been used repeatedly for the union of the M fired rule consequent sets.

A super-script rule notation has been included in order to distinguish each of the fired rule consequent sets.

Each of the super-scripted LMFs and UMFs can be computed using T1 FS mathematics as described in the earlier slides. The exact formulas to use depend on the kind of fuzzification and number of antecedents.

Again, for more details see Ref. 1.



It is now time to turn our attention to the *Output Processing* block.

Recall, that if this was a T1 FLS only defuzzification would be needed in order to map the fired rule consequent sets (or their union) into a number.

In an IT2 FLS, Output Processing consists of two stages, *type-reduction* and then *defuzzification*.

Type-reduction maps the fired rule consequent sets (or their union) into a T1 FS.

Type-Reduction: 1

- **Type-reduction begins with the interval T2 output of the inference engine of an IT2 FLS**

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Because there are many type-1 defuzzification methods, there are a comparable number of type-reduction methods. See Ref. 3 for discussions about some of them.

Type-Reduction: 1

- **Type-reduction begins with the interval T2 output of the inference engine of an IT2 FLS**
- **It reduces an IT2 FS to a T1 FS, which we call the type-reduced set**

63

Type-reduction (TR) is the process that leads to the *type-reduced set*.

Type-Reduction: 1

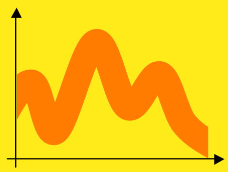
- **Type-reduction begins with the interval T2 output of the inference engine of an IT2 FLS**
- **It reduces an IT2 FS to a T1 FS, which we call the type-reduced set**
- **Type-reduction methods are "extended" versions of T1 defuzzification methods, and compute the centroid of the T2 output FS**

64

The word "extended" refers to Zadeh's *Extension Principle*. The earliest derivation of the type-reduced set relied on it.

As a result of the Representation Theorem, it is no longer necessary to use it.

Type-Reduction: 2

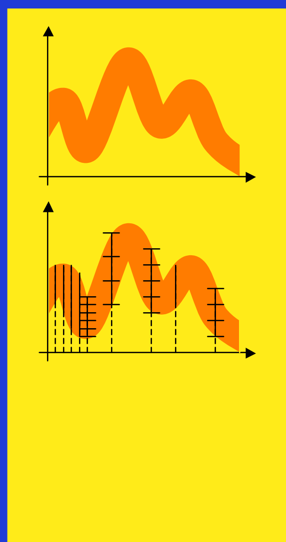


- FOU for an interval T2 FS

65

Shown on this slide is a generic FOU, representative of the FOU of unioned fired rule consequent sets.

Type-Reduction: 2



- FOU for an interval T2 FS
- Sampled primary variable and primary memberships leading to discretized FOU

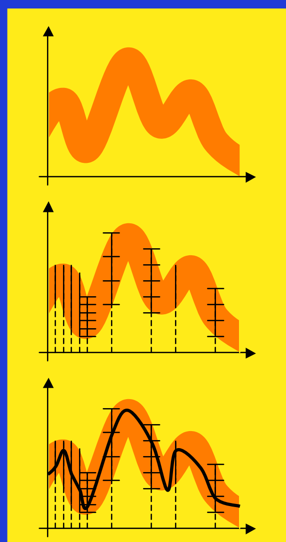
66

Whenever computations are performed for a FS using a digital computer the primary variable and the primary membership must be discretized (sampled).

Sampling does not have to be uniform.

Representative sampling is shown in the second figure of this slide.

Type-Reduction: 2



- FOU for an interval T2 FS
- Sampled primary variable and primary memberships leading to discretized FOU
- The solid curve is one *embedded* type-1 set

67

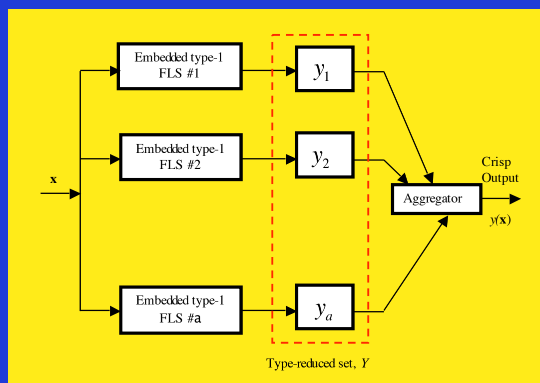
Recall that the FOU is the union of all of its embedded T1 FSs.

A representative embedded T1 FS is shown in the third figure on this slide.

Because each embedded set is a type-1 FS, its center of gravity (COG) can be computed using existing and well-known type-1 COG formulas.

Type-Reduction: 3

- We have already shown that an IT2 FLS can be interpreted as a collection of a very large number of embedded T1 FLSs



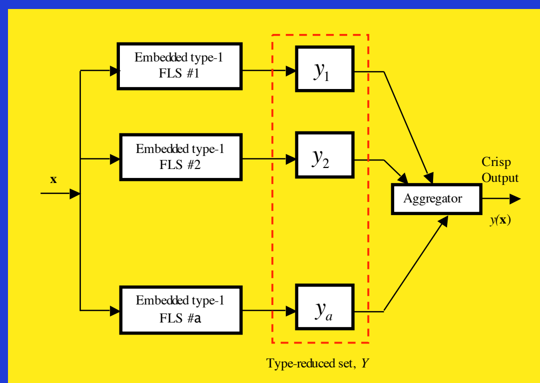
68

This figure provides an interpretation for an IT2 FLS that lets us easily communicate what an IT2 FLS is to others—a collection of a very large number of embedded T1 FLSs.

Of course, because the number of embedded T1 FSs for the FOU of an IT2 FS is astronomical, you would never think of actually computing using all of these embedded T1 FSs.

Type-Reduction: 3

- We have already shown that an IT2 FLS can be interpreted as a collection of a very large number of embedded T1 FLSs



- Compute the centroid for all embedded T1 FLSs
- Fuse the centroids using union

69

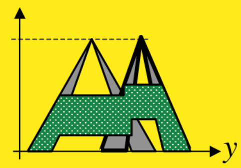
Contained within the dashed red rectangle are the defuzzified outputs of all of the embedded T1 FLSs.

Visualize them as points on a horizontal axis. These points will have a smallest and a largest value. This collection of points is the TR set.

Aggregation refers to defuzzification of the type-reduced set.

Type-Reduction: 4

- $C_{\tilde{B}} = 1/\{c_l, \ldots, c_r\}$



70

The inserted FOU shows the union (shaded green) of two IT2 fired rule consequent sets.

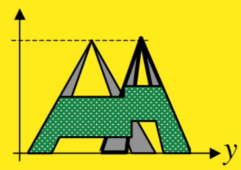
Picture in you mind the Representation Theorem applied to this FOU.

Even though there may be an astronomical number of embedded T1 FSs in the FOU, there will be embedded T1 FSs which have the smallest and largest centroid; hence, the centroid of an IT2 FS is an IT1 FS. The membership value for all points in this interval equals 1, because the secondary grades for an IT2 FS all equal 1.

The smallest centroid is denoted c_l and the largest centroid is denoted c_r.

Type-Reduction: 4

- $C_{\tilde{B}} = 1/\{c_l, \ldots, c_r\}$



- $c_l = \min\limits_{\forall \theta_i \in \left[\underline{\mu}_{\tilde{B}}(y_i), \bar{\mu}_{\tilde{B}}(y_i)\right]} \left[\sum_{i=1}^{N} y_i \theta_i \Big/ \sum_{i=1}^{N} \theta_i \right]$

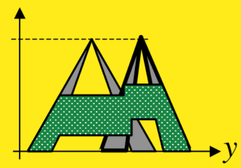
71

c_l is given by the second formula on this slide. It is a mathematical version of what has just been stated on Slide 70.

What makes the computation of c_l challenging are the interval sets that appear in both its numerator and denominator.

Because the same interval sets appear in numerator and denominator, standard interval arithmetic cannot be used to compute c_l.

Type-Reduction: 4



- $C_{\tilde{B}} = 1/\{c_l, \ldots, c_r\}$
- $c_l = \min\limits_{\forall \theta_i \in \left[\underline{\mu}_{\tilde{B}}(y_i), \bar{\mu}_{\tilde{B}}(y_i)\right]} \left[\sum_{i=1}^{N} y_i \theta_i \Big/ \sum_{i=1}^{N} \theta_i \right]$
- $c_r = \max\limits_{\forall \theta_i \in \left[\underline{\mu}_{\tilde{B}}(y_i), \bar{\mu}_{\tilde{B}}(y_i)\right]} \left[\sum_{i=1}^{N} y_i \theta_i \Big/ \sum_{i=1}^{N} \theta_i \right]$

72

c_r is given by the third formula on this slide. It is also a mathematical version of what has just been stated on Slide 70.

What makes the computation of c_r also challenging are the interval sets that appear in both its numerator and denominator.

Because the same interval sets appear in numerator and denominator, standard interval arithmetic also cannot be used to compute c_r.

Type-Reduction: 5

- $$c_l = \frac{\sum_{i=1}^{L} y_i \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} y_i \underline{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{B}}(y_i)}$$

73

Karnik and Mendel have shown that c_l can be expressed as shown on this slide.

Observe that the first summation in both the numerator and denominator use the UMF of the combined fired rule output set, whereas the second summation in both the numerator and denominator use the LMF of the combined fired rule output set.

The key is to determine the *switch point L*.

Type-Reduction: 5

- $$c_l = \frac{\sum_{i=1}^{L} y_i \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} y_i \underline{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{B}}(y_i)}$$
- $$c_r = \frac{\sum_{i=1}^{R} y_i \underline{\mu}_{\tilde{B}}(y_i) + \sum_{i=R+1}^{N} y_i \bar{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{B}}(y_i) + \sum_{i=R+1}^{N} \bar{\mu}_{\tilde{B}}(y_i)}$$

Karnik and Mendel have also shown that c_r can be expressed as shown on this slide.

Observe that the first summations in both the numerator and denominator use the LMF of the combined fired rule output set, whereas the second summation in both the numerator and denominator use the UMF of the combined fired rule output set.

The key is to determine the switch point R.

In general, $R \neq L$.

Type-Reduction: 5

- $$c_l = \frac{\sum_{i=1}^{L} y_i \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} y_i \underline{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{B}}(y_i)}$$
- $$c_r = \frac{\sum_{i=1}^{R} y_i \underline{\mu}_{\tilde{B}}(y_i) + \sum_{i=R+1}^{N} y_i \bar{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{B}}(y_i) + \sum_{i=R+1}^{N} \bar{\mu}_{\tilde{B}}(y_i)}$$

- **Switch points *L* and *R* can be computed using the Karnik-Mendel algorithms**

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Switch points L and R can be computed using algorithms that were developed by Karnik and Mendel. There are no closed form formulas for them.

The Karnik-Mendel (KM) algorithms are iterative and each of the two algorithms can be run in parallel.

The original references for the KM algorithms are: N. Karnik and J. M. Mendel, "Centroid of a Type-2 Fuzzy Set," *Information Sciences*, vol. 132, pp. 195-220, 2001. See , also Ref, 3, Chapter 9.

Type-Reduction: 5

- $$c_l = \frac{\sum_{i=1}^{L} y_i \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} y_i \underline{\mu}_{\tilde{B}}(y_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{B}}(y_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{B}}(y_i)}$$
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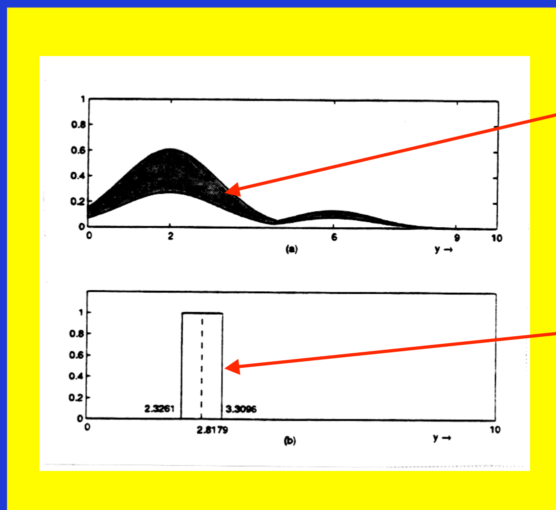
- **Switch points *L* and *R* can be computed using the Karnik-Mendel algorithms**
- **Monotonic and super-exponentially convergent**

Recently, it was proved that these algorithms converge to their exact answer monotonically and super-exponentially fast (really fast!).

The algorithms are very easy to derive and implement, and they are very widely used by practitioners of IT2 FLSs.

See, also, Ref. 1.

Type-Reduction: 6



- FOU at output of an IT2-FLS
- Type-reduced set

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The upper figure is a generic FOU at the output of an IT2 FLS.

The KM algorithms were used to compute the left and right end-points of the type-reduced set that is shown in the bottom figure on this slide.

Again, the type-reduced set is an IT1 FS. Its center of gravity is shown dashed.

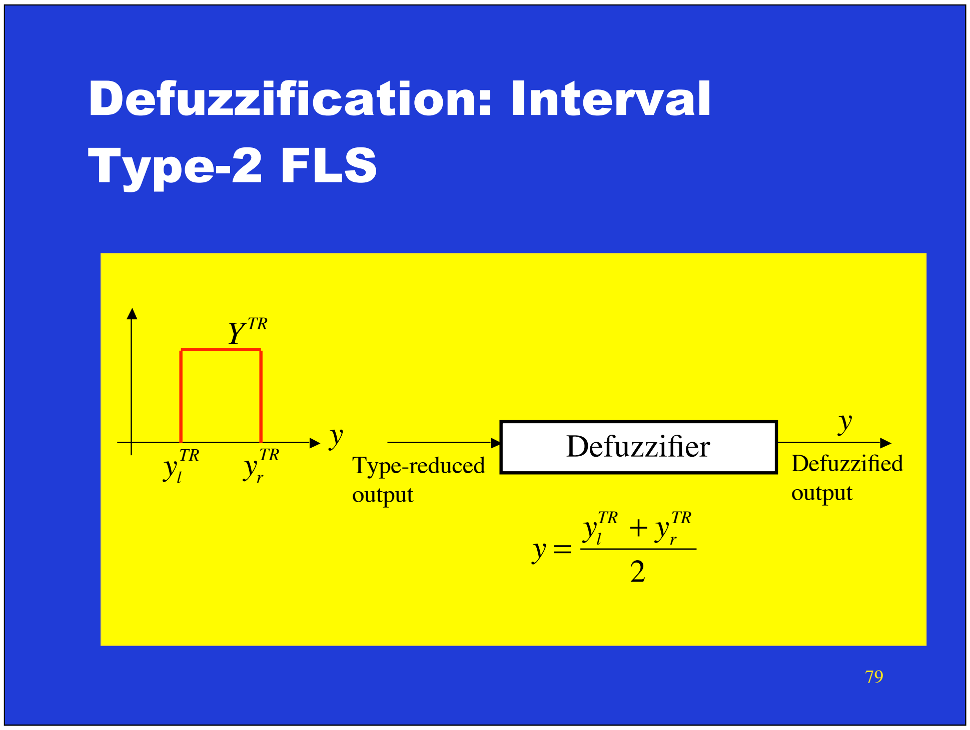
Intuitively, we expect the width of the type-reduced set to increase as the area of the FOU increases and to decrease as the area of the FOU decreases.

In fact, if all uncertainty disappears then the FOU becomes a curve—a T1 FS—and the type-reduced set is a single number, i.e. the IT2 FLS reduces to a T1 FLS.

Defuzzification

- We defuzzify the type-reduced set to get a crisp output from a type-2 FLS
- The most natural way of doing this seems to be by finding the centroid of the type-reduced set

No additional comments are needed about this slide.

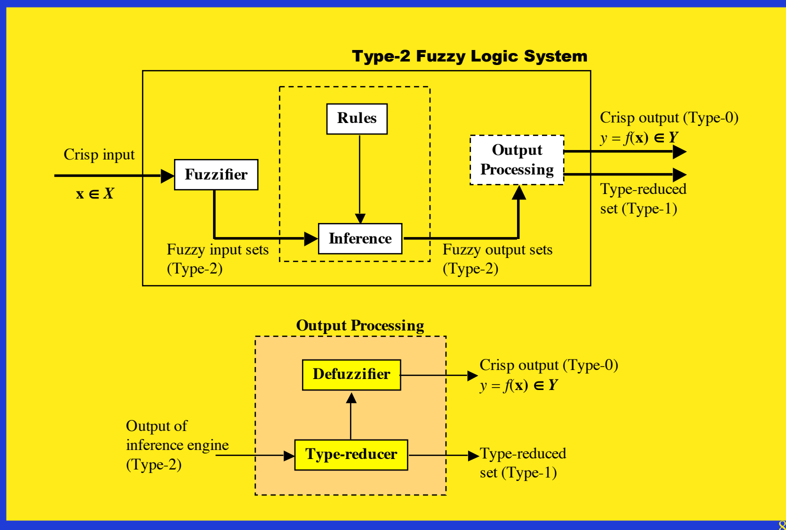


A representative type-reduced set is shown at the left.

Once the type-reduced set has been computed, defuzzification is trivial. Just compute the average of the left and right end-points of the type-reduced set.

The formula for the defuzzified output is shown on this slide.

Summary



All of the computations needed to implement an IT2 FLS have now been presented.

Only T1 FS mathematics has been used to do this.

Pretty simple :).

Feel free to use this presentation and Ref. 1 to teach about IT2 FLSs.

Conclusions

- We have shown that all of the results that are needed to implement an IT2 FLS can be obtained using T1 FS mathematics

Conclusions

- We have shown that all of the results that are needed to implement an IT2 FLS can be obtained using T1 FS mathematics
- The key to doing this is the Representation Theorem for an IT2 FS that lets us express the FOU as a union of T1 wavy-slices

Conclusions (Cont'd.)

- Although a huge number of T1 FSs appear in the derivations, only two are needed in the final calculations, and they are associated with lower and upper MFs of the FOUs of all IT2 FSs

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Conclusions (Cont'd.)

- Although a huge number of T1 FSs appear in the derivations, only two are needed in the final calculations, and they are associated with lower and upper MFs of the FOUs of all IT2 FSs
- The third dimension of general T2 FSs is not needed for an IT2 FS
- It is only the FOU that is needed for an IT2 FS

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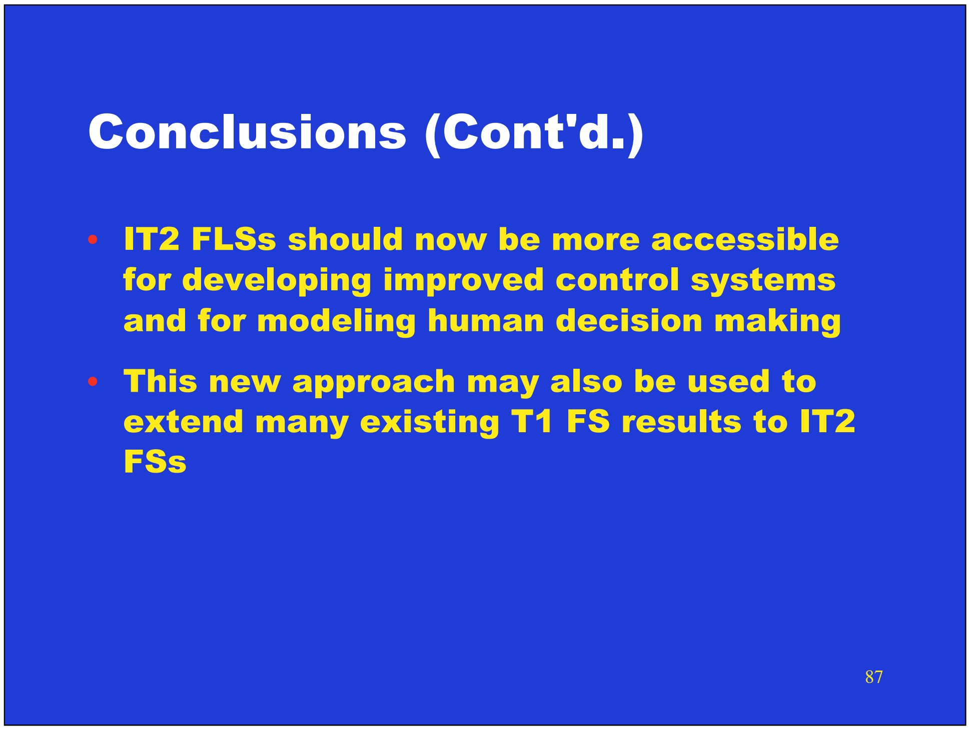
Conclusions (Cont'd.)

- IT2 FLSs should now be more accessible for developing improved control systems and for modeling human decision making



The Representation Theorem is the starting point for establishing such things as similarity, fuzziness, skew, etc. for IT2 FSs.

It leads to solution-structures quickly.

Then, practical algorithms have to be established to compute the solutions (e.g., as in the centroid of an IT2 FS, and the KM algorithms).

Conclusions (Cont'd.)

- IT2 FLSs should now be more accessible for developing improved control systems and for modeling human decision making
- This new approach may also be used to extend many existing T1 FS results to IT2 FSs
- Whether or not comparable results can be obtained for general T2 FSs is also an open question

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As mentioned earlier the Representation Theorem was initially developed for a general T2 FS.

The possibility exists that its use for such sets and general T2 FLSs will also lead to useful results.

Much work remains to be done.

References

- J. M. Mendel, R. I. John and F. Liu, "Interval type-2 fuzzy logic systems made simple, *IEEE Trans. on Fuzzy Systems*, 2006, in press.
- J. M. Mendel and R. I. John, "Type-2 fuzzy sets made simple," *IEEE Trans. on Fuzzy Systems*, vol. 10, pp. 117-127, April 2002.
- J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*, Prentice-Hall, 2001.

Announcement

- **"Standard Background Material About Interval Type-2 Fuzzy Logic Systems that can be Used by All Authors," J. M. Mendel, H. Hagras and R. I. John**
  - **http://ieee-cis.org/_files/standards.t2.win.pdf**
  - **MAC version and non-pdf versions also available on the web-site**
- **As a service to the fuzzy logic community we give all authors permission to use any or all of this material in their articles, as long as they reference this Standard.**

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Many people are now publishing articles about IT2 FLSs.

Because most people still do not know about IT2 FSs and FLSs, these articles require some background material about them.

No sense re-inventing the wheel.

Feel free to cut and paste from this on-line article, which is available in different formats.