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Introduction

1. Problem Formulation and Definitions of Optimality

Finding Efficient Solutions – Scalarization

2. The Idea of Scalarization
3. Scalarization Techniques and Their Properties

Multiobjective Linear Programming

4. Formulation and the Fundamental Theorem
5. Solving MOLPs in Decision and Objective Space

Multiobjective Combinatorial Optimization

6. Definitions Revisited and Characteristics
7. Solution Methods

Applications

6. Commercials
Overview

1. Introduction
   - Problem Formulation and Definitions of Optimality

2. Finding Efficient Solutions – Scalarization
   - The Idea of Scalarization
   - Scalarization Techniques and Their Properties

3. Multiobjective Linear Programming
   - Formulation and the Fundamental Theorem
   - Solving MOLPs in Decision and Objective Space

4. Multiobjective Combinatorial Optimization
   - Definitions Revisited and Characteristics
   - Solution Methods

5. Applications

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Mathematical Formulation

\[ \begin{align*}
\min f(x) \\
{\text{subject to}} \quad g(x) &\leq 0 \\
x &\in \mathbb{R}^n
\end{align*} \]

\[ x \in \mathbb{R}^n \quad \rightarrow \quad n \text{ variables, } i = 1, \ldots, n \]

\[ g : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \rightarrow \quad m \text{ constraints, } j = 1, \ldots, m \]

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^p \quad \rightarrow \quad p \text{ objective functions, } k = 1, \ldots, p \]
Mathematical Formulation

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\begin{align*}
\min & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0 \\
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\end{align*}
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**Mathematical Formulation**

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\begin{align*}
\min f(x) \\
\text{subject to } g(x) & \leq 0 \\
\quad x & \in \mathbb{R}^n
\end{align*}
\]
Feasible Sets

- \( X = \{ x \in \mathbb{R}^n : g(x) \leq 0 \} \)
  feasible set in decision space

- \( Y = f(X) = \{ f(x) : x \in X \} \)
  feasible set in objective space
Feasible Sets

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- \( Y = f(X) = \{ f(x) : x \in X \} \)
  feasible set in objective space
Notation

- $y^1 \leq y^2 \iff y_k^1 \leq y_k^2$ for $k = 1, \ldots, p$
- $y^1 < y^2 \iff y_k^1 < y_k^2$ for $k = 1, \ldots, p$
- $y^1 \leq y^2 \iff y^1 \leq y^2$ and $y^1 \neq y^2$
- $\mathbb{R}^p_\geq = \{ y \in \mathbb{R}^p : y \geq 0 \}$
- $\mathbb{R}^p_> = \{ y \in \mathbb{R}^p : y > 0 \}$
- $\mathbb{R}^p_\geq = \{ y \in \mathbb{R}^p : y \geq 0 \}$
Notation

- \( y^1 \leq y^2 \iff y^1_k \leq y^2_k \) for \( k = 1, \ldots, p \)
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- \( y^1 \leq y^2 \iff y^1 \leq y^2 \) and \( y^1 \neq y^2 \)
- \( \mathbb{R}^p_{\geq} = \{ y \in \mathbb{R}^p : y \geq 0 \} \)
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Notation

- $y^1 \leq y^2 \iff y^1_k \leq y^2_k$ for $k = 1, \ldots, p$
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Lexicographic Optimality

- Individual minima
  \[ f_k(\hat{x}) \leq f_k(x) \text{ for all } x \in X \]
- Lexicographic optimality (1)
  \[ f(\hat{x}) \leq_{\text{lex}} f(x) \text{ for all } x \in X \]
- Lexicographic optimality (2)
  \[ f^\pi(\hat{x}) \leq_{\text{lex}} f^\pi(x) \text{ for all } x \in X \]
  and some permutation \( f^\pi \) of \((f_1, \ldots, f_p)\)
Lexicographic Optimality

- **Individual minima**
  \[ f_k(\hat{x}) \leq f_k(x) \text{ for all } x \in X \]

- **Lexicographic optimality (1)**
  \[ f(\hat{x}) \leq_{\text{lex}} f(x) \text{ for all } x \in X \]

- **Lexicographic optimality (2)**
  \[ f^\pi(\hat{x}) \leq_{\text{lex}} f^\pi(x) \text{ for all } x \in X \]
  and some permutation \( f^\pi \) of \( (f_1, \ldots, f_p) \)
Lexicographic Optimality

- Individual minima
  \[ f_k(\hat{x}) \leq f_k(x) \text{ for all } x \in X \]
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  and some permutation \( f^\pi \) of \((f_1, \ldots, f_p)\)
(Weakly) Efficient Solutions

- Weakly efficient solutions $X_{WE}$
  
  There is no $x$ with $f(x) < f(\tilde{x})$
  
  $f(\tilde{x})$ is weakly nondominated

  $Y_{wN} := f(X_{wN})$

- Efficient solutions $X_E$
  
  There is no $x$ with $f(x) \leq f(\tilde{x})$
  
  $f(\tilde{x})$ is nondominated

  $Y_N := f(X_E)$
(Weakly) Efficient Solutions

- Weakly efficient solutions $X_{WE}$
  There is no $x$ with $f(x) < f(\hat{x})$
  $f(\hat{x})$ is weakly nondominated
  $Y_{wN} := f(X_{wN})$

- Efficient solutions $X_E$
  There is no $x$ with $f(x) \leq f(\hat{x})$
  $f(\hat{x})$ is nondominated
  $Y_N := f(X_E)$
**Properly Efficient Solutions**

- Properly efficient solutions $X_{pE}$
  - $\hat{x}$ is efficient
  - There is $M > 0$ such that for each $k$ and $x$ with $f_k(x) < f_k(\hat{x})$ there is $l$ with $f_l(\hat{x}) < f_l(x)$ and
    \[
    \frac{f_k(\hat{x}) - f_k(x)}{f_l(x) - f_l(\hat{x})} \leq M
    \]

$f(\hat{x})$ is properly nondominated

$Y_{pN} := f(X_{pE})$
Properly Efficient Solutions

- Properly efficient solutions $X_{pE}$
  - $\hat{x}$ is efficient
  - There is $M > 0$ such that for each $k$ and $x$ with $f_k(x) < f_k(\hat{x})$ there is $l$ with $f_l(\hat{x}) < f_l(x)$ and
    \[
    \frac{f_k(\hat{x}) - f_k(x)}{f_l(x) - f_l(\hat{x})} \leq M
    \]
  - $f(\hat{x})$ is properly nondominated
  - $Y_{pN} := f(X_{pE})$
Properly Efficient Solutions

- Properly efficient solutions $X_{pE}$
  - $\hat{x}$ is efficient
  - There is $M > 0$ such that for each $k$ and $x$ with $f_k(x) < f_k(\hat{x})$ there is $l$ with $f_l(\hat{x}) < f_l(x)$ and
    \[
    \frac{f_k(\hat{x}) - f_k(x)}{f_l(x) - f_l(\hat{x})} \leq M
    \]
  - $f(\hat{x})$ is properly nondominated

$Y_{pN} := f(X_{pE})$
Existence

- $Y_N \neq \emptyset$ if for some $y^0 \in Y$ the section \((y^0 - \mathbb{R}_{\geq}) \cap Y \neq \emptyset\) is compact
- $X_E \neq \emptyset$ if $X$ is compact and $f$ is $(\mathbb{R}_{\geq}$-semi-)continuous
Existence

- \( Y_N \neq \emptyset \) if for some \( y^0 \in Y \) the section \( (y^0 - \mathbb{R}_\geq) \cap Y \neq \emptyset \) is compact
- \( X_E \neq \emptyset \) if \( X \) is compact and \( f \) is \((\mathbb{R}_\geq\text{-semi-})\)continuous
Relationships of Solution Sets

\[ X_{pE} \subseteq X_E \subseteq X_{wE} \]
\[ Y_{pN} \subseteq Y_N \subseteq Y_{wN} \]

It is possible that
\[ Y_N = Y \text{ but } Y_{pN} = \emptyset \]
\[ Y = \left\{ (y_1, y_2) : y_2 = \frac{1}{y_1}, y_1 < 0 \right\} \]
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Finding Efficient Solutions – Scalarization
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Problem Formulation and Definitions of Optimality

Relationships of Solution Sets

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\[ Y = \left\{ (y_1, y_2) : y_2 = \frac{1}{y_1}, y_1 < 0 \right\} \]
Ideal and Nadir Points

**Ideal point** $y^I$
- $y_k^I = \min \{y_k : y \in Y\}$

**Nadir point** $y^N$
- $y_k^N = \min \{y_k : y \in Y_N\}$

**Anti-ideal point** $y^A$
- $y_k^A = \max \{y_k : y \in Y\}$

**Utopia point** $y^U$
- $y_k^U = y_k^I - \varepsilon_k$
Ideal and Nadir Points

Ideal point $y^I$
- $y^I_k = \min\{y_k : y \in Y\}$

Nadir point $y^N$
- $y^N_k = \min\{y_k : y \in Y_N\}$

Anti-ideal point $y^{AI}$
- $y^{AI}_k = \max\{y_k : y \in Y\}$

Utopia point $y^U$
- $y^U_k = y^I_k - \varepsilon_k$
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Problem Formulation and Definitions of Optimality
Ideal and Nadir Points

Ideal point $y^I$
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Anti-ideal point $y^{AI}$
- $y^{AI}_k = \max\{y_k : y \in Y\}$

Utopia point $y^U$
- $y^U_k = y^I_k - \varepsilon_k$
General Assumptions

- $X_E$ is non-empty
- $y^I \neq y^N$
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Principle of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations

- Correctness: Optimal solutions are (weakly, properly) efficient
- Completeness: All (weakly, properly) efficient solutions can be found
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Three Ideas for Scalarization

- Aggregate objectives
- Convert objectives to constraints
- Minimize distance to ideal point
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Let $\lambda \geq 0$

$$\min \left\{ \sum_{k=1}^{p} \lambda_k f_k(x) : x \in X \right\} \quad (1)$$
The Weighted Sum Method

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The Weighted Sum Method

Let \( \lambda \geq 0 \)

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\min \left\{ \sum_{k=1}^{p} \lambda_k f_k(x) : x \in X \right\}
\]  \hspace{1cm} (1)
The Weighted Sum Method

Let $\lambda \geq 0$

$$\min \left\{ \sum_{k=1}^{p} \lambda_k f_k(x) : x \in X \right\}$$

(1)
The Weighted Sum Method: Results

Theorem

Let \( \hat{x} \) be an optimal solution of (1).

1. If \( \lambda \geq 0 \) then \( \hat{x} \in X_{wE} \).
2. If \( \lambda \geq 0 \) and \( f(\hat{x}) \) is unique then \( \hat{x} \in X_E \).
3. If \( \lambda > 0 \) then \( \hat{x} \in X_{pE} \).

Proof.

1. By contradiction
2. By contradiction
3. Construct \( M \) so that larger tradeoff would contradict optimality of \( \hat{x} \)
The Weighted Sum Method: Results

**Theorem**

Let \( \hat{x} \) be an optimal solution of (1).

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**Proof.**

1. By contradiction
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3. Construct \( M \) so that larger tradeoff would contradict optimality of \( \hat{x} \)
The Weighted Sum Method: Results

Theorem (Geoffrion 1968)

Let $X$ and $f$ be such that $Y = f(X)$ is convex.

1. If $\hat{x} \in X_{WE}$ then there is $\lambda \geq 0$ such that $\hat{x}$ is an optimal solution to (1).

2. If $\hat{x} \in X_{PE}$ then there is $\lambda > 0$ such that $\hat{x}$ is an optimal solution to (1).

Proof.

1. Apply separation theorem to $(Y + \mathbb{R}^p_{\subseteq} - \hat{y})$ and $-\mathbb{R}^p$

2. Apply separation theorem to $(\text{cl cone } Y + \mathbb{R}^p_{\subseteq} - \hat{y})$ and $-\mathbb{R}^p$

   to show that weights are positive

3. If $X$ and $f$ are convex use properties of convex functions
The Weighted Sum Method: Results

Theorem (Geoffrion 1968)

Let $X$ and $f$ be such that $Y = f(X)$ is convex.

1. If $\hat{x} \in X_{wE}$ then there is $\lambda \geq 0$ such that $\hat{x}$ is an optimal solution to (1).
2. If $\hat{x} \in X_{pE}$ then there is $\lambda > 0$ such that $\hat{x}$ is an optimal solution to (1).

Proof.

1. Apply separation theorem to $(Y + \mathbb{R}_\geq^p - \hat{y})$ and $-\mathbb{R}_>^p$
2. Apply separation theorem to $(\text{cl cone } Y + \mathbb{R}_\geq^p - \hat{y})$ and $-\mathbb{R}_>^p$
   to show that weights are positive
3. If $X$ and $f$ are convex use properties of convex functions
Nondominated and Properly Nondominated Points

\[ X_{sE} := \{ x \in X : x \text{ is optimal solution to (1) for some } \lambda > 0 \} \]

**Theorem**

*Assume that* \( Y + \mathbb{R}^p \) *is closed and convex. Then*

\[ Y_{pN} = f(X_{sE}) \subseteq Y_N \subseteq \text{closure } f(X_{sE}) = \text{closure } Y_{pN} \]
Nondominated and Properly Nondominated Points

\[ X_{sE} := \{ x \in X : x \text{ is optimal solution to (1) for some } \lambda > 0 \} \]

**Theorem**

Assume that \( Y + \mathbb{R}^p \) is closed and convex. Then

\[ Y_{pN} = f(X_{sE}) \subseteq Y_N \subseteq \text{closure } f(X_{sE}) = \text{closure } Y_{pN} \]
Nondominated and Properly Nondominated Points

\[ X_{SE} := \{ x \in X : x \text{ is optimal solution to (1) for some } \lambda > 0 \} \]

**Theorem**

Assume that \( Y + \mathbb{R}^p \) is closed and convex. Then

\[ Y_{pN} = f(X_{SE}) \subseteq Y_N \subseteq \text{closure } f(X_{SE}) = \text{closure } Y_{pN} \]
Supported efficient solutions are efficient solutions with $f(x)$ on the convex hull of $Y$.

$$\text{conv}(f(X)) + \mathbb{R}^2_{\geq}$$
Let $\varepsilon \in \mathbb{R}^p$

$$\min f_l(x) \quad \text{s.t.} \quad f_k(x) \leq \varepsilon_k \quad k \neq l \quad (2)$$
$$g_j(x) \leq 0 \quad j = 1, \ldots, m$$
The $\varepsilon$-constraint Method

Let $\varepsilon \in \mathbb{R}^p$

$$\begin{align*}
&\min f_l(x) \\
&\text{s.t. } f_k(x) \leq \varepsilon_k \quad k \neq l \\
&\quad g_j(x) \leq 0 \quad j = 1, \ldots, m
\end{align*}$$

(2)
The $\varepsilon$-constraint Method

Theorem (Chankong and Haimes 1983)

1. If $\hat{x}$ is an optimal solution to (2) then $\hat{x} \in X_{wE}$.
2. If $\hat{x}$ is an optimal solution to (2) and $f(\hat{x})$ is unique then $\hat{x} \in X_E$.
3. $\hat{x} \in X_E$ if and only if there is $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution to (2) for all $l = 1, \ldots, p$.

Proof.

By contradiction and using $\varepsilon_l = f_l(\hat{x})$
### The $\varepsilon$-constraint Method

**Theorem (Chankong and Haimes 1983)**

1. If $\hat{x}$ is an optimal solution to (2) then $\hat{x} \in X_{WE}$.
2. If $\hat{x}$ is an optimal solution to (2) and $f(\hat{x})$ is unique then $\hat{x} \in X_E$.
3. $\hat{x} \in X_E$ if and only if there is $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution to (2) for all $l = 1, \ldots, p$.

**Proof.**

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Multiobjective Optimization
The $\varepsilon$-constraint Method

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2. If $\hat{x}$ is an optimal solution to (2) and $f(\hat{x})$ is unique then $\hat{x} \in X_E$.
3. $\hat{x} \in X_E$ if and only if there is $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution to (2) for all $l = 1, \ldots, p$.

**Proof.**

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The $\varepsilon$-constraint Method

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1. If $\hat{x}$ is an optimal solution to (2) then $\hat{x} \in X_{WE}$.

2. If $\hat{x}$ is an optimal solution to (2) and $f(\hat{x})$ is unique then $\hat{x} \in X_E$.

3. $\hat{x} \in X_E$ if and only if there is $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution to (2) for all $l = 1, \ldots, p$.

Proof. By contradiction and using $\varepsilon_l = f_l(\hat{x})$
The Hybrid Method

Let $\lambda \in \mathbb{R}_+^p$ and $\varepsilon \in \mathbb{R}^p$

$$\min \sum_{k=1}^p \lambda_k f_k(x)$$ (3)

s.t. $f_k(x) \leq \varepsilon_k \quad k = 1, \ldots, p$

$$g_j(x) \leq 0 \quad j = 1, \ldots, m$$

Theorem (Guddat et al. 1985)

$\hat{x}$ is efficient if and only if there are $\lambda \geq 0$ and $\varepsilon$ such that $\hat{x}$ is an optimal solution to (3).
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Let $\lambda \in \mathbb{R}^p_\geq$ and $\varepsilon \in \mathbb{R}^p$

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subject to:

$$f_k(x) \leq \varepsilon_k \quad k = 1, \ldots, p$$

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Compromise Solutions

Let $\lambda \in \mathbb{R}^p_+$ and $1 \leq q < \infty$

$$\min_{x \in X} \left( \sum_{k=1}^{p} \lambda_k (f_k(x) - y_k^l)^q \right)^{\frac{1}{q}}$$  \hspace{1cm} (4)

Let $\lambda \in \mathbb{R}^p_+$

$$\min_{x \in X} \max_{k=1,\ldots,p} \lambda_k (f_k(x) - y_k^l)$$  \hspace{1cm} (5)
Compromise Solutions

Let \( \lambda \in \mathbb{R}^p \geq \) and \( 1 \leq q < \infty \)

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\min_{x \in X} \left( \sum_{k=1}^{p} \lambda_k (f_k(x) - y_k^l)^q \right)^{\frac{1}{q}}
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Let \( \lambda \in \mathbb{R}^p \geq \)

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Let $\lambda \in \mathbb{R}^p_\geq$ and $1 \leq q < \infty$

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Theorem

1. If \( \hat{x} \) is a unique optimal solution to (4) or if \( \lambda > 0 \) then \( \hat{x} \) is efficient.

2. If \( \hat{x} \) is an optimal solution to (5) and \( \lambda > 0 \) then \( \hat{x} \) is weakly efficient.

3. If \( \hat{x} \) is a unique optimal solution to (5) and \( \lambda > 0 \) then \( \hat{x} \) is efficient.
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Compromise Solutions

- For $q = 1$ (4) is the weighted sum scalarization
- If $y^I$ is replaced by $y^U$ in (4) stronger results follow
  Solutions obtained are properly efficient, and $Y_N$ is contained in the closure of the set of all solutions obtained (Sawaragi et al. 1985)
- True without convexity assumption, value of $q$ indicates “degree of non-convexity”
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More General Concepts

- $l_q$ norms can be replaced by more general distance functions
- Ideal point can be replaced by a reference point and the distance function by a ((strictly, strongly) increasing) achievement function $\mathbb{R}^p \rightarrow \mathbb{R}$ (Wierzbicki 1986)

\[
\min \{ s_R(f(x)) : x \in X \} \\
s_R(y) = \max_{k=1,...,p} \{ \lambda_k (y_k - y_k^R) \} + \rho \sum_{k=1}^{p} (y_k - y_k^R)
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   • Problem Formulation and Definitions of Optimality

2. Finding Efficient Solutions – Scalarization
   • The Idea of Scalarization
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   • Formulation and the Fundamental Theorem
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MOLP Formulation

- $f(x) = Cx$ where $C \in \mathbb{R}^{p \times n}$
- Constraints $Ax = b$ where $A \in \mathbb{R}^{m \times n}$
- Nonnegativity $x \geq 0$

$$\min \{ Cx : Ax = b, x \geq 0 \}$$

- $X$ and $Y$ are convex
- Weakly and properly efficient solutions are found by weighted sum method
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MOLP Example

\[ C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \]
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Theorem (Isermann 1974)

A feasible solution $\hat{x} \in X$ is efficient if and only if there is $\lambda > 0$ such that $\lambda^T C\hat{x} \leq \lambda^T x$ for all $x \in X$.

Proof.

- If $\hat{x}$ is efficient, $\max\{e^T z : Ax = b, Cx + lz = C\hat{x}; x, z \geq 0\}$ has optimal solution $\hat{z} = 0$.
- By duality $\min\{u^T b + w^T C\hat{x} : u^T A = w^T C \geq 0 : w \geq e\}$ has optimal solution $(\hat{u}, \hat{w})$ with $\hat{u}^T b = -\hat{w}^T C\hat{x}$.
- $\hat{u}$ is optimal solution of $\min\{u^T b : u^T A \geq -\hat{w}^T C\}$.
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- $\hat{x}$ is an optimal solution of this LP.
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Multiobjective Simplex Algorithm

- Phase I: Feasibility
  MOLP is feasible if and only if
  \[
  \min \{ e^T z : Ax + lz = b; x, z \geq 0 \}
  \]
  has optimal value 0
  Let \((x^0, \hat{z})\) be optimal solution

- Phase II: First efficient solution
  If \(\min \{ u^T b + w^T Cx^0 : u^T A = w^T C \geq 0; w \geq e \}\)
  is infeasible then \(X_E = \emptyset\)
  Let \(\hat{w}\) be optimal solution
  Optimal solution \(\hat{x}\) to \(\min \{ \hat{w}^T Cx : Ax = b, x \geq 0 \}\) is efficient

- Phase III: Explore efficient solutions by identifying entering variables
Multiobjective Simplex Algorithm

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  MOLP is feasible if and only if
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Multiobjective Simplex Algorithm

- Reduced cost matrix
  \[ R := (C - C_B A_B^{-1} A)_N \]

- \( x_j \) is efficient nonbasic variable if there is \( \lambda > 0 \) such that
  \[ \lambda^T R \geq 0 \text{ and } \lambda^T r^j = 0 \]

- At every efficient basis there exists an efficient nonbasic variable and every feasible pivot leads to another efficient basis

Theorem (Evans and Steuer 1973)

Nonbasic variable \( x_j \) is efficient if and only if the LP

\[ \max\{e^T v : Rz - r^j \delta + lv = 0, z\delta, v \geq 0\} \]

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Theorem

- The set of all efficient bases is connected by pivots with efficient entering variables.
- The set of all efficient extreme points of $X_E$ is connected by efficient edges.
- The set of all nondominated extreme points of $Y_N$ is connected by nondominated edges.
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• Phase II: Optimal weight
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• Phase II: First efficient solution
  \( x^2 = (0, 3) \)

• Phase III: Efficient entering variables \( s^1, x^2 \)

• Phase III: Efficient solutions
  \( x^1 = (0, 0), x^3 = (3, 3) \)

• Phase III: No more efficient entering variables
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Solving MOLPs in Objective Space

(Benson 1998)
- Degeneracy causes problems for simplex algorithm
- Decisions based on objective function values
- Usually $\text{dim } Y \leq p \ll \text{dim } X$
- Assume $X$ is bounded

**Theorem (Benson 1998)**

The dimension of $Y + \mathbb{R}^p_\geq$ is $p$ and $(Y + \mathbb{R}^p_\geq)_N = Y_N$.

\[ Y' := (Y + \mathbb{R}^p_\geq) \cap (y^{AI} - \mathbb{R}^p_\geq) \]
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$$Y' := (Y + \mathbb{R}_\geq^p) \cap (y^{AI} - \mathbb{R}_\geq^p)$$
Let \( \hat{\rho} \in \text{int } Y' \)

Let \( \hat{y} \) be solution of
\[
\min \{ e^T y : y \in Y \}
\]

Simplex \( S^0 \) such that \( Y' \subseteq S^0 \)
defined by axes-parallel hyperplanes and supporting hyperplane with normal \( e \) at \( \hat{y} \)

While \( S^k \neq Y' \)

- Find vertex \( y^k \) of \( S^{k-1} \) with \( s^k \notin Y' \)
- Find \( \alpha^k > 0 \) such that \( \alpha y^k + (1 - \alpha) \hat{\rho} \) is on the boundary of \( Y' \)
- Find supporting hyperplane to \( Y' \) through boundary point

![Diagram](image-url)
Let $\hat{p} \in \text{int } Y'$
Let $\hat{y}$ be solution of $\min\{e^T y : y \in Y\}$
Simplex $S^0$ such that $Y' \subseteq S^0$
defined by axes-parallel hyperplanes and supporting hyperplane with normal $e$ at $\hat{y}$
While $S^k \neq Y'$
  Find vertex $y^k$ of $S^{k-1}$ with $s^k \notin Y'$
  Find $\alpha^k > 0$ such that $\alpha y^k + (1 - \alpha)\hat{p}$ is on the boundary of $Y'$
  Find supporting hyperplane to $Y'$ through boundary point

Matthias Ehrgott  Multiobjective Optimization
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Mathematical Formulation

\[ \min z(x) = Cx \]
subject to \( Ax = b \)
\[ x \in \{0, 1\}^n \]

\[ x \in \{0, 1\}^n \quad \rightarrow \quad n \text{ variables, } i = 1, \ldots, n \]
\[ C \in \mathbb{Z}^{p \times n} \quad \rightarrow \quad p \text{ objective functions, } k = 1, \ldots, p \]
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Feasible Sets

- \( X = \{ x \in \{0, 1\}^n : Ax = b \} \)
  feasible set in decision space
- \( Y = z(X) = \{ Cx : x \in X \} \)
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- \( \text{conv}(Y) + \mathbb{R}_\geq^p \)
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Lexicographic Optimality

- Individual minima
  \[ z_k(\hat{x}) \leq z_k(x) \] for all \( x \in X \)

- Lexicographic optimality (1)
  \[ z(\hat{x}) \leq_{lex} z(x) \] for all \( x \in X \)

- Lexicographic optimality (2)
  \[ z^\pi(\hat{x}) \leq_{lex} z^\pi(x) \] for all \( x \in X \)
  and some permutation \( z^\pi \) of 
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  There is no $x$ with $z(x) < z(\hat{x})$
  $z(\hat{x})$ is weakly nondominated
  $Y_{wN} := z(X_{wN})$

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Supported efficient solutions
$X_{sE}$: There is $\lambda > 0$ with
$\lambda^T C\hat{x} \leq \lambda^T Cx$ for all $x \in X$
- $C\hat{x}$ is extreme point of
  $\text{conv}(Y) + \mathbb{R}^p_\geq \rightarrow X_{sE1}$
- $C\hat{x}$ is in relative interior of
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Non-supported efficient solutions
$X_{nE}$: $C\hat{x}$ is in interior of
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Efficient Solutions

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  \[ X_{nE} : C\hat{x} \text{ is in interior of } \text{conv}(Y) + \mathbb{R}^p_{\geq} \]
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Hansen 1979:

- $x^1, x^2 \in X_E$ are equivalent if $Cx^1 = Cx^2$
- Complete set: $\hat{X} \subset X_E$ such that for all $y \in Y_N$ there is $x \in \hat{X}$ with $z(x) = y$
- Minimal complete set contains no equivalent solutions
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Theorem

Multiobjective combinatorial optimization problems are NP-hard, \#P-complete, and intractable.

Examples:
- Shortest path (Hansen 1979, Serafini 1986)
- Assignment (Serafini 1986, Neumayer 1994)
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Empirically often

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- $|X_{sE}|$ grows polynomially with instance size

But this depends on numerical values of $C$
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Bi-KP Problems: proportion de supportées

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Solving MOCO Problems

- **Scalarization**
  - Single objective problem polynomially solvable and algorithm can be directly extended to multiple objectives
  - Single objective problem polynomially solvable and ranking algorithm exists: The 2 Phase Method
  - Single objective problem NP-hard: General integer programming methods
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Principle and Properties of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations: (Wierzbicki 1984)

- Correctness: Optimal solutions are (weakly) efficient
- Completeness: All efficient solutions can be found
- Computability: Scalarization is not harder than single objective version of problem (theory and practice)
- Linearity: Scalarization has linear formulation
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Scalarization Methods

- **Weighted sum:**
  \[
  \min_{x \in X} \left\{ \lambda^T z(x) \right\}
  \]

- **\(\varepsilon\)-constraint:**
  \[
  \min_{x \in X} \left\{ z_l(x) : z_k(x) \leq \varepsilon_k, \ k \neq l \right\}
  \]

- **Weighted Chebychev:**
  \[
  \min_{x \in X} \left\{ \max_{k=1,\ldots,p} \nu_k(z_k(x) - y_k^l) \right\}
  \]

\[\lambda^T = \left( \frac{3}{7}, \frac{4}{7} \right)\]
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Matthias Ehrgott
Multiobjective Optimization
General Formulation

\[
\min_{x \in X} \left\{ \max_{k=1}^{p} \left[ \nu_k (c_k x - \rho_k) \right] + \sum_{k=1}^{p} [\lambda_k (c_k x - \rho_k)] \right\}
\]

subject to

\[c_k x \leq \varepsilon_k \quad k = 1, \ldots, p\]

<table>
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Matthias Ehrgott  Multiobjective Optimization
General Formulation

\[
\min_{x \in X} \left\{ \max_{k=1}^{p} [\nu_k (c_k x - \rho_k)] + \sum_{k=1}^{p} [\lambda_k (c_k x - \rho_k)] \right\}
\]

subject to
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Theorem (Ehrgott 2005)

1. The general scalarization is NP-hard.
2. An optimal solution of the Lagrangian dual of the linearized general scalarization is a supported efficient solution.
Method of Elastic Constraints

\[
\min_{x \in X} c_i x + \sum_{k \neq l} \mu_k w_k \\
\text{s.t. } c_k x + v_k - w_k \leq \varepsilon_k \quad k \neq l \\
v_k, w_k \geq 0 \quad k \neq l
\]
Method of Elastic Constraints

\[
\begin{align*}
\min_{x \in X} & \quad c_l x + \sum_{k \neq l} \mu_k w_k \\
\text{s.t.} & \quad c_k x + v_k - w_k = \varepsilon_k \quad k \neq l \\
& \quad v_k, w_k \geq 0 \quad k \neq l
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Method of Elastic Constraints

**Theorem (Ehrgott and Ryan 2002)**

*The method of elastic constraints*

- *is correct and complete*,
- *contains the weighted sum and $\varepsilon$-constraint method as special cases*,
- *is NP-hard*.

... but (often) solvable in practice because

- it “respects” problem structure
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Integer Programming Duality

Theorem (Klamroth et al. 2004)

- \( \hat{x} \in X_E \) if and only if there is \( \hat{F} \in \mathcal{F} := \{ F : \mathbb{R}^{m+p-1} \rightarrow \mathbb{R} \) nondecreasing\} such that \( \hat{x} \) is an optimal solution to

\[
\max \left\{ c_j x - \hat{F} \left( (c_k x)_{k \neq j}, b \right) : Ax \leq b, x \geq 0, x \text{ integer} \right\}.
\]

- \( \hat{F} \) can be chosen as an optimal solution of the IP dual

\[
\min \left\{ F(-e, b) : F(( -c_k x)_{k \neq j}, Ax) \geq c_j x \ \forall x \in \mathbb{Z}^n_\geq, F \in \mathcal{F} \right\}
\]

of \( \max \{ c_l x : c_k x \geq \varepsilon_k, k \neq l, Ax = b, x \in \mathbb{Z}^n_\geq \} \)

- The level curve of the objective function of the composite IP at level 0 defines an upper bound on \( Y_N \).
Direct Application of Single Objective Method

- The Shortest Path Problem
  - Shortest path from node $s$ to node $t$ in a directed graph
  - Labels are vectors, each node has set of labels
  - New labels deleted if dominated by another label
  - Labels dominated by new label dominated

- More general: Dynamic Programming

- The Spanning Tree Problem
  - Generalizations of Prim's and Kruskal's algorithms
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The Two Phase Method

- **Phase 1: Compute** $X_{sE}$
  1. Find lexicographic solutions
  2. Recursively:
     - Calculate $\lambda$
     - Solve $\min_{x \in X} \lambda^T Cx$
- **Phase 2: Compute** $X_{nE}$
  1. Solve by triangle
  2. Use neighborhood (wrong)
  3. Use constraints (bad)
  4. Use variable fixing (possible)
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- **Finding maximal complete set:**
  - Enumeration to find all optimal solutions of \( \min_{x \in X} \lambda^T Cx \)
  - Enumeration to find all \( x \in X_{nE} \) with \( Cx = y \in Y_{nD} \)

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(Przybylski et al. 2004)

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  - **Dichotomic search impossible** since normal defined by three nondominated extreme points need not define positive weights
  - \( y^1 = (11, 11, 14), y^2 = (15, 9, 17), y^3 = (19, 14, 10) \) are three nondominated extreme points, normal is \((-1, 40, 28)\)
  - Nondominated extreme point \( y^4 = (13, 16, 11) \) not found

- Phase 2:
  - Search by triangle impossible due to lack of natural order of points
  - \( y^1 = (22, 42, 25), y^2 = (38, 33, 27), y^3 = (39, 31, 30) \) are three nondominated extreme points
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  - $y^4 = (30, 38, 37)$ is not “below” $(39, 41, 30)$
The 2 Phase Method for 3 Objectives

- **Phase 1:**
  - **Dichotomic search impossible** since normal defined by three nondominated extreme points need not define positive weights
  - \( y^1 = (11, 11, 14), y^2 = (15, 9, 17), y^3 = (19, 14, 10) \) are three nondominated extreme points, normal is \((-1, 40, 28)\)
  - Nondominated extreme point \( y^4 = (13, 16, 11) \) not found

- **Phase 2:**
  - **Search by triangle impossible** due to lack of natural order of points
  - \( y^1 = (22, 42, 25), y^2 = (38, 33, 27), y^3 = (39, 31, 30) \) are three nondominated extreme points
  - \( y^4 = (30, 38, 37) \) is not “below” \((39,41,30)\)
Weight Set Decomposition

\[ W^0 := \left\{ \lambda > 0 : \lambda_p = 1 - \sum_{k=1}^{p} \lambda_k \right\} \]

\[ W^0(y) := \left\{ \lambda \in W^0 : \lambda^T y = \min \{ \lambda^T y : y \in Y \} \right\} \]

**Theorem**

- If \( y \) is a nondominated extreme point of \( Y \) then \( \dim W^0(y) = p - 1 \).
- \( W^0(y) = \bigcup_{y \in Y_{sN1}} W^0(y) \).
- \( \dim W^0(y) + \dim F(y) = p - 1 \) for all \( y \in Y_{sN} \), where \( F(y) \) is the maximal nondominated face of \( Y \) containing \( y \).
Weight Set Decomposition

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Finding Nondominated Extreme Points

- For $p = 1, \ldots, p$ find $y^k$ minimizing the $k$-th objective
- $S := \{y^1, \ldots, y^p\}$ and $W^0_p(y^k) = \{\lambda \in W^0 : \lambda^T y = \min \{\lambda^T y : y \in S\}\}$
- Facets of $W^0_p(y^k)$ define biobjective problems
- Solve biobjective problems for all facets for all $y^k$ to find new nondominated extreme points added to $S$
- Stop if $W^0_p(y) = W^0(y)$ for all $y \in S$
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- Stop if \( W^0_p(y) = W^0(y) \) for all \( y \in S \)
Finding Supported Nondominated Points

- Relevant weights
  - Intersection points of at least three sets $W^0(y)$
  - Points in the interior of faces where two sets $W^0(y)$ intersect
- Enumerate all optimal solutions of weighted sum problems
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  - Points in the interior of faces where two sets $W^0(y)$ intersect
- Enumerate all optimal solutions of weighted sum problems
Finding Nonsupported Nondominated Points

The search area

\[ A = \left( (\text{conv } Y_{sN})_N + \mathbb{R}^p_\geq \right) \setminus \left( Y_{sN} + \mathbb{R}^p_\geq \right) \]
\[ = \left( (\text{conv } Y_{sN})_N + \mathbb{R}^p_\geq \right) \cap \left( \bigcup_{u \in D(Y_{sN})} u - \mathbb{R}^p_\geq \right) \]

- Procedure to calculate \( D(Y_{sN}) \)
- For each \( u \in D(Y_{sN}) \) find closest nondominated facet of \( Y \)
- Apply ranking procedure to enumerate solutions between facet of \( Y \) and parallel plane through \( u \)
Finding Nonsupported Nondominated Points

The search area

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Procedure to calculate \( D(Y_{sN}) \)

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Finding Nonsupported Nondominated Points

The search area

\[
A = \left( (\text{conv } Y_{sN})_N + \mathbb{R}_\geq^p \right) \setminus \left( Y_{sN} + \mathbb{R}_\geq^p \right) \\
= \left( (\text{conv } Y_{sN})_N + \mathbb{R}_\geq^p \right) \cap \left( \bigcup_{u \in D(Y_{sN})} u - \mathbb{R}_\geq^p \right)
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○ Procedure to calculate \( D(Y_{sN}) \)

○ For each \( u \in D(Y_{sN}) \) find closest nondominated facet of \( Y \)

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- Apply ranking procedure to enumerate solutions between facet of \( Y \) and parallel plane through \( u \)
Finding Nonsupported Nondominated Points

The search area

\[
A = \left( (\text{conv } Y_{SN})_N + \mathbb{R}^p_\geq \right) \setminus \left( Y_{SN} + \mathbb{R}^p_\geq \right)
= \left( (\text{conv } Y_{SN})_N + \mathbb{R}^p_\geq \right) \cap \left( \bigcup_{u \in D(Y_{SN})} u - \mathbb{R}^p_\geq \right)
\]

- Procedure to calculate \( D(Y_{SN}) \)
- For each \( u \in D(Y_{SN}) \) find closest nondominated facet of \( Y \)
- Apply ranking procedure to enumerate solutions between facet of \( Y \) and parallel plane through \( u \)
# Results for Three-Objective Assignment Problem

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Multicriteria Branch and Bound

- Ulungu and Teghem 1997, Mavrotas and Diakoulaki 2002
- Branching: As in single objective case
- Bounding: Ideal point of problem at node is dominated by efficient solution
- Branching may be very ineffective
- Use lower and upper bound sets

\[
N_0 \quad \emptyset \quad \emptyset \\
N_1 \quad 1, 2, 3 \quad \emptyset \\
N_2 \quad 1, 2 \quad 3 \\
N_3 \quad 1, 2, 4 \quad 3 \\
N_4 \quad 1, 2 \quad 3, 4 \\
N_5 \quad 1 \quad 2 \\
N_6 \quad 1, 3 \quad 2 \\
N_7 \quad 1 \quad 2, 3 \\
N_8 \quad 1, 4, 5 \quad 2, 3 \\
L = \{(23, 27)\} \\
L = \{(23, 27), (29, 16)\} \\
L = \{(23, 27), (29, 16), (27, 18)\}
\]
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Bound Sets

Ehrgott and Gandibleux 2005:

1. **Lower bound set** $L$
   - is $\mathbb{R}_\geq^p$-closed
   - is $\mathbb{R}_\geq^p$-bounded
   - $Y_N \subset L + \mathbb{R}_\geq^p$
   - $L \subset \left( L + \mathbb{R}_\geq^p \right)_N$

2. **Upper bound set** $U$
   - is $\mathbb{R}_\geq^p$-closed
   - is $\mathbb{R}_\geq^p$-bounded
   - $Y_N \in \overline{\left( U + \mathbb{R}_\geq^p \right)}$
   - $U \subset \left( U + \mathbb{R}_\geq^p \right)_N$
Bound Sets

Ehrgott and Gandibleux 2005:

1. **Lower bound set** $L$
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   - $U \subset \left( U + \mathbb{R}^p_{\geq} \right)_N$
Markowitz 1952 with cardinality constraint, e.g. Chang et al. 2000

\[
\begin{align*}
\max z_1(x) &= \mu^T x \\
\min z_2(x) &= x^T \sigma x \\
\text{subject to } e^T x &= 1 \\
x_i &\leq u_i y_i \\
x_i &\geq l_i y_i \\
e^T y &= k \\
y &\in \{0, 1\}^n
\end{align*}
\]
Portfolio Selection

Markowitz 1952 with cardinality constraint, e.g. Chang et al. 2000

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\begin{align*}
\text{max } z_1(x) &= \mu^T x \\
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Airline Crew Scheduling

Partition flights into set of pairings, but minimizing cost can cause delays ...

and be very expensive
Airline Crew Scheduling

Partition flights into set of pairings, but minimizing cost can cause delays ... and be very expensive

Passengers with low-cost airline Easyjet are suffering delays after 19 flights in and out of Britain were cancelled. The company blamed the move - which comes a week after passengers staged a protest sit-in at Nice airport - on crewing problems stemming from technical hitches with aircraft. Crews caught up in the delays worked up to their maximum hours and then had to be allowed home to rest. Mobilising replacement crews has been a problem as it takes time to bring people to airports from home. Standby crews were already being used and other staff are on holiday.
Airline Crew Scheduling

Model 1: Minimize cost and minimize non-robustness (Ehrgott and Ryan 2002)

\[ a_{ij} = \begin{cases} 1 & \text{pairing } j \text{ includes flight } i \\ 0 & \text{otherwise} \end{cases} \]

\[
\begin{align*}
\min z_1(x) &= c^T x \\
\min z_2(x) &= r^T x \\
\text{subject to} \quad Ax &= e \\
Mx &= b \\
x &\in \{0, 1\}^n
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Introduction
Finding Efficient Solutions – Scalarization
Multiobjective Linear Programming
Multiobjective Combinatorial Optimization
Applications
Commercials

Matthias Ehrgott  Multiobjective Optimization
Radiotherapy Treatment Design

Choose beam directions and intensities to destroy tumour and spare healthy organs (e.g. Holder 2004)

$$\begin{align*}
\text{min}(z_T, z_S, z_N) \\
\text{subject to } A_T x + z_T e &\geq l_T \\
A_T x &\leq u_T \\
A_S x - z_S e &\leq u_S \\
A_N x - z_N e &\leq u_N \\
z_S &\geq -u_S \\
z_N &\geq 0 \\
x &\geq 0 \\
x &\geq Mye \\
y &\in \{0, 1\}^n
\end{align*}$$
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& A_T x \leq u_T \\
& A_S x - z_S e \leq u_S \\
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& z_S \geq -u_S \\
& z_N \geq 0 \\
& x \geq 0 \\
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\end{align*}
$$
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19th International Conference on
Multiple Criteria Decision Making
MCDM for Sustainable Energy and Transportation Systems

7th – 12th January 2008
The University of Auckland, Auckland, New Zealand

Deadline for abstract submission September 30, 2007
Deadline for early registration October 15, 2007

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mcdm2008@esc.auckland.ac.nz

Matthias Ehrgott
Multiobjective Optimization