Hybrid Random-Structured Coding

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Random vs structured codes: Point-to-point systems

Random vs. structured coding: Multiterminal systems
- Slepian-Wolf coding
- Körner & Marton’s binary two-help-one problem

The Gaussian two-help-one problem
- Berger-Tung (BT) random coding
- Krithivasan-Pradhan (KP) structured coding
- Hybrid random-structured coding
  - Partial sum-rate tightness
  - Gap to optimal sum-rate

Conclusions
1 Random vs structured codes: Point-to-point systems

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4 Conclusions
Random codes for point-to-point systems

What are they?

Codes with randomness
- Invented by Shannon in 1948

Key theoretical tool
- In proving Shannon’s classic source and channel coding theorems

Recent invention/rediscovery of codes based on random graphs
- Turbo/LDPC codes
- Fulfilled Shannon’s prophecy (after 50 years)

What do we learn from Shannon?
- Random codes are optimal for point-to-point systems
Structured codes for point-to-point systems

What are they?

Codes with structure

- Linear/lattice/trellis codes

Widely used in communication systems

- For their low complexity
- Potentially limit-approaching (as promised by info theory)
  - Achievable capacity
    \[ \frac{1}{2} \log(\text{SNR}) \rightarrow \frac{1}{2} \log(1 + \text{SNR}) \]
    for AWGN channels
  - Optimal for at least some point-to-point systems as well
  - Structure “comes for free”

Can structured codes exceed the info-theoretic limits?

- No, at least for memoryless point-to-point systems
Outline

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Random codes for multiterminal systems

Slepian-Wolf coding’73

- $X, Y$: two correlated sources with finite alphabet
- Separate encoding and joint decoding
- Near-lossless decoding, i.e., $\lim_{n \to \infty} P_e((X^n, Y^n) \neq (\hat{X}^n, \hat{Y}^n)) = 0$

Diagram:

- Encoder $X^n$ to $W_X$ with rate $R_1$
- Encoder $Y^n$ to $W_Y$ with rate $R_2$
- Joint Decoder
- Output $\hat{X}^n$ and $\hat{Y}^n$
Slepian-Wolf theorem

- Slepian-Wolf rate region (proved by using random coding/binning)

\[
R_1 \geq H(X|Y) \\
R_2 \geq H(Y|X) \\
R_1 + R_2 \geq H(X,Y)
\]

- No rate loss compared to joint encoding of $X$ and $Y$
Slepian-Wolf coding example

Assumptions:
- $X, Y \in \{0, 1\}^3 \rightarrow$ binary triplets
- $H(X) = H(Y) = 3$ bits $\rightarrow$ equiprobable triplets
- Correlation between $X$ and $Y$
  - $x$ and $y$ differ at most in one position
  - Hamming distance $d_H(x,y) \leq 1$
  - $H(X|Y) = 2$ bits

Question:
- If $y$ is perfectly known at the decoder but not at the encoder, is it possible
  - to send 2 bits instead of 3 for $x$ and
  - reconstruct $x$ without loss?
Slepian-Wolf coding example

Solution:
- Form 4 cosets and send 2-bit index of the coset $x$ belongs to:
  - coset $Z_00$: $\{000, 111\}$ ← codewords of rate-$\frac{1}{3}$ repetition code
  - coset $Z_{01}$: $\{001, 110\}$
  - coset $Z_{10}$: $\{010, 101\}$
  - coset $Z_{11}$: $\{011, 100\}$
- In each set: 2 members at $d_H = 3$
- Joint decoder: in the set indexed by $Z$:
  - Using $y$, pick $\hat{x}$ s.t. $d_H(\hat{x}, y) \leq 1$
  - This guarantees correct/lossless decoding
    - Example: $y=[000]$, index 00 from encoder, $\hat{x}=[000]$
      - index 01 from encoder, $\hat{x}=[001]$
      - index 10 from encoder, $\hat{x}=[010]$
      - index 11 from encoder, $\hat{x}=[100]$
- Separate encoding as efficient as joint encoding!
Equivalent way of viewing last example from a syndrome concept

- Form parity-check matrix of rate-$\frac{1}{3}$ repetition code

$$
H = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
$$

- Syndrome = coset/bin index
  - Coset/bin 0: \{000,111\} has syndrome $Z = 00$
  - Coset/bin 1: \{001,110\} has syndrome $Z = 01$
  - Coset/bin 2: \{010,101\} has syndrome $Z = 10$
  - Coset/bin 3: \{011,100\} has syndrome $Z = 11$

- All 4 cosets preserve the distance properties of the repetition code
- Encoding corresponds to matrix multiplication $Hx$
  - Compression 3:2
- Separate encoding as efficient as joint encoding!

1. Random coding optimal for this symmetric Slepian-Wolf coding example
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Structured codes for multiterminal systems

The binary two-help-one problem

How to encode the mod-2 sum of binary sources? Introduced by Körner & Marton in 1979

- $Y_1$ and $Y_2$ are doubly symmetric binary sources
- Only the first two encoders are allowed to transmit, the decoder reconstructs $Z = Y_1 \oplus Y_2$ losslessly
  - This is combined compression and inference (for big data)
  - Goes beyond Slepian-Wolf coding
Structured codes for multiterminal systems

The rate region of Körner-Marton coding

- Slepian-Wolf coding before forming $Z = Y_1 \oplus Y_2$ certainly works
- Structured coding strictly improves random (Slepian-Wolf) coding!

Using the same linear code, encoder $i$ transmits $H y_i$
- So both encoders use the same rate $H(Z)$
- The decoder forms $H y_1 \oplus H y_2 = H (y_1 \oplus y_2) = H z$ before recovering $z$
  - by picking the element with $d_H \leq 1$ in the coset indexed by $z$
Körner-Marton coding example

Recall 4 cosets at each encoder (from the Slepian-Wolf coding example)

- coset $Z_{00}$: \{000, 111\}; coset $Z_{01}$: \{001, 110\}
- coset $Z_{10}$: \{010, 101\}; coset $Z_{11}$: \{011, 100\}

Suppose $y_1=[101]$, encoder 1 transmits index 10

- $y_2=[101]$, encoder 2 transmits 10, decoder forms 00 before reconstructing $z$ as [000]
- $y_2=[001]$, encoder 2 transmits 01, decoder forms 11 before reconstructing $z$ as [100]
- $y_2=[111]$, encoder 2 transmits 00, decoder forms 10 before reconstructing $z$ as [010]
- $y_2=[100]$, encoder 2 transmits 11, decoder forms 01 before reconstructing $z$ as [001]

First instance of linear coding beating the best-known random coding

2. Linear coding optimal for this symmetric Körner-Marton coding example
### Structured vs. random codes

It was long held that random codes are optimal for many comm. systems

- Körner and Marton’s work showed advantage of structured code for some multiterminal systems
- Partially responsible for their recent back-to-back Shannon awards
- Structured coding for multiterminal systems currently a very active area of research

#### Reprise

- Linear coding optimal for symmetric Körner-Marton coding
  - Symmetry means $H(Y_1|Y_2) = H(Y_2|Y_1)$ or $P(Y_1 = 0|Y_2 = 1) = P(Y_1 = 1|Y_2 = 0)$

What about the general asymmetric case?

- $Y_1$ and $Y_2$ are binary sources with joint PMF ($p_{01} \neq p_{10}$ in general)

<table>
<thead>
<tr>
<th>$Y_1 \backslash Y_2$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$p_{00}$</td>
<td>$p_{01}$</td>
</tr>
<tr>
<td>1</td>
<td>$p_{10}$</td>
<td>$p_{11}$</td>
</tr>
</tbody>
</table>

**Q:** Which coding scheme is better? **A:** Neither
How to encode the mod-2 sum of binary sources?

**Ahlswede-Han coding**

- Introduced by Ahlswede & Han in 1983
- **Random coding**: First quantize $Y_1^n$ and $Y_2^n$ as $U_1^n$ and $U_2^n$
- **Structured coding**: Then apply Körner-Marton coding on $Y_1^n$ and $Y_2^n$ with $U_1^n$ and $U_2^n$ as decoder side info
- Achievable rate pairs

\[
R_1 \geq I(Y_1; U_1|U_2) + H(Z|U_1, U_2)
\]
\[
R_2 \geq I(Y_2; U_2|U_1) + H(Z|U_1, U_2)
\]
\[
R_1 + R_2 \geq I(Y_1, Y_2; U_1, U_2) + 2H(Z|U_1, U_2)
\]

- Performs no worse than Slepian-Wolf and Körner-Marton coding for general correlation
How to encode the mod-2 sum of binary sources?

Ahlswede-Han coding example

- Expanded rate region when

<table>
<thead>
<tr>
<th>$Y_1 \setminus Y_2$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00392</td>
<td>0.97608</td>
</tr>
<tr>
<td>1</td>
<td>0.01992</td>
<td>0.00008</td>
</tr>
</tbody>
</table>

- Due to asymmetry, **time sharing** between Slepian-Wolf and Körner-Marton always exists
- Ahlswede-Han coding achieves points (e.g., P) **outside** the time-sharing region
  - Optimal coding scheme **not known**
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4 Conclusions
The **Gaussian** two-help-one problem

**Definition**
- Separate compression and joint decompression of a linear combination $Z = Y_1 - cY_2$ of jointly Gaussian sources $Y_1$ and $Y_2$ subject to an MSE distortion constraint on $Z$
- Problem characterized by the linear coefficient $c$, the source correlation coefficient $\rho$, and the MSE distortion constraint $D$ on $Z$

**Motivation**
- Arises in many practical video surveillance applications, e.g., reconstructing the motion difference between two video sequences
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The \textbf{Gaussian} two-help-one problem

\textbf{Berger-Tung’s generic random coding scheme 1977}

- Independent quantization of
  \begin{itemize}
  \item $Y_1$ to $U_1$ s.t. $U_1 = Y_1 + Q_1$ with $Q_1 \sim \mathcal{N}(0, q_1)$
  \item $Y_2$ to $U_2$ s.t. $U_2 = Y_2 + Q_2$ with $Q_2 \sim \mathcal{N}(0, q_2)$
  \end{itemize}

- Followed by Slepian-Wolf compression (or binning) of $U_1$ and $U_2$ leads to rate region

\[
\mathcal{R}^{BT}(q_1, q_2) = \left\{ \begin{array}{l}
R_1 \geq H(U_1|U_2) \\
R_2 \geq H(U_2|U_2) \\
R_1 + R_2 \geq H(U_1, U_2)
\end{array} \right\}
\]

- Berger-Tung (BT) achievable rate region

\[
\mathcal{R}_{BT}(D) = \text{conv}\left( \bigcup_{(U_1, U_2) \in \mathcal{U}(Y_1, Y_2)} \left\{ (R_1, R_2) : R_1 \geq I(Y_1; U_1|U_2), R_2 \geq I(Y_2; U_2|U_1), R_1 + R_2 \geq I(Y_1, Y_2; U_1, U_2) \right\} \right)
\]
The **Gaussian** two-help-one problem

When $c \cdot \rho \leq 0$, e.g., $\rho > 0$, $c = -1$

- Referred to as the $\mu$-sum problem (e.g., with $Z = Y_1 + Y_2$)
- Considered by Wagner et al. in 2005
- Berger-Tung (or random QB) coding is optimal

![Graph showing sum-rate-distortion function and sum-rate of the Berger-Tung scheme](image_url)
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The **Gaussian** two-help-one problem

When $c \cdot \rho > 0$, e.g., $\rho > 0$, $c = 1$

- Referred to as the $\mu$-difference problem (e.g., with $Z = Y_1 - Y_2$)
- Considered by Krithivasan-Pradhan in 2009 using structured/lattice codes
- Two lattices $(\Lambda_1, \Lambda_2)$ for SC; one lattice $\Lambda_C$ with $\Lambda_C \subseteq \Lambda_i$ for CC
  - Independent lattice quantizers $(\Lambda_1, \Lambda_2)$ on $Y_1$ and $cY_2$
  - Encoders send quantized versions modulo the same lattice $\Lambda_C$
  - Transmission rates
    \[
    R_i = \log_2 \frac{\sigma^2(\Lambda_C)}{\sigma^2(\Lambda_i)}
    \]
- Krithivasan-Pradhan (KP) achievable rate region

\[
R_{KP}(D) = \left\{ (R_1, R_2) : 2^{-2R_1} + 2^{-2R_2} \leq \frac{D}{1 + c^2 - 2c\rho} \right\}
\]
The **Gaussian** two-help-one problem

When \( c \cdot \rho > 0, \) e.g., \( \rho > 0, c = 1 \)

- Referred to as the \( \mu \)-difference problem (e.g., with \( Z = Y_1 - Y_2 \))
- Considered by Krithivasan-Pradhan in 2009 using structured/lattice codes
- Smaller sum-rate than the random Berger-Tung scheme for certain \((\rho, c)\) pairs
  - Structured codes beat random codes!

![Graph showing sum-rate comparison between Krithivasan & Pradhan's scheme and Berger-Tung scheme.](image-url)
The Gaussian two-help-one problem

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The **Gaussian** two-help-one problem

When \( c \cdot \rho > 0 \), e.g., \( \rho > 0, c = 1 \)

- For the \( \mu \)-difference problem, can we do better?
- New hybrid random-structured coding scheme

  - Inspired by Ahlswede & Han’s for encoding the mod-2 sum of binary sources
  - Hybrid random BT coding (layer I) and structured KP coding (layer II)

**Diagram**

Encoder 1

- Layer II
  - \( Q_{\Lambda_1}(T_1^n) \)
  - \( V_1^n \)
  - mod \( \Lambda_C \)
- Layer I
  - \( Y_1^n \)
  - Random Quantizer I
  - \( U_1^n \)
  - Slepian-Wolf Encoder I

Encoder 2

- Layer II
  - \( Q_{\Lambda_2}(T_2^n) \)
  - \( V_2^n \)
  - mod \( \Lambda_C \)
- Layer I
  - \( Y_2^n \)
  - Random Quantizer II
  - \( U_2^n \)
  - Slepian-Wolf Encoder II

Decoder

- Layer I
  - \( \hat{U}_1^n \)
  - \( \lambda_1 \)
  - \( \hat{U}_2^n \)
  - \( \lambda_2 \)
  - \( \hat{V}^n \)
  - \( \lambda_3 \)
  - \( \hat{Z}^n \)
- Layer II
  - mod \( \Lambda_C \)
The Gaussian two-help-one problem

When \( c \cdot \rho > 0 \), e.g., \( \rho > 0, c = 1 \)

- Theorem 1: Achievable rate region of hybrid random-structured coding

\[
R_{\text{new}}(D) = \text{conv} \left( \bigcup_{(U_1, U_2) \in \mathcal{U}(c, D)} \{ (R_1, R_2) : \right.
\]
\[
R_1 \geq I(Y_1; U_1|U_2) + I(V_1 - V_2; Y_1, V_2|U_1, U_2),
\]
\[
R_2 \geq I(Y_2; U_2|U_1) + I(V_1 - V_2; V_1, Y_2|U_1, U_2),
\]
\[
R_1 + R_2 \geq I(Y_1, Y_2; U_1, U_2) + I(V_1 - V_2; Y_1, V_2|U_1, U_2)
\]
\[
+ I(V_1 - V_2; V_1, Y_2|U_1, U_2) \left. \right) \right)
\]

\[
\mathcal{U}(c, D) \Delta \left\{ (U_1, U_2, V_1, V_2) : U_i = Y_i + P_i, V_1 = Y_1 + Q_1, V_2 = cY_2 + Q_2,
\]
\[
P_i \sim \mathcal{N}(0, p_i), Q_i \sim \mathcal{N}(0, q_i), i = 1, 2, \text{ indep. of each other and (}Y_1, Y_2), \text{ such that } E\left[ (Z - E(Z|U_1, U_2, V_1 - V_2))^2 \right] \leq D \right\}
\]
The Gaussian two-help-one problem

When $c \cdot \rho > 0$, e.g., $\rho > 0, c = 1$

- Comparison between $\mathcal{R}_{new}(D)$, $\mathcal{R}_{KP}(D)$ and $\mathcal{R}_{BT}(D)$
  - Hybrid coding always subsumes structured KP coding

\[
\mathcal{R}_{new}(D) \supseteq \mathcal{R}_{KP}(D)
\]

- It becomes a QB random coding with additional rate of $\frac{1}{2} \log_2 \frac{1}{\alpha}$ and $\frac{1}{2} \log_2 \frac{1}{\beta}$ (with $\alpha + \beta = 1$) at the two encoders

- Lemma 1:

\[
\mathcal{R}_{new}(D) \supseteq \mathcal{R}_{KP}(D) \cup \left[ \mathcal{R}_{BT}(D) \cup \left\{ \left( \frac{1}{2} \log_2 \frac{1}{\alpha}, \frac{1}{2} \log_2 \frac{1}{\beta} \right) : \alpha + \beta = 1 \right\} \right]
\]

New rate region $\mathcal{R}_{new}(D)$ strictly improves the time-sharing region between $\mathcal{R}_{KP}(D)$ and $\mathcal{R}_{BT}(D)$
The **Gaussian** two-help-one problem

When \( c \cdot \rho > 0 \), e.g., \( \rho > 0, c = 1 \)

- Comparison between \( \mathcal{R}_{\text{new}}(D) \), \( \mathcal{R}_{\text{KP}}(D) \) and \( \mathcal{R}_{\text{BT}}(D) \)
The Gaussian two-help-one problem

When \( c \cdot \rho > 0 \), e.g., \( \rho > 0, c = 1 \)

- We look deeper at the minimum achievable sum-rate

\[
R_{\text{new}}(D) \triangleq \min \left\{ R_1 + R_2 : (R_1, R_2) \in \mathcal{R}_{\text{new}}(D) \right\}
\]

- Theorem 2: Achievable sum-rate of hybrid random-structured coding

\[
R_{\text{new}}(D) = \begin{cases} 
\frac{1}{2} \log_2 \frac{16c^2(1-\rho^2)(1-c\rho)^2}{D^2}, & \text{if } c \leq \frac{1}{\rho + \sqrt{1-\rho^2}} \text{ and } D \leq 2c^2(1-\rho^2) \\
\min \left( \log_2 \frac{2\sigma_Z^2}{D}, \frac{1}{2} \log_2 \frac{16c((1-\rho^2)c-\rho D)}{D^2} \right), & \text{if } c > \frac{1}{\rho + \sqrt{1-\rho^2}} \text{ and } D \leq \frac{2c^2(1-\rho^2)}{1+c\rho} \\
\min \left( \log_2^+ \frac{2\sigma_Z^2}{D}, \frac{1}{2} \log_2^+ \frac{4(1-c\rho)^2}{D-c^2(1-\rho^2)} \right), & \text{otherwise}
\end{cases}
\]
The **Gaussian** two-help-one problem

**When** $c \cdot \rho > 0$, e.g., $\rho > 0$, $c = 1$

- **Comparison** between $R_{new}(D)$, $R_{KP}(D)$ and $R_{BT}(D)$
  - **Corollary** to Lemma 1:
    \[
    R_{new}(D) \leq \min \left( R_{KP}(D), R_{BT}(D) + 1 \right)
    \]

- **When can** $R_{new}(D)$ strictly improve both $R_{KP}(D)$ and $R_{BT}(D)$?

- **Lemma 2:**
  \[
  R_{new}(D) < \min \left( R_{KP}(D), R_{BT}(D) \right)
  \]
  if either
  \[
  \frac{1}{2\rho} < c < \min \left( \frac{\sqrt{3}}{2\rho}, \frac{1}{\rho + \sqrt{1 - \rho^2}} \right) \quad \& \quad D < \frac{c(1 - \rho^2)(3 - 2c\rho)(2c\rho - 1)}{\rho}
  \]
  or
  \[
  \frac{\sqrt{3}}{2\rho} < c < \frac{1}{\rho + \sqrt{1 - \rho^2}} \quad \& \quad D < 4(2 - \sqrt{3})c^2(1 - \rho^2)
  \]
The Gaussian two-help-one problem

When $c \cdot \rho > 0$, e.g., $\rho > 0$, $c = 1$

- Comparison between $R_{new}(D)$, $R_{KP}(D)$ and $R_{BT}(D)$
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**Partial sum-rate tightness**

- Sum-rate lower bound needed — to compared with achievable sum-rate upper bound from hybrid coding
  - Consider a new and more general problem of Gaussian two-terminal SC problem with covariance matrix $\Sigma_Y = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ and covariance matrix distortion constraint $\mathcal{D} = \begin{bmatrix} k_1^2 & \theta k_1 k_2 \\ \theta k_1 k_2 & k_2^2 \end{bmatrix}$
  - **Fact:** If the minimum sum-rate for the above problem is $R(\mathcal{D})$, then $R(D) = \min_{\mathcal{D} \in \mathcal{Y}(\rho, c, D)} R(\mathcal{D})$,

where $\mathcal{Y}(\rho, c, D)$ contains all real $2 \times 2$ symmetric matrices $\mathcal{D}$ such that $0 \leq \mathcal{D} \leq \Sigma_Y$ and $[1 - c] \mathcal{D} [1 - c]^T \leq D$
The Gaussian two-help-one problem

Partial sum-rate tightness

- **Lemma 3:** A new sum-rate lower bound \( R(\mathcal{D}) \geq \max \left( R^\dagger(\mathcal{D}), R^\ddagger(\mathcal{D}) \right) \), where

\[
R^\dagger(\mathcal{D}) \triangleq \begin{cases} 
\frac{1}{2} \log_2 \left[ \frac{1-\rho^2+2\rho k_1 k_2 (1+\theta)}{(1+\theta)^2 k_1^2 k_2^2} \right], & \theta \leq \theta^* \\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2}{(1-\theta)^2 k_1^2 k_2^2 (1-\rho^2+2\rho k_1 k_2 (1+\theta))} \right], & \theta > \theta^* 
\end{cases}
\]

\[
R^\ddagger(\mathcal{D}) \triangleq \begin{cases} 
\frac{1}{2} \log_2 \left[ \frac{1-\rho^2+2\rho k_1 k_2 (1+\theta)}{(1+\theta)^2 k_1^2 k_2^2} \right], & \theta \leq \theta^* \\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2}{(1-\theta)^2 k_1^2 (4k_i^2 \rho^2-4\rho \theta k_1 k_2 + k_j^2)} \right], & \theta^* < \theta \leq \theta^\ddagger \\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2 (1-\rho^2-k_j^2 (1-\theta^2))}{(1-\theta^2)^2 k_1^2 k_2^2 ((1-\rho^2)^2-k_j (k_j-2k_i \rho \theta) (1-\rho^2)-k_1^2 k_2^2 \rho^2 (1-\theta^2))} \right], & \theta^\ddagger < \theta 
\end{cases}
\]

with \( k_i = \min(k_1, k_2) \), \( k_j = \max(k_1, k_2) \), and

\[
\theta^* \triangleq \frac{1}{2\rho k_1 k_2} \left( \sqrt{(1-\rho^2)^2 + 4\rho^2 k_1^2 k_2^2} - (1-\rho^2) \right)
\]

\[
\theta^\ddagger \triangleq \frac{1}{2\rho k_1 k_2} \left( \sqrt{(1-\rho^2)^2 + 4\rho^2 k_1^2 k_2^2 - 8k_i^2 \rho^2 (1-\rho^2) + (1-\rho^2)} \right)
\]
The Gaussian two-help-one problem

Partial sum-rate tightness

Lemma 3: A new sum-rate lower bound $R(D) \geq \max\left( R^+(D), R^\ddagger(D) \right)$, where

$$R^+(D) \triangleq \left\{ \begin{array}{ll}
\frac{1}{2} \log_2 \left[ \frac{1-\rho^2 + 2\rho k_1 k_2 (1+\theta)}{(1+\theta)^2 k_1^2 k_2^2} \right], & \theta \leq \theta^*\\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2}{(1-\theta)^2 k_1^2 k_2^2 (1-\rho^2 + 2\rho k_1 k_2 (1+\theta))} \right], & \theta > \theta^*
\end{array} \right.$$

$$R^\ddagger(D) \triangleq \left\{ \begin{array}{ll}
\frac{1}{2} \log_2 \left[ \frac{1-\rho^2 + 2\rho k_1 k_2 (1+\theta)}{(1+\theta)^2 k_1^2 k_2^2} \right], & \theta \leq \theta^*\\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2}{(1-\theta)^2 k_1^2 k_2^2 (4k_1^2 \rho^2 - 4\rho \theta k_1 k_2 + k_j^2)} \right], & \theta^* < \theta \leq \theta^\ddagger\\
\frac{1}{2} \log_2 \left[ \frac{(1-\rho^2)^2 (1-\rho^2 - k_j^2 (1-\theta^2))}{(1-\theta^2)^2 k_1^2 k_2^2 \left( (1-\rho^2)^2 - k_j (k_j - 2k_i \rho \theta) (1-\rho^2) - k_1^2 k_2^2 \rho^2 (1-\theta^2) \right)} \right], & \theta^\ddagger < \theta
\end{array} \right.$$

The 1st lower bound $R^+(D)$ proved in Xiong’13 using the estimation-theoretic approach of Wang’10

The 2nd lower bound $R^\ddagger(D)$ newly obtained by combining the approach of Wang’10 and the technique in Wagner’11, which exploits stochastic degradedness of the channel $Y_1 \rightarrow Y_2$ with respect to $Y_1 \rightarrow Z$
The Gaussian two-help-one problem

Partial sum-rate tightness

- **Theorem 3**: A new sum-rate lower bound

\[ R(D) \geq R(D) \triangleq \min_{D \in \mathcal{Y}(\rho, c, D)} \max\left( R^+(D), R^-(D) \right) \]

- Comparison among sum-rate lower and upper bounds

![Graph showing comparison among sum-rate lower and upper bounds](image-url)
The **Gaussian** two-help-one problem

**Partial sum-rate tightness**

- **Theorem 4:** First partial sum-rate tightness result

\[ R_{QB}(D) = R(D) = \overline{R(D)} \]

If \( \rho \in (0, 1) \), \( 0 \leq c \leq \frac{1}{1+2\rho} \), \( D \geq \frac{2c^2(1-\rho^2)(1-2c\rho)}{1-3c\rho} \) or \( 0 \leq c \leq \frac{2\rho}{1+2\rho^2} \), \( D \geq \frac{c(1-\rho^2)(1-2c^2\rho^2)}{\rho(2-3c\rho)} \)

![Graph showing the boundaries of sum-rate tightness and suboptimality](image_url)
The Gaussian two-help-one problem

Partial sum-rate tightness

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Outline

1 Random vs structured codes: Point-to-point systems

2 Random vs. structured coding: Multiterminal systems
   - Slepian-Wolf coding
   - Körner & Marton’s binary two-help-one problem

3 The Gaussian two-help-one problem
   - Berger-Tung (BT) random coding
   - Krithivasan-Pradhan (KP) structured coding
   - Hybrid random-structured coding
     - Partial sum-rate tightness
     - Gap to optimal sum-rate

4 Conclusions
The Gaussian two-help-one problem

Gap to optimal sum-rate

- **Theorem 5**: For any \((\rho, c, D)\) triple, it holds that

\[
R_{\text{new}}(D) - R(D) \leq R_{\text{new}}(D) - R(D) \leq 2
\]
The Gaussian two-help-one problem

Gap to optimal sum-rate

- **Theorem 5**: For any \((\rho, c, D)\) triple, it holds that

\[
R_{\text{new}}(D) - R(D) \leq R_{\text{new}}(D) - R(D) \leq 2
\]

The diagram shows the comparison of different sum-rate bounds and schemes. The black line represents the sum-rate of Krithivasan & Pradhan’s scheme \(R_{KP}(D)\), the magenta line with circles represents the BT sum-rate bound \(R_{BT}(D)\), the blue line with crosses represents the sum-rate of the new scheme \(R_{\text{new}}(D)\), and the red line with squares represents the lower sum-rate bound \(R_{\text{lb}}(D)\).

The y-axis represents the sum-rate in bits per second (b/s), and the x-axis represents the ratio \(D/\sigma^2_z\). The graph illustrates the sum-rate gap, with the BT sum-rate being suboptimal and the lower sum-rate bound being tight.
Lemma 4: If $c = 1$ or $c = \rho$, it holds that

$$R_{\text{new}}(D) - R(D) \leq R_{\text{new}}(D) - \frac{R(D)}{} \leq 1$$

QB sum-rate is tight
Conclusions

Intellectual merits:

- Hybrid scheme conceptually brings together two different worlds
- Philosophically the right approach (with better performance)
Conclusions

**Intellectual merits:**
- Hybrid scheme conceptually brings together two different worlds
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**Problem addressed:**
- Very timely and interesting
  - Thanks to Körner & Marton, Ahlswede & Han, and other IT gurus
  - Combined SC and inference for big data
  - The more general many-help-one problem
- Results significant and intriguing
  - First partial sum-rate tightness results
  - Many open issues (e.g., optimality of hybrid coding)
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Broader impact:
- Hybrid approach applicable to many other network comm. scenarios
  - Cooperative networks: The two-way relay channel
  - The interference channels
References


