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Analyses of Spherical antennas

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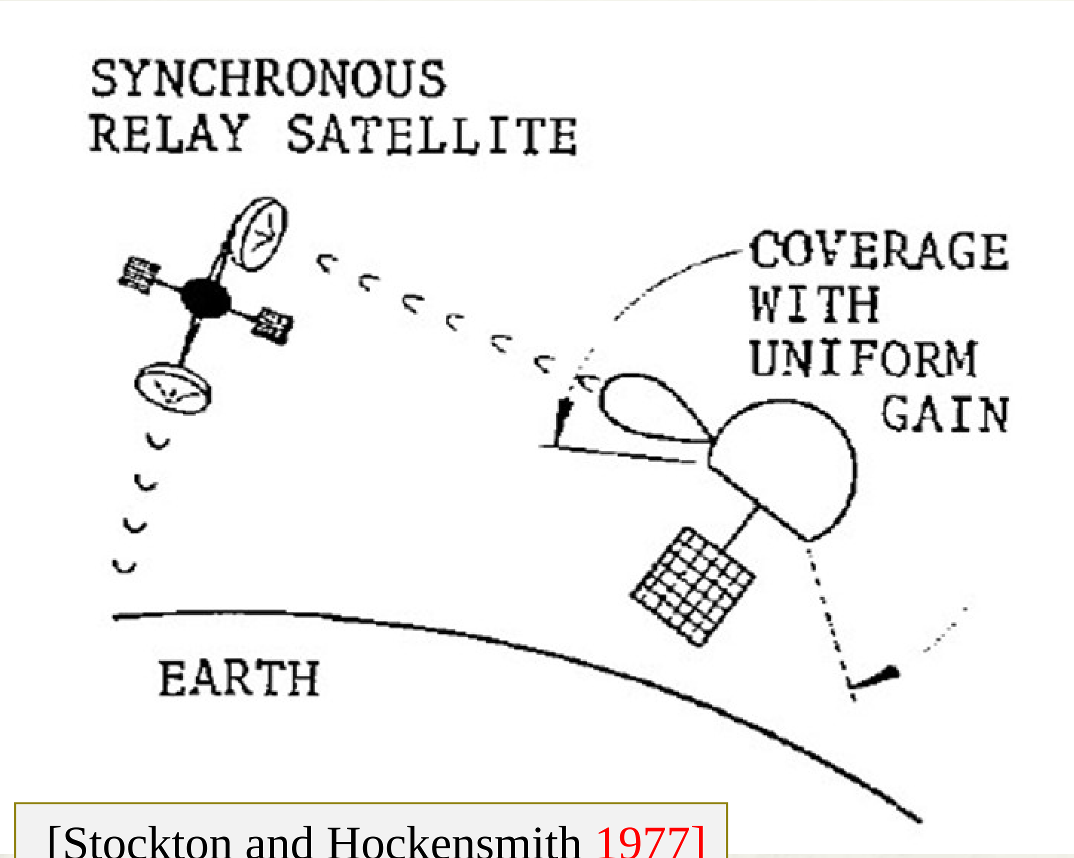
Characteristics of spherical array

- ◆Array elements distributed on a sphere surface.
- ◆ Conformal, low profile, light weight, easy to install on aircraft surfaces
- ◆ Wide angular coverage by activating different array elements
- ◆ Stable antenna gain and radiation pattern for scanning

Spherical array application



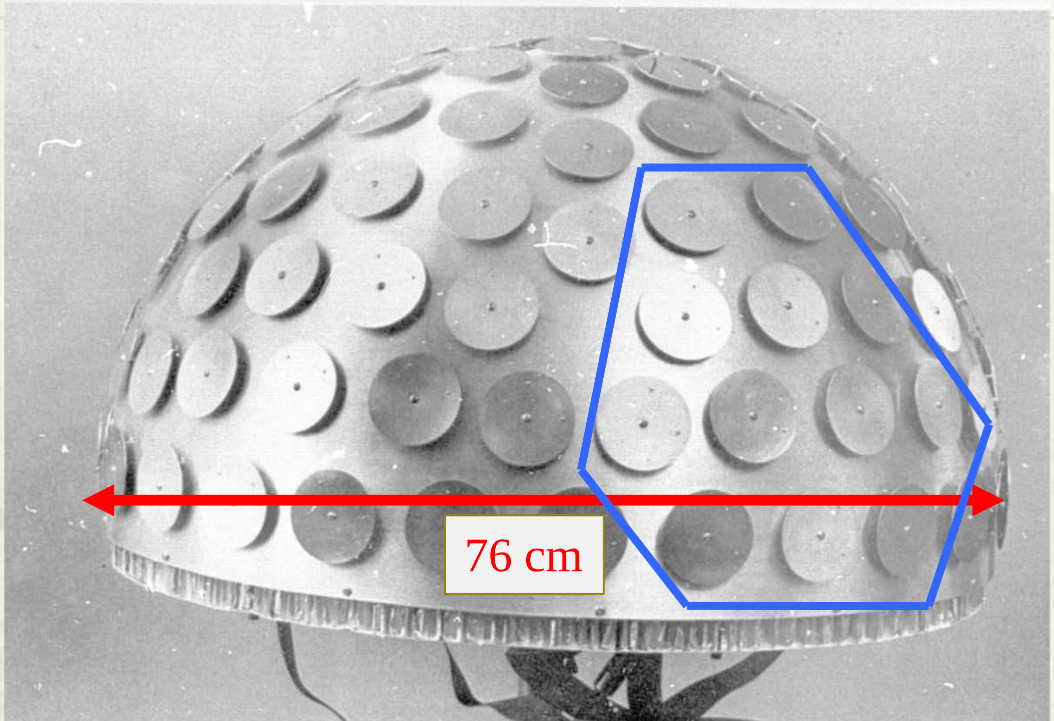
[Stockton and Hockensmith 1977]

Earth-orbiting scientific satellite system

The user satellite is using the spherical array

Stockton & Hockensmith, "Application of spherical arrays -- A simple approach," Antennas and Propagation Society International Symposium, 1977 , vol.15, pp. 202- 205, June 1977.

Hemispherical patch array for satellite data link:
Electronic Switching Spherical Array (ESSA)



[Stockton 1978]

- 2.0-2.3 GHz
- 551 pre-programed Beams
- 3-dB Beam width: 26°
- Max. VSWR: 1.5

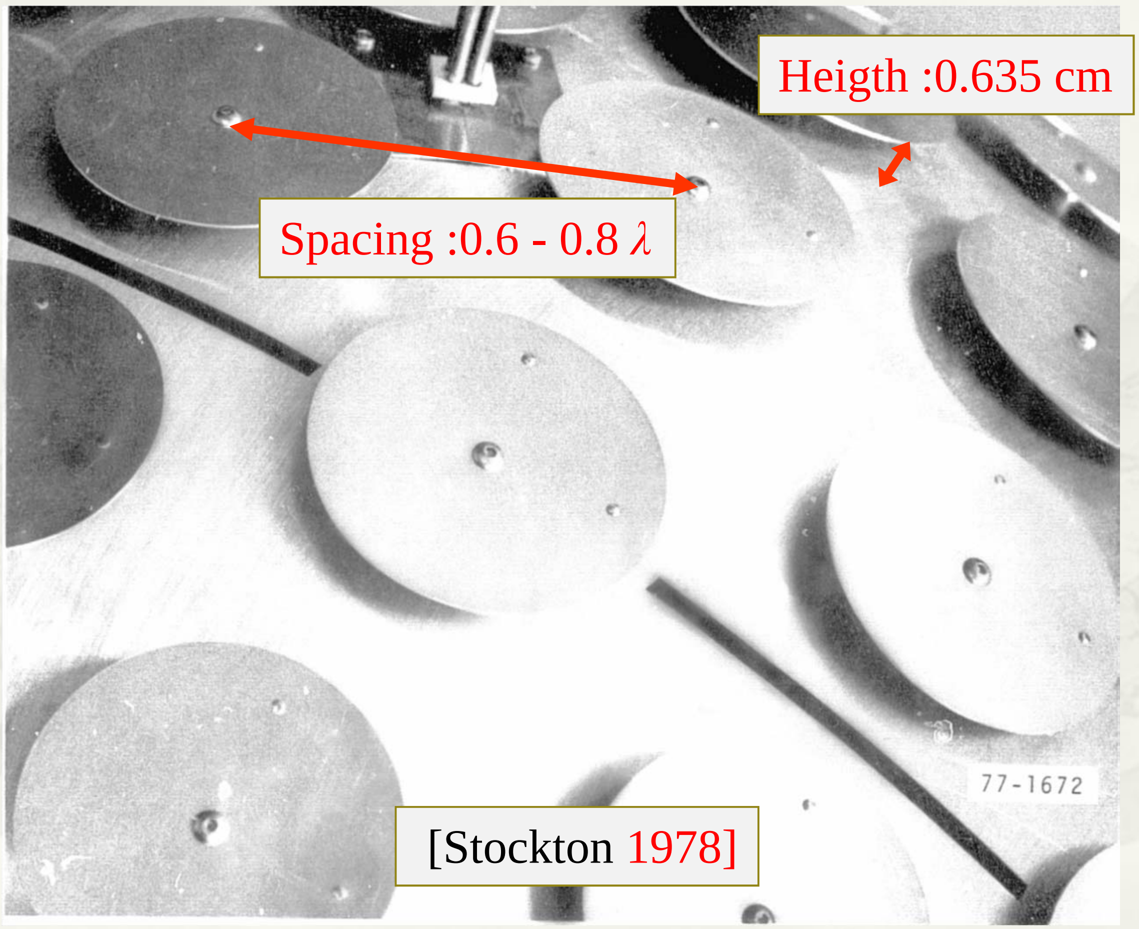
◆ A cluster of 12 patch elements out of a total 120 are activated at a time.

◆ Beam steering was accomplished by shifting the active part in small steps.

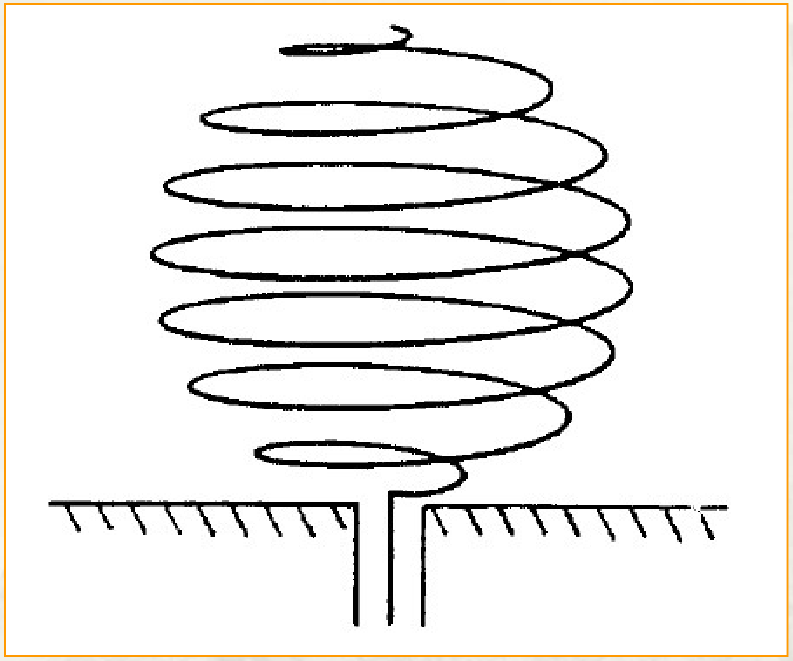
◆ Hemispherical coverage with a moderate gain for steered beams (~13 dBic).

R. Stockton, "Electronic switching spherical array antenna", NASA (Unspecified Center), Apr. 1, 1978.

Elements of the ESSA: RHCP Microstrip Patch Antenna



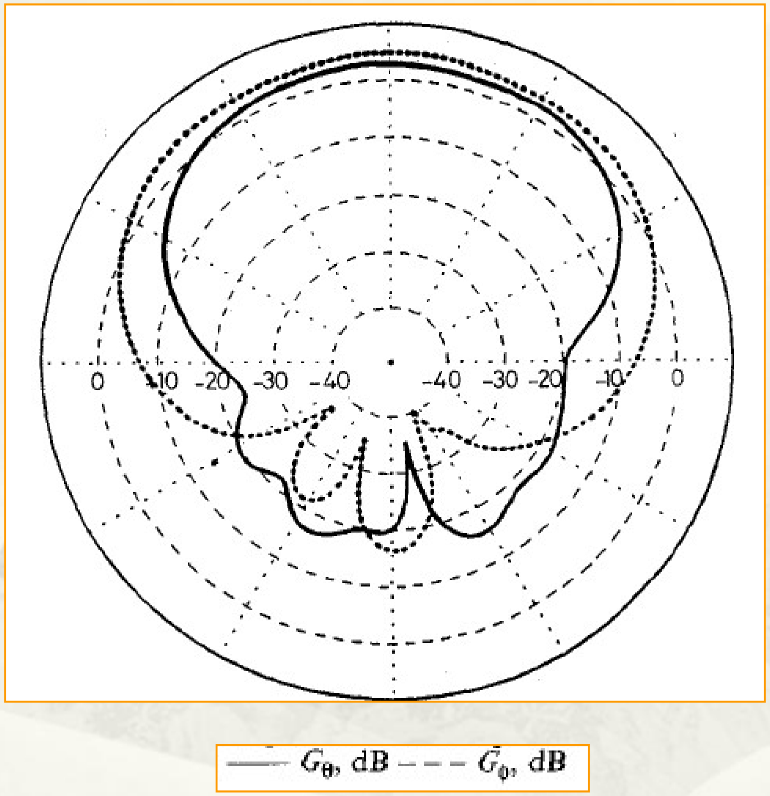
Spherical helical antenna



- Proposed by Mei
- Radiate circularly polarized fields over a wide beamwidth.
- Suitable for systems requiring wide-beamwidths (e.g., GPS).

K. K. Mei and M. Meyer, "Solutions to spherical anisotropic antennas," *IEEE Trans. Antennas Progagat.*, vol. 12, pp.459-463, 1964.

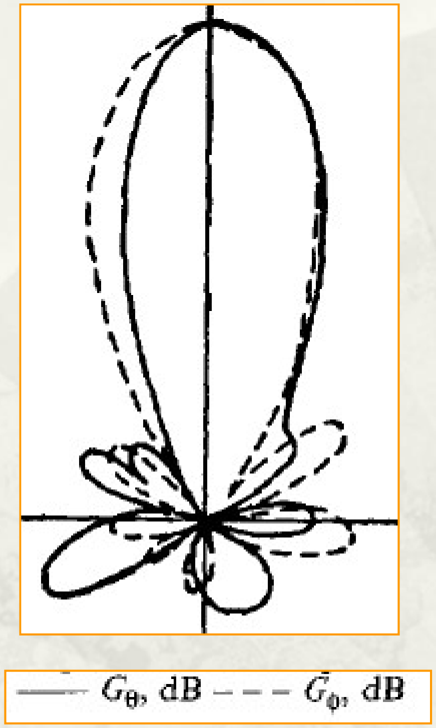
Radiation patterns of spherical helical antenna



- Radiation pattern of a 7-turn spherical helical antenna
- 3dB beamwidths: 60^{o}

A. Safaai-Jazi and J. C. Cardoso,"Radiation characteristics of a spherical helical antenna," *IEE Proc. Microw. Antennas Propag.*, vol. 143, no. I, Feb. 1996

Comparison:radiation pattern of the cylindrical helical antenna



- Measured electric field patterns of the 6-turn cylindrical helical antenna
- 3dB Beamwidth: ~ 40^o

J. D. Kraus,"Helical Beam Antennas for Wide-Band Applications," *Proceedings of the IRE*,vol. 36, no.10, pp1236-1242,1948.

Spherical Solutions

- No edge-shaped boundaries as found in cylindrical and rectangular structures
- Closed-form Green's functions obtainable
- Exact solutions used as references for checking the accuracy of numerical or approximation techniques

Helmholtz Equation in Spherical Cooridnates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Psi}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial \phi^2}+k^2\Psi=0$$

Solution

$$\Psi_{m,n}=b_n(kr)P_m^n(\cos\theta)e^{jm\phi}$$

where $b_n(kr)$ is the spherical Bessel function

$P_n^m(\cos\theta)$ is the Associated Legendre function of the first kind

$e^{jm\phi}$ is sinusoidal function.

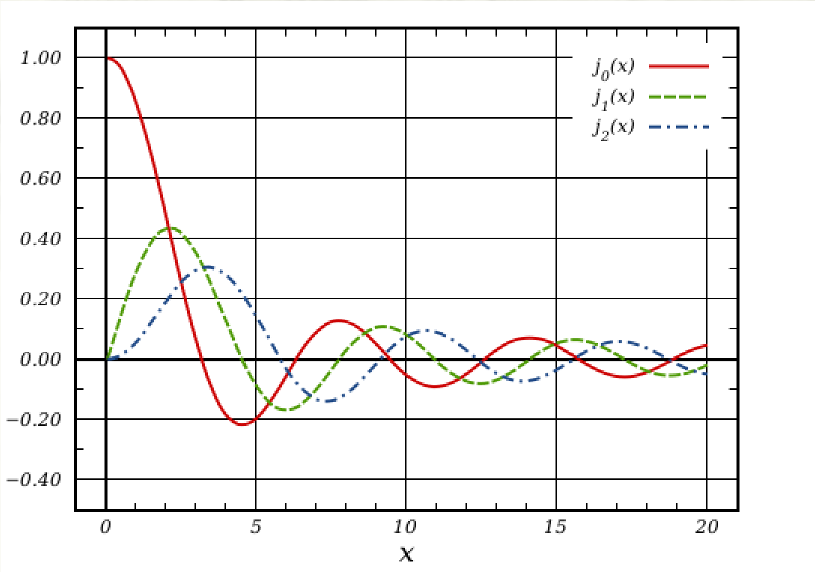
Remark: Since the associated Legendre function of the second kind, $Q_n^m(\cos\theta)$ is singular at $\theta = 0$ or π, it is generally not used for engineering EM problems.

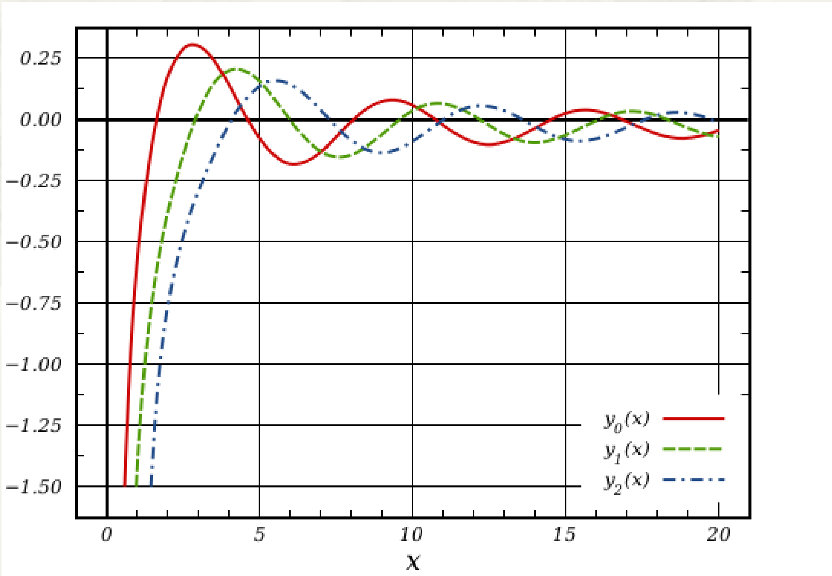
Spherical Bessel Functions

$$b_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+1/2}(kr)$$

where $B_{n+1/2}(kr)$ is the ordinary (cylindrical) Bessel function

$$b_n(kr) \sim j_n(kr), y_n(kr)$$





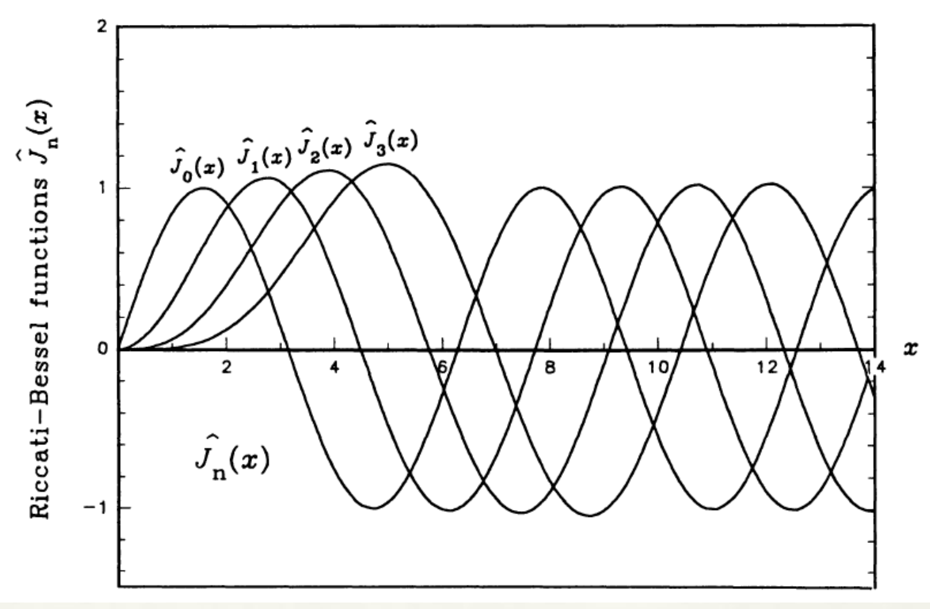
Riccati-Bessel Functions (Schelkunoff-type Spherical Bessel Function)

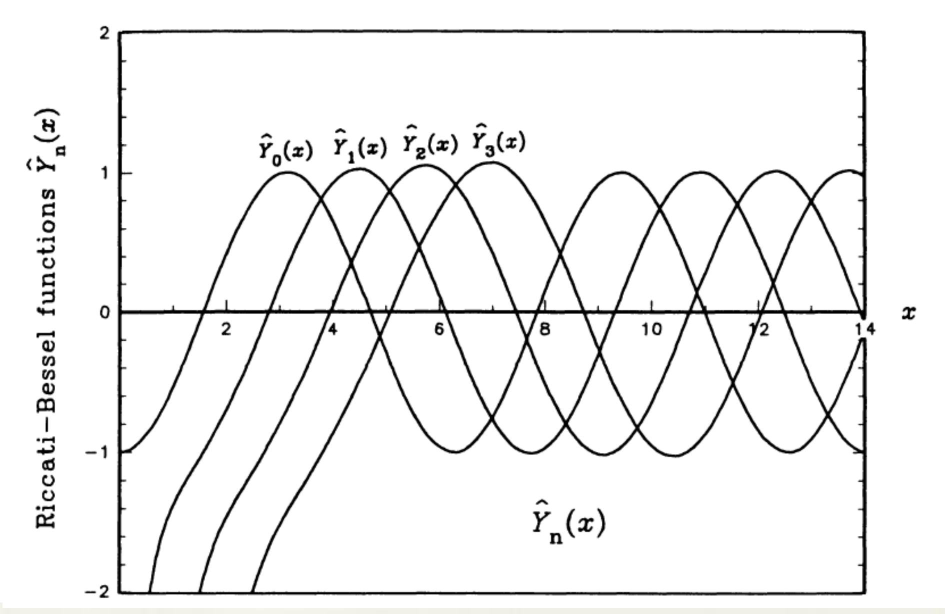
- All EM fields can be found from 2 potential functions
- Define $\vec{A} = A_r \hat{r}$ and $\vec{F} = F_r \hat{r}$
- But A_r, F_r, are not solutions of Helmholtz equation
- Instead, A_r/r, F_r/r are solutions of Helmholtz equation
- Define Riccati-Bessel function

$$\hat{B}_n(kr) = krb_n(kr) = \sqrt{\frac{\pi kr}{2}} B_{n+1/2}(kr)$$

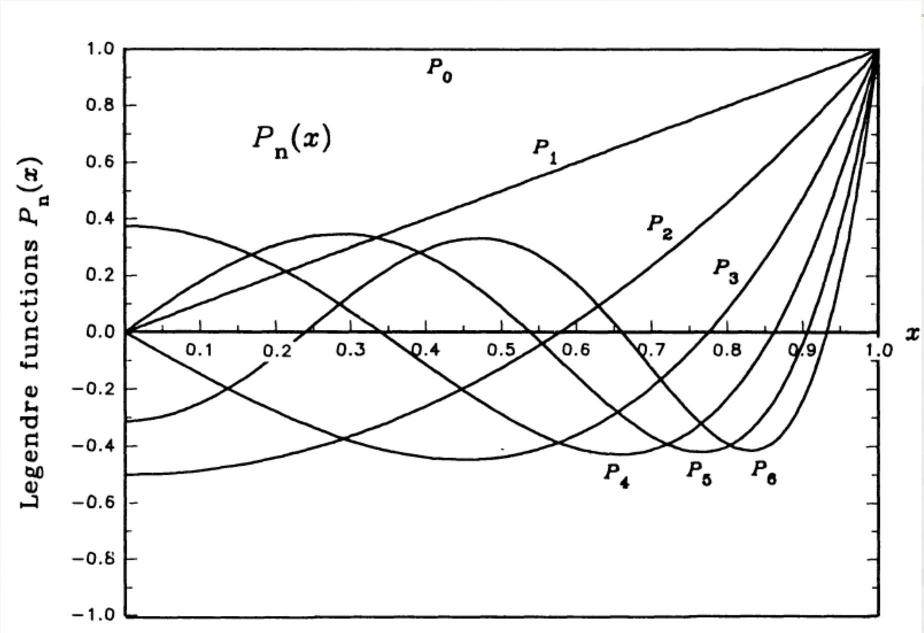
General Solutions of Sphericl Potential Functions

$$A_r, F_r, \sim \sum \hat{B}_n(kr) P_n^m(\cos\theta) e^{jm\phi}$$

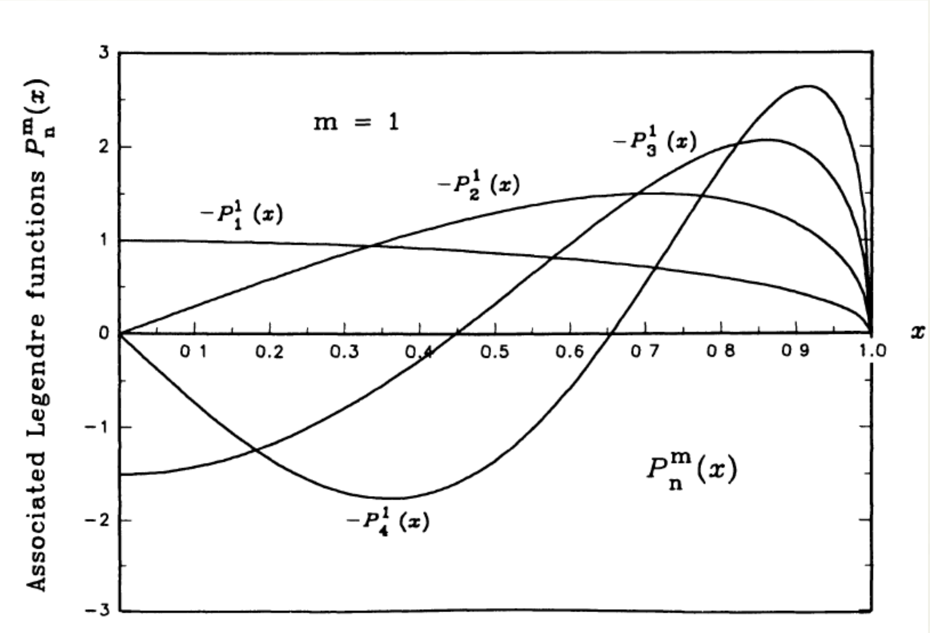




Legendre Functions



$m = 0$



$m = 1$

E & H fields in Electromagnetics

$$E_r = \frac{1}{j\omega\varepsilon u_0}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r$$

$$E_\theta = \frac{-1}{\varepsilon r sin\theta}\frac{\partial F_r}{\partial \Phi} + \frac{1}{j\omega\varepsilon u_0 r}\frac{\partial A_r}{\partial r\partial\theta}$$

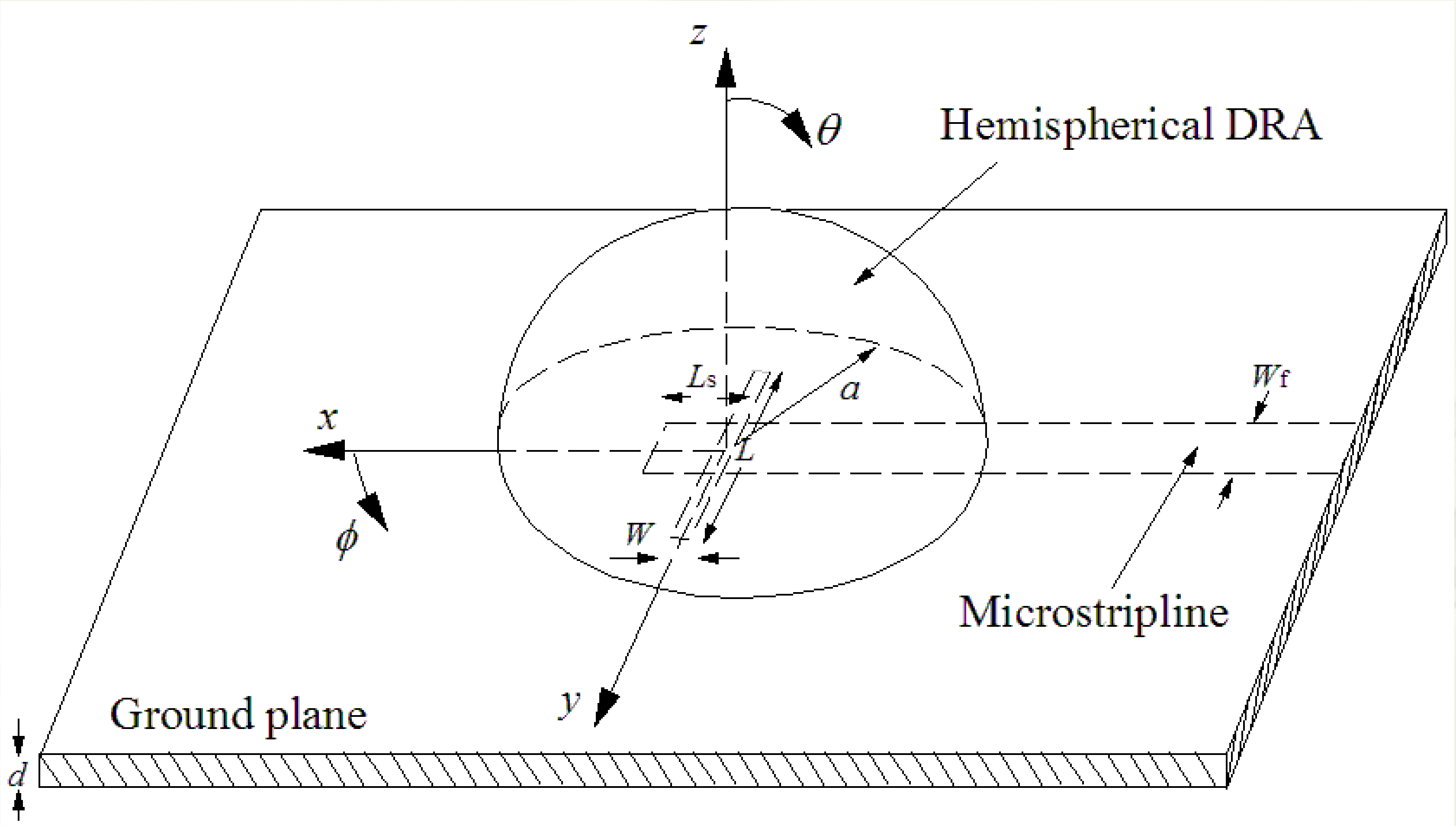
$$E_\emptyset = \frac{-1}{\varepsilon r}\frac{\partial F_r}{\partial \theta} + \frac{1}{j\omega\varepsilon u_0 r sin\theta}\frac{\partial A_r}{\partial r\partial\emptyset}$$

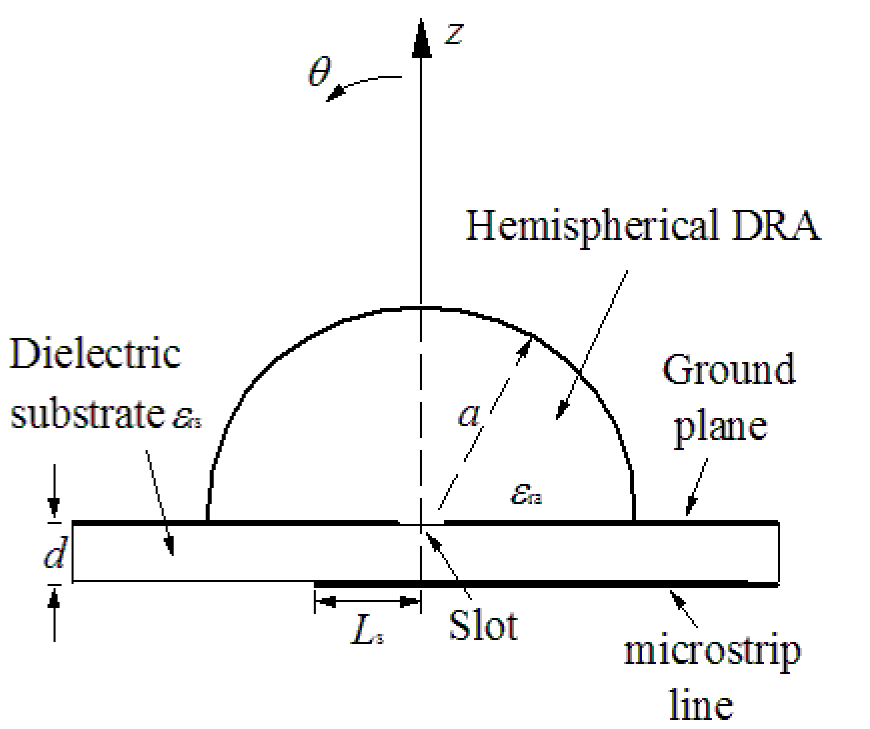
$$H_r = \frac{1}{j\omega\varepsilon u_0}\left(\frac{\partial^2}{\partial r^2} + k^2\right)F_r$$

$$H_\theta = \frac{-1}{u_0 r sin\theta}\frac{\partial A_r}{\partial \Phi} + \frac{1}{j\omega\varepsilon u_0 r}\frac{\partial F_r}{\partial r\partial\theta}$$

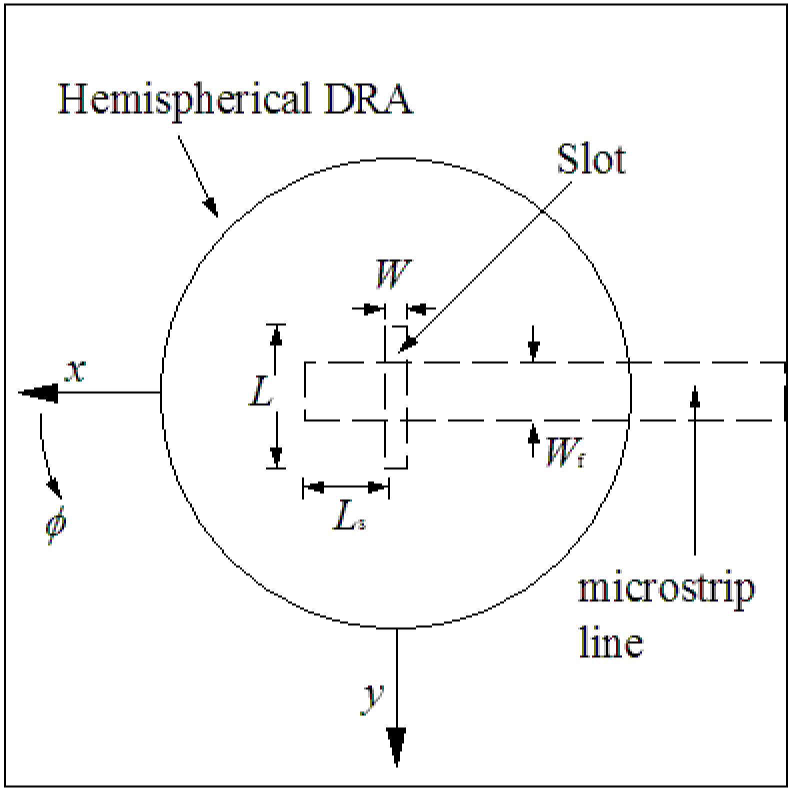
$$H_\emptyset = \frac{-1}{u_0 r}\frac{\partial A_r}{\partial \theta} + \frac{1}{j\omega\varepsilon u_0 r sin\theta}\frac{\partial F_r}{\partial r\partial\emptyset}$$

Grounded Spherical Hemispherical Dielectric Resonator Antenna: Embedded Magnetic Source

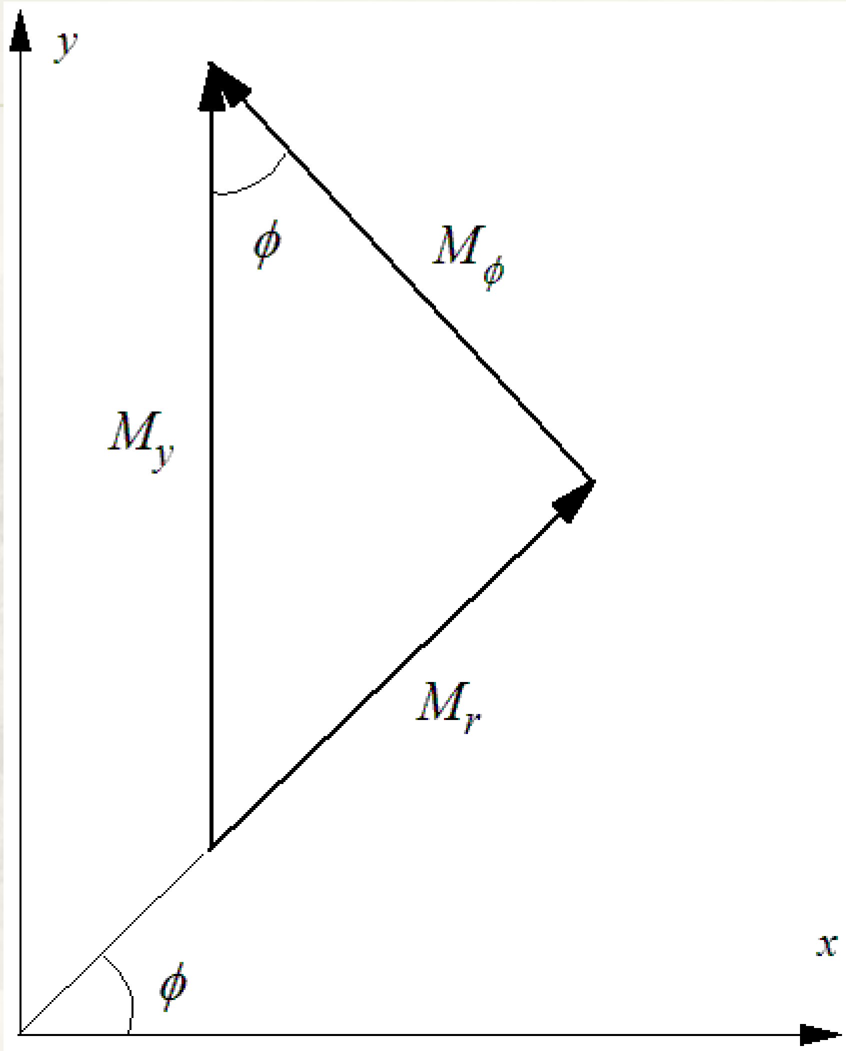




(a) Side View



(b) Top View


y
ϕ
M_ϕ
M_y
M_r
x
ϕ

Electric & Magnetic Green's Functions Due to M_ϕ

$$G_{M_\phi}^{F_{rp}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm} P_n^m(\cos\theta) e^{jm\phi} \cdot \begin{cases} \hat{J}_n'(kr')\hat{H}_n^{(2)}(kr) & r > r' \\ \hat{H}_n^{(2)}{}'(kr')\hat{J}_n(kr) & r < r' \end{cases}$$

$$G_{M_\phi}^{F_{rh}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(\cos\theta) e^{jm\phi} \cdot \begin{cases} B_{nm}\hat{J}_n(kr) & r \le a \\ C_{nm}\hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

$$G_{M_\phi}^{A_{rp}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{nm} P_n^m(\cos\theta) e^{jm\phi} \cdot \begin{cases} \hat{J}_n(kr')\hat{H}_n^{(2)}(kr) & r > r' \\ \hat{H}_n^{(2)}(kr')\hat{J}_n(kr) & r < r' \end{cases}$$

$$G_{M_\phi}^{A_{rh}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(\cos\theta) e^{jm\phi} \cdot \begin{cases} E_{nm}\hat{J}_n(kr) & r \le a \\ F_{nm}\hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

$$k = \sqrt{\varepsilon_r}\, k_0$$

$P_n^m(x)$: Associated Legendre function of the first kind (order m,degree n)

$\hat{J}_n(x)$: Spherical Bessel function of the first kind (Schelkunoff type)

$\hat{H}_n^{(2)}(x)$: Spherical Hankel function of the second kind (Schelkunoff type)

Particular solutions obtained by matching the boundary conditions at the source point ($r = r'$)

$$E_\theta^+ - E_\theta^- = -M_{\phi s}$$

$$E_\phi^+ - E_\phi^- = 0$$

$$H_\theta^+ - H_\theta^- = 0$$

$$H_\phi^+ - H_\phi^- = 0$$

$$A_{nm} = \frac{-\varepsilon r'}{4\pi} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!} \cdot m \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} M_{\phi s} P_n^m(\cos\theta) e^{-jm\phi} d\phi d\theta$$

$$D_{nm} = \frac{\omega\mu_o \varepsilon r'}{4\pi k} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} M_{\phi s} \sin\theta \frac{d}{d\theta} P_n^m(\cos\theta) e^{-jm\phi} d\phi d\theta$$

For a point source $M_{\phi s} = \dfrac{\delta(\theta-\theta')\delta(\phi-\phi')}{r'^2 \sin\theta}$

$$G_{M_\phi}^{F_{rp}} = \frac{1}{r'\sin\theta'}\sum_{n=1}^{\infty}\sum_{m=1}^{n} a_{nm} P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\cdot\begin{cases}\hat{J}_n'(kr')\hat{H}_n^{(2)}(kr) & r>r' \\ \hat{H}_n^{(2)}{}'(kr')\hat{J}_n(kr) & r<r'\end{cases}$$

$$G_{M_\phi}^{A_{rp}} = \frac{1}{r'}\sum_{n=1}^{\infty}\sum_{m=0}^{n} d_{nm}\frac{d}{d\theta'}P_n^m(\cos\theta')P_n^m(\cos\theta)\cos m(\phi-\phi')\cdot\begin{cases}\hat{J}_n(kr')\hat{H}_n^{(2)}(kr) & r>r' \\ \hat{H}_n^{(2)}(kr')\hat{J}_n(kr) & r<r'\end{cases}$$

where

$$a_{nm} = \frac{-j\varepsilon}{2\pi}\cdot\frac{2n+1}{n(n+1)}\cdot\frac{(n-m)!}{(n+m)!}\cdot m$$

$$d_{nm} = \frac{\omega\mu_o\varepsilon}{\Delta_m 2\pi k}\cdot\frac{2n+1}{n(n+1)}\cdot\frac{(n-m)!}{(n+m)!}$$

$$\Delta_m = \begin{cases}1 & \text{for } m>0 \\ 2 & \text{for } m=0\end{cases}$$

Homogenous solutions obtained by matching the boundary conditions at the dielectric surface ($r = a$)

$$E_\theta^+ - E_\theta^- = 0, \quad E_\phi^+ - E_\phi^- = 0, \quad H_\theta^+ - H_\theta^- = 0, \quad H_\phi^+ - H_\phi^- = 0$$

$$G_{M_\phi}^{F_{rh}} = \frac{1}{r'\sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} P_n^m(\cos\theta') P_n^m(\cos\theta) \sin m(\phi - \phi') \hat{J}_n'(kr') \cdot \begin{cases} b_{nm} \hat{J}_n(kr) & r \le a \\ c_{nm} \hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

$$G_{M_\phi}^{A_{rh}} = \frac{1}{r'} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{d}{d\theta'} P_n^m(\cos\theta') P_n^m(\cos\theta) \cos m(\phi - \phi') \hat{J}_n(kr') \cdot \begin{cases} e_{nm} \hat{J}_n(kr) & r \le a \\ f_{nm} \hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

where

$$b_{nm} = \frac{-a_{nm}}{\Delta_n^{TE}} \left[\hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}{}'(k_o a) - \frac{k}{k_o} \hat{H}_n^{(2)}{}'(ka) \hat{H}_n^{(2)}(k_0 a) \right]$$

$$e_{nm} = \frac{-d_{nm}}{\Delta_n^{TM}} \left[\hat{H}_n^{(2)}{}'(ka) \hat{H}_n^{(2)}(k_o a) - \frac{k}{k_o} \hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}{}'(k_0 a) \right]$$

$$c_{nm} = -j \frac{k_o}{k} \cdot \frac{a_{nm}}{\Delta_n^{TE}} , \qquad f_{nm} = j \frac{d_{nm}}{\Delta_n^{TM}}$$

Magnetic Green's Function Due to M_r

$$(\nabla^2 + k^2)\frac{G_{M_r}^{F_{rp}}}{r} = \frac{-\varepsilon}{r}\frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta}$$

Particular solution

$$G_{M_r}^{F_{rp}} = \frac{1}{r'^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n} g_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi')\cdot\begin{cases}\hat{J}_n(kr')\hat{H}_n^{(2)}(kr) & r > r' \\ \hat{H}_n^{(2)}(kr')\hat{J}_n(kr) & r < r'\end{cases}$$

where
$$g_{nm} = \frac{-j\varepsilon}{2\pi\Delta_m k}\frac{(n-m)!}{(n+m)!}(2n+1)$$

Homogenous solution

$$G_{M_r}^{F_{rh}} = \frac{1}{r'^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi')\hat{J}_n(kr')\cdot\begin{cases} h_{nm}\hat{J}_n(kr) & r \le a \\ i_{nm}\hat{H}_n^{(2)}(k_0 r) & r \ge a\end{cases}$$

where
$$h_{nm} = \frac{-g_{nm}}{\Delta_n^{TE}}\left[\hat{H}_n^{(2)}(ka)\hat{H}_n^{(2)}{}'(k_0 a) - \frac{k}{k_0}\hat{H}_n^{(2)}{}'(ka)\hat{H}_n^{(2)}(k_0 a)\right]; \quad i_{nm} = -j\frac{k}{k_0}\cdot\frac{g_{nm}}{\Delta_n^{TE}}$$

Define 3 Dyadic Green's Function $G_{M_r}^{H_r}\hat{r}\hat{r},\quad G_{M_r}^{H_\theta}\hat{\theta}\hat{r},\quad G_{M_r}^{H_\phi}\hat{\phi}\hat{r}$

$$\overline{\overline{G}}_{M_r}^{\vec{H}} = G_{M_r}^{H_r}\hat{r}\hat{r} + G_{M_r}^{H_\theta}\hat{\theta}\hat{r} + G_{M_r}^{H_\phi}\hat{\phi}\hat{r}$$

$$\overline{\overline{G}}_{M_\phi}^{\vec{H}} = G_{M_\phi}^{H_r}\hat{r}\hat{\phi} + G_{M_\theta}^{H_\phi}\hat{\theta}\hat{\phi} + G_{M_\phi}^{H_\phi}\hat{\phi}\hat{\phi}$$

Total H-field due to M_r & M_ϕ is given by

$$\vec{H} = \iint_{S_o} [\overline{\overline{G}}_{M_r}^{\vec{H}} \cdot (M_r'\hat{r}) + \overline{\overline{G}}_{M_\phi}^{\vec{H}} \cdot (M_\phi'\hat{\phi})] dS'$$

$$\vec{H} = \iint_{S_o} (G_{M_r}^{H_r} M_r'\hat{r} + G_{M_r}^{H_\theta} M_r'\hat{\theta} + G_{M_r}^{H_\phi} M_r'\hat{\phi}) + (G_{M_\phi}^{H_r} M_\phi'\hat{r} + G_{M_\phi}^{H_\theta} M_\phi'\hat{\theta} + G_{M_\phi}^{H_\phi} M_\phi'\hat{\phi}) dS'$$

$$= \iint_{S_o} (G_{M_r}^{H_r} M_r' + G_{M_\phi}^{H_r} M_\phi')\hat{r} + (G_{M_r}^{H_\theta} M_r' + G_{M_\phi}^{H_\theta} M_\phi')\hat{\theta} + (G_{M_r}^{H_\theta} M_r' + G_{M_\phi}^{H_\phi} M_\phi')\hat{\phi} dS'$$

Since

$$H_y = H_r \sin\phi + H_\phi \cos\phi$$

$$M_r' = M_y' \sin\phi'$$

$$M_\phi' = M_y' \cos\phi'$$

we have

$$\vec{H} = \iint_{S_o} [(G_{M_r}^{H_r} \sin\phi' + G_{M_\phi}^{H_r} \cos\phi')\sin\phi + (G_{M_r}^{H_\phi} \sin\phi' + G_{M_\phi}^{H_\phi} \cos\phi')\cos\phi] M_y' dS'$$

$$= \iint_{S_o} G_{M_y}^{H_y} M_y' dS'$$

Therefore, the required Green's function is given by

$$G_{M_y}^{H_y} = (G_{M_r}^{H_r} \sin\phi' + G_{M_\phi}^{H_r} \cos\phi')\sin\phi + (G_{M_r}^{H_\phi} \sin\phi' + G_{M_\phi}^{H_\phi} \cos\phi')\cos\phi$$

Expressing the Green's function in G_P & G_H:

$$G_{M_y}^{H_y} = G_P + G_H$$

where

$$G_P = \frac{1}{j\omega\mu_0\varepsilon}\cdot\frac{\sin\phi'\sin\phi}{r^2 r'^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n} n(n+1) g_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi')\Phi_n(kr')\Psi_n(kr)$$

$$+\frac{1}{j\omega\mu_0\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{r^2 r'}\sum_{n=1}^{\infty}\sum_{m=0}^{n} n(n+1) a_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta)\sin m(\phi-\phi')\Phi_n'(kr')\Psi_n(kr)$$

$$-\frac{k}{j\omega\mu_0\varepsilon}\cdot\frac{\sin\phi'\cos\phi}{r^2 r'^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n} m g_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta)\sin m(\phi-\phi')\Phi_n(kr')\Psi_n'(kr)$$

$$-\frac{1}{\mu_0}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty}\sum_{m=1}^{n} d_{nm}\frac{d}{d\theta'}P_n^m(\cos\theta')\frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\Phi_n(kr')\Psi_n(kr)$$

$$+\frac{k}{j\omega\mu_0\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty}\sum_{m=0}^{n} m a_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi')\Phi_n'(kr')\Psi_n'(kr)$$

in which

$$\Phi_n(kr') = \begin{cases}\hat{J}_n(kr'), & r > r' \\ \hat{H}_n^{(2)}(kr'), & r < r'\end{cases} \qquad \Psi_n(kr) = \begin{cases}\hat{H}_n^{(2)}(kr), & r > r' \\ \hat{J}_n(kr), & r < r'\end{cases}$$

$$G_H = \frac{1}{j\omega\mu_0\varepsilon}\cdot\frac{\sin\phi'\sin\phi}{r^2 r'^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n} n(n+1)h_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\cos m(\phi-\phi')\hat{J}_n(kr')\hat{J}_n(kr)$$

$$+\frac{1}{j\omega\mu_0\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{r^2 r'}\sum_{n=1}^{\infty}\sum_{m=0}^{n} n(n+1)b_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\hat{J}_n'(kr')\hat{J}_n(kr)$$

$$-\frac{k}{j\omega\mu_0\varepsilon}\cdot\frac{\sin\phi'\cos\phi}{rr'^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n} mh_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\hat{J}_n(kr')\hat{J}_n'(kr)$$

$$-\frac{1}{\mu_0}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty}\sum_{m=1}^{n} e_{nm}\frac{d}{d\theta'}P_n^m(\cos\theta')\frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\hat{J}_n(kr')\hat{J}_n(kr)$$

$$+\frac{k}{j\omega\mu_0\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty}\sum_{m=0}^{n} mb_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\cos m(\phi-\phi')\hat{J}_n'(kr')\hat{J}_n'(kr)$$

Converting double summations to single summation

By applying the addition theorem for Legendre polynomials

$$P_n(\cos\xi) = \sum_{m=o}^{\infty} \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi')$$

where

$$\cos\xi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$G_{M_y}^{H_y} = G_P + G_H$$

$$G_P = \frac{-1}{4\pi\omega\mu_0 k} \cdot \frac{\sin\phi'\sin\phi}{r^2 r'^2} \sum_{n=1}^{\infty} n(n+1)(2n+1) P_n(\cos(\phi - \phi'))\Phi_n(kr')\Psi_n(kr)$$

$$- \frac{1}{4\pi\omega\mu_0} \cdot \frac{\cos\phi'\sin\phi}{r^2 r'} \sum_{n=1}^{\infty} (2n+1) \frac{\partial}{\partial\phi'} P_n(\cos(\phi - \phi'))\Phi_n'(kr')\Psi_n(kr)$$

$$- \frac{1}{4\pi\omega\mu_0} \cdot \frac{\sin\phi'\cos\phi}{r r'^2} \sum_{n=1}^{\infty} (2n+1) \frac{\partial}{\partial\phi} P_n(\cos(\phi - \phi'))\Phi_n(kr')\Psi_n'(kr)$$

$$- \frac{\omega\varepsilon}{4\pi k} \cdot \frac{\cos\phi'\cos\phi}{r r'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n'(\cos(\phi - \phi'))\Phi_n(kr')\Psi_n(kr)$$

$$- \frac{k}{4\pi\omega\mu_o} \cdot \frac{\cos\phi'\cos\phi}{r r'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \frac{\partial^2}{\partial\phi\partial\phi'} P_n'(\cos(\phi - \phi'))\Phi_n'(kr')\Psi_n'(kr)$$

where we have used the fact that $\theta = \theta' = \pi/2$.

$$G_H = \frac{-1}{4\pi\omega\mu_0 k}\cdot\frac{\sin\phi'\sin\phi}{r^2 r'^2}\sum_{n=1}^{\infty} b_n n(n+1)(2n+1)P_n(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n(kr)$$

$$-\frac{1}{4\pi\omega\mu_0}\cdot\frac{\cos\phi'\sin\phi}{r^2 r'}\sum_{n=1}^{\infty} b_n(2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\hat{J}_n'(kr')\hat{J}_n(kr)$$

$$-\frac{1}{4\pi\omega\mu_0}\cdot\frac{\sin\phi'\cos\phi}{rr'^2}\sum_{n=1}^{\infty} b_n(2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n'(kr)$$

$$-\frac{\omega\varepsilon}{4\pi k}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty} e_n\frac{2n+1}{n(n+1)}P_n'(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n(kr)$$

$$-\frac{k}{4\pi\omega\mu_o}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty} b_n\frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n'(\cos(\phi-\phi'))\hat{J}_n'(kr')\hat{J}_n'(kr)$$

where

$$b_n = \frac{-1}{\Delta_n^{TE}}\left[\hat{H}_n^{(2)}(ka)\hat{H}_n^{(2)}{}'(k_o a) - \frac{k}{k_o}\hat{H}_n^{(2)}{}'(ka)\hat{H}_n^{(2)}(k_o a)\right]$$

$$e_n = \frac{-1}{\Delta_n^{TM}}\left[\hat{H}_n^{(2)}{}'(ka)\hat{H}_n^{(2)}(k_o a) - \frac{k}{k_o}\hat{H}_n^{(2)}(ka)\hat{H}_n^{(2)}{}'(k_o a)\right]$$

Numerical Problem for G_P

- G_P is an infinite summation over n
- Hankel functions of very high orders have very large ampliudes
- Difficult to handle numerically

Solution

Recall that physically G_p represents a z-directed electric field excited by a z-directed point current, therefore

$$G_P = \frac{-j}{\omega\mu_o}\left[\frac{\partial^2}{\partial y^2} + k^2\right]\frac{e^{-jkR}}{4\pi R}$$

where

$$R = \sqrt{(r\cos\phi - r'\cos\phi')^2 + (y - r'\sin\phi')^2}$$

Mathematical Identity

Based on this fact, a mathematical identity can be established:

$$\frac{-j}{\omega\mu_o}\left[\frac{\partial^2}{\partial y^2}+k^2\right]\frac{e^{-jkR}}{4\pi R} \qquad \left(R=\sqrt{(r\cos\phi - r'\cos\phi')^2+(y-r'\sin\phi')^2}\right)$$

$$= \frac{-1}{4\pi\omega\mu_0 k}\cdot\frac{\sin\phi'\sin\phi}{r^2 r'^2}\sum_{n=1}^{\infty} n(n+1)(2n+1)P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr)$$

$$- \frac{1}{4\pi\omega\mu_0}\cdot\frac{\cos\phi'\sin\phi}{r^2 r'}\sum_{n=1}^{\infty}(2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n(kr)$$

$$- \frac{1}{4\pi\omega\mu_0}\cdot\frac{\sin\phi'\cos\phi}{r r'^2}\sum_{n=1}^{\infty}(2n+1)\frac{\partial}{\partial\phi}P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n'(kr)$$

$$- \frac{\omega\varepsilon}{4\pi k}\cdot\frac{\cos\phi'\cos\phi}{r r'}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n'(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr)$$

$$- \frac{k}{4\pi\omega\mu_o}\cdot\frac{\cos\phi'\cos\phi}{r r'}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n'(kr)$$

Finally, the required Green's function is given as follows:

$$G_{M_y}^{H_y} = \frac{-j}{\omega\mu_o}\left[\frac{\partial^2}{\partial y^2} + k^2\right]\frac{e^{-jkR}}{4\pi R}$$

$$= \frac{-1}{4\pi\omega\mu_0 k}\cdot\frac{\sin\phi'\sin\phi}{r^2 r'^2}\sum_{n=1}^{\infty} b_n n(n+1)(2n+1)P_n(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n(kr)$$

$$-\frac{1}{4\pi\omega\mu_0}\cdot\frac{\cos\phi'\sin\phi}{r^2 r'}\sum_{n=1}^{\infty} b_n(2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\hat{J}_n'(kr')\hat{J}_n(kr)$$

$$-\frac{1}{4\pi\omega\mu_0}\cdot\frac{\sin\phi'\cos\phi}{rr'^2}\sum_{n=1}^{\infty} b_n(2n+1)\frac{\partial}{\partial\phi}P_n(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n'(kr)$$

$$-\frac{\omega\varepsilon}{4\pi k}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty} e_n\frac{2n+1}{n(n+1)}P_n'(\cos(\phi-\phi'))\hat{J}_n(kr')\hat{J}_n(kr)$$

$$-\frac{k}{4\pi\omega\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty} b_n\frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n'(\cos(\phi-\phi'))\hat{J}_n'(kr')\hat{J}_n'(kr)$$

Method-of-Moments Solution

Expand the equivalent magnetic current of the slot:

$$M(y) = \sum_{n=1}^{N} V_n f_n(y)$$

where

$$f_n(x, y) = f_u(x) f_p(y - y_n)$$

$$f_u(x) = \begin{cases} \dfrac{1}{W} & |x| < W/2 \\ 0 & |x| > W/2 \end{cases}$$

$$f_p(y) = \begin{cases} \dfrac{\sin k_e (h - |y|)}{\sin k_e h} & |y| < h \\ 0 & |y| > h \end{cases}$$

MoM Admittance of the Grounded Hemispherical Dielectric Resonator Antenna

$$Y_{mn}^{a} = Y_{mn}^{p} + Y_{mn}^{H}$$

where

$$Y_{mn}^{P} = -2\iint_{S_0}\iint_{S_0} f_m(x,y) G_p f_n(x',y') dS' dS$$

$$Y_{mn}^{H} = -2\iint_{S_0}\iint_{S_0} f_m(x,y) G_H f_n(x',y') dS' dS$$

However, G_P is singular

$\Rightarrow$ Difficult to integrate Y_{mn}^{p} numerically

Solution: Use the reduced kernel (Richmond form)

$$Y_{mn}^{p} = \frac{-2}{\eta^2}\left[\frac{-j\eta}{4\pi k}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} f_p(y-y_m)\frac{e^{-jk\zeta_e}}{\zeta_e^5}\left[(1+jk\zeta_e)(2\zeta_e{}^2-3a_e{}^2)+a_e{}^2k^2\zeta_e{}^2\right]f_p(y'-y_n)dy'dy\right]$$

where

$$\zeta_e = \sqrt{(y-y')^2 + a_e{}^2}$$

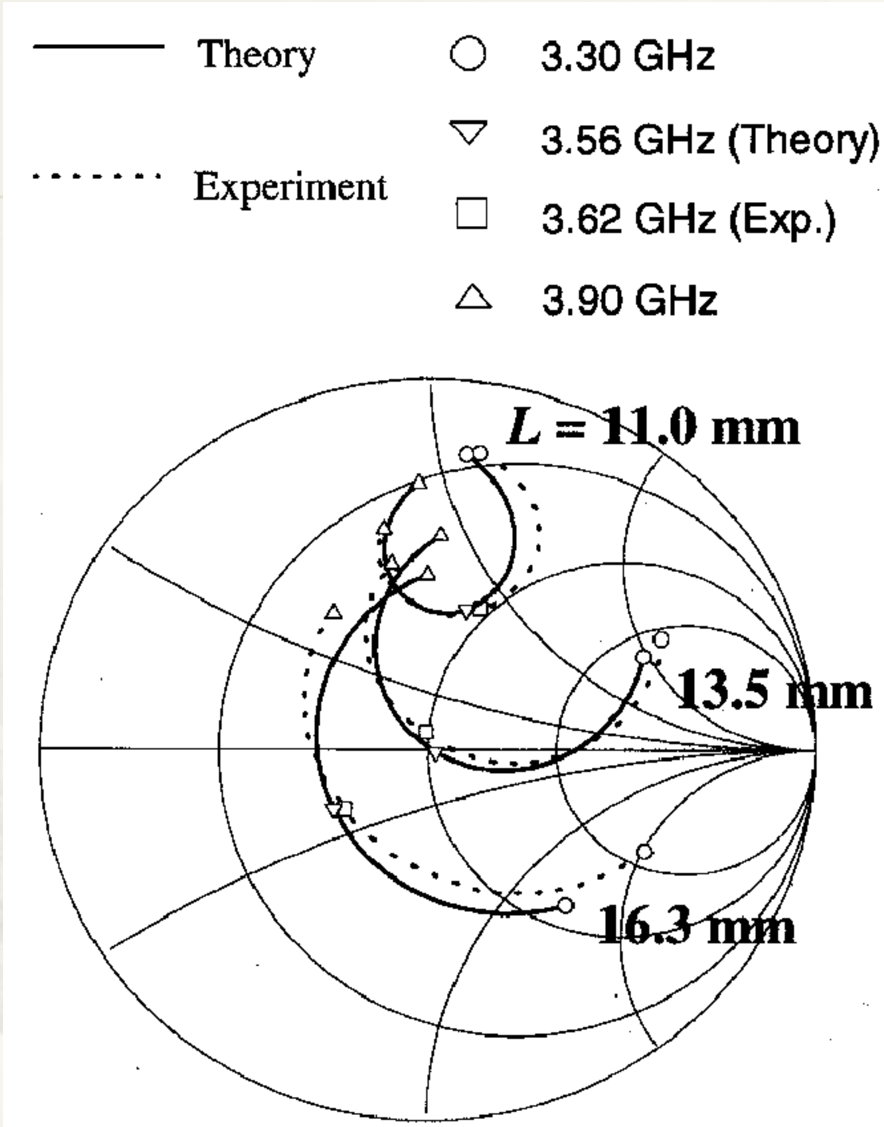
with $a_e = W/4$ being the equivalent radius of the slot.

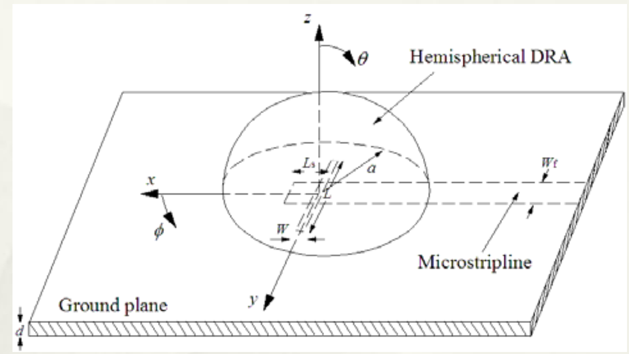
Formulation of the Microstrip Feed network

- Apply the reciprocity approach as done by David Pozar*
- Not repeated here.

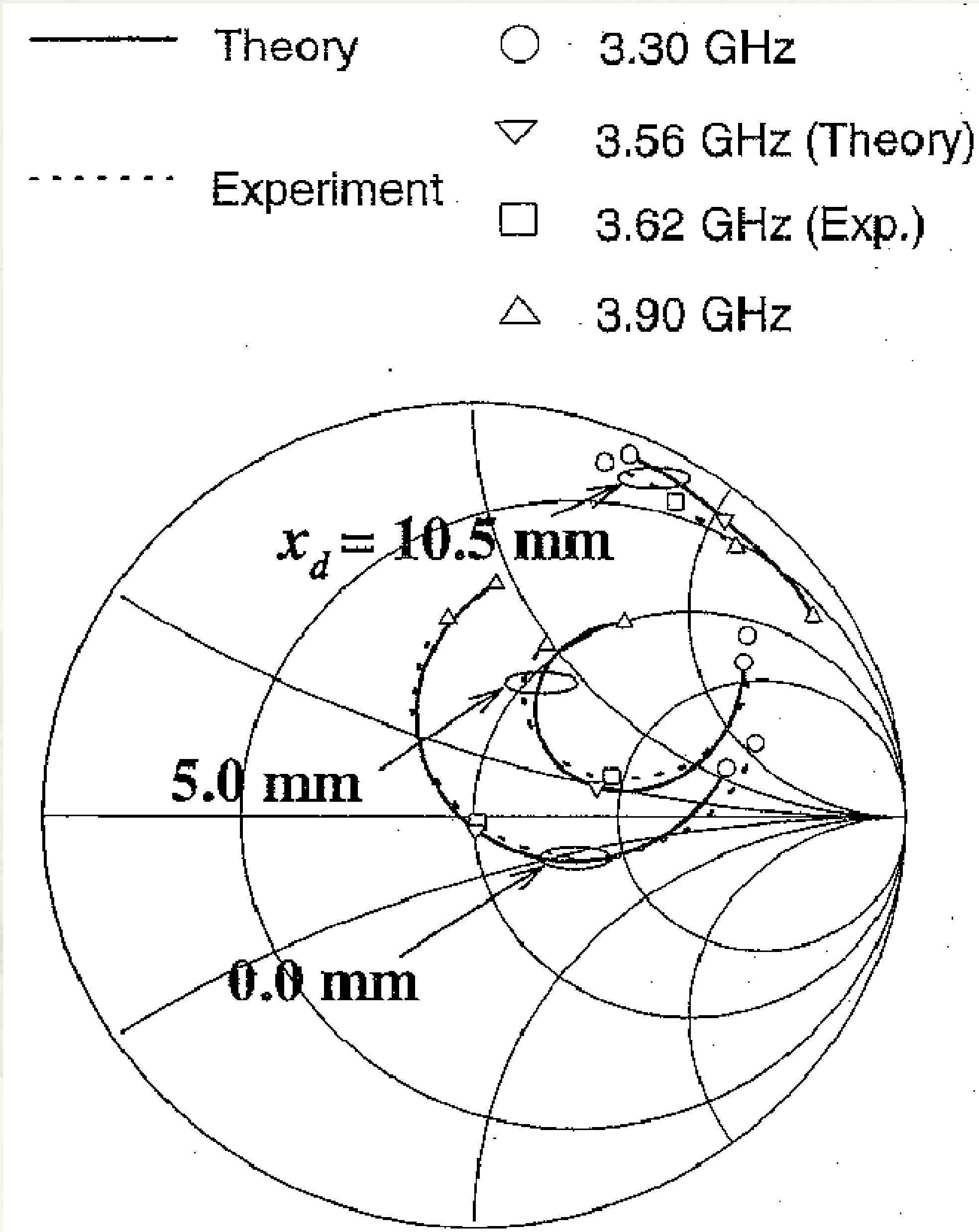
* D. M. Pozar, "A reciprocity method of analysis for printed slot and slot-coupled microstrip antennas," *IEEE Trans. Antennas Progagat.*, vol.34, pp. 1439-1446, Dec. 1986.

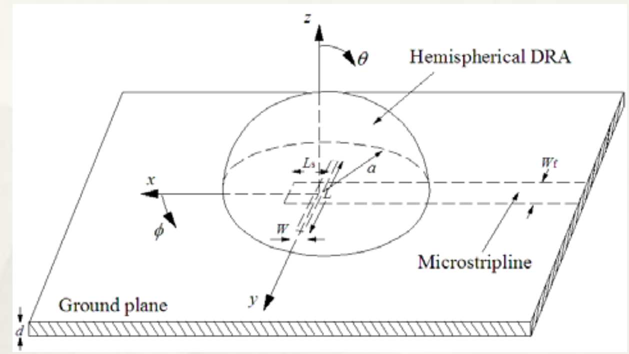
Results



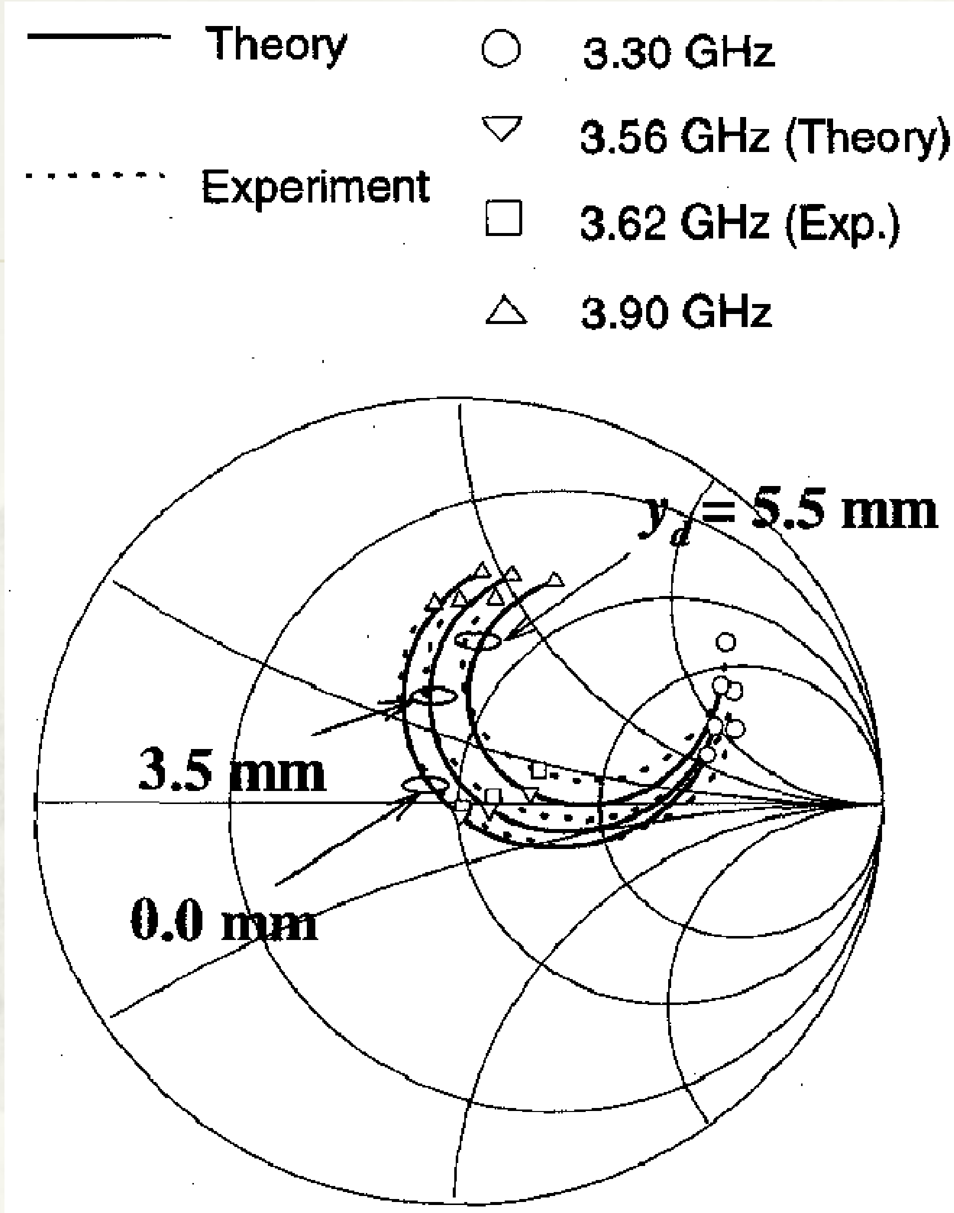


At the reference plane (slot position): a = 12.5 mm, ε_{ra} = 9.5, $x_d = y_d$ = 0.0, W = 0.9 mm, W_f = 1.45 mm, d = 0.635 mm, ε_{rs} = 2.96, L_s = 13.6 mm .



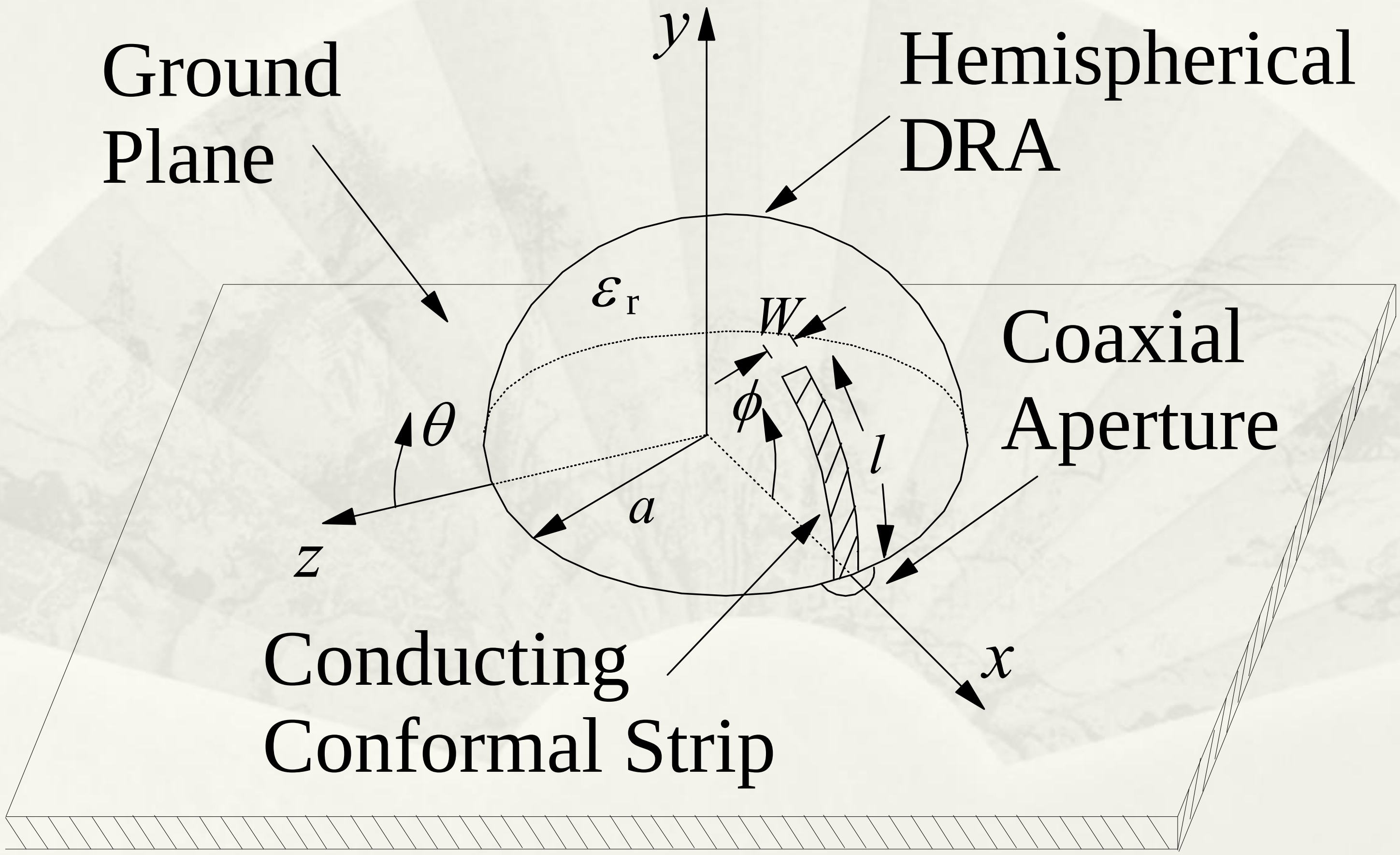


At the reference plane (slot position): a = 12.5 mm, ε_{ra} = 9.5, L = 13.5mm, y_d = 0.0, W = 1.3 mm, W_f = 1.45 mm, d = 0.635 mm, ε_{rs} = 2.96, L_s = 13.6 mm .



At the reference plane (slot position): a = 12.5 mm, ε_{ra} = 9.5, L = 13.5mm, x_d = 0.0, W = 1.3 mm, W_f = 1.45 mm, d = 0.635 mm, ε_{rs} = 2.96, L_s = 13.6 mm .

Grounded Spherical Hemispherical Dielectric Resonator Antenna: Surface Electric Source



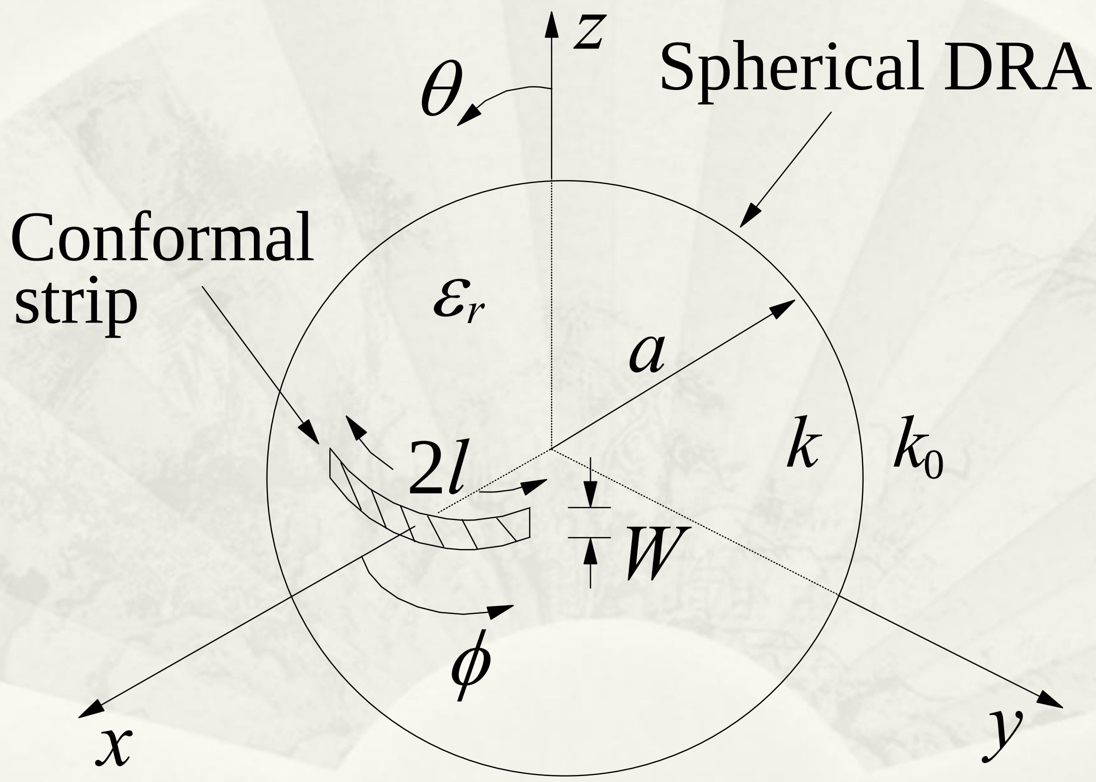
Advantages of Conformal-Strip Method

- Compatible with the probe-feed method
- Drilling hole not required
- Very convenient post manufacturing trimmings
  - Cut shorter without leaving an air hole
  - Extended longer without the need for deepening the hole
- It is conformal

Hemispherical DRA for Demonstration

- Analytical closed-form Green function can be obtained
- Efficient in numerical implementation
- Excited in the fundamental broadside TE_{111} mode

Co-ordinate system in the analysis



DRA Green's Function

Inside the DRA ($r < a$)

$$F_{r1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} \hat{J}_n(kr) P_n^m(\cos\theta) e^{jm\phi} \qquad (1)$$

$$A_{r1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{nm} \hat{J}_n(kr) P_n^m(\cos\theta) e^{jm\phi} \qquad (2)$$

Outside the DRA ($r > a$)

$$F_{r2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{nm} \hat{H}_n^{(2)}(k_0 r) P_n^m(\cos\theta) e^{jm\phi} \qquad (3)$$

$$A_{r2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} D_{nm} \hat{H}_n^{(2)}(k_0 r) P_n^m(\cos\theta) e^{jm\phi} \qquad (4)$$

Boundary Conditions

At the DRA-air interface:

$$\hat{r} \times (\vec{E}_2 - \vec{E}_1) = 0 \qquad (5)$$

$$\hat{r} \times \left(\vec{H}_2 - \vec{H}_1\right) = J_{\phi s}\, \hat{\phi} \qquad (6)$$

where $J_{\phi s}$ is the conformal strip current.

On the DRA surface, we have $r = r' = a$ and $G_1 = G_2 = G$, which is given by:

$$G = \frac{j\eta_0}{4\pi a^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{2n+1}{n(n+1)\Delta_n^{\mathrm{TE}}}\hat{J}_n(ka)\hat{H}_n^{(2)}(k_0 a)$$

$$\cdot\frac{2}{\Delta_m}\cdot\frac{(n-m)!}{(n+m)!}\cdot\frac{dP_n^m(\cos\theta)}{d\theta}\cdot\frac{dP_n^m(\cos\theta')}{d\theta'}\cos m(\phi-\phi')$$

$$-\frac{j\eta_0}{4\pi a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{2n+1}{n(n+1)\Delta_n^{\mathrm{TM}}}\hat{J}_n'(ka)\hat{H}_n^{(2)\prime}(k_0 a) \qquad (7)$$

$$\cdot 2m^2\cdot\frac{(n-m)!}{(n+m)!}\cdot\frac{P_n^m(\cos\theta)}{\sin\theta}\cdot\frac{P_n^m(\cos\theta')}{\sin\theta'}\cos m(\phi-\phi')$$

where $$\Delta_n^{\mathrm{TE}} = \hat{J}_n(ka)\hat{H}_n^{(2)\prime}(k_0 a) - \frac{k}{k_0}\hat{J}_n'(ka)\hat{H}_n^{(2)}(k_0 a) \qquad (8)$$

$$\Delta_n^{\mathrm{TM}} = \hat{J}_n'(ka)\hat{H}_n^{(2)}(k_0 a) - \frac{k}{k_0}\hat{J}_n(ka)\hat{H}_n^{(2)\prime}(k_0 a) \qquad (9)$$

$$\Delta_m = \begin{cases} 1, & m > 0 \\ 2, & m = 0 \end{cases} \qquad (10)$$

and η_0 is the wave impedance in vacuum.

MoM Solution for the Strip Current

Using the MoM:

$$I(\phi) = \sum_{q=1}^{N} I_q f_q(\phi) \qquad (14)$$

where

$$f_q(\phi) = \begin{cases} \dfrac{\sin k_e(h - a|\phi - \phi_q|)}{\sin k_e h}, & a|\phi - \phi_q| < h \\ 0, & a|\phi - \phi_q| \geq h \end{cases} \qquad (15)$$

in which

$$h = \frac{L}{N+1}, \quad \phi_q = \frac{1}{a}\left(\frac{-L}{2} + qh\right), \quad k_e = \sqrt{(\varepsilon_r + 1)/2}\, k_0 \qquad (16\text{-}18)$$

MoM Solution for the Strip Current (Cont')

By Galerkin's procedure, the following matrix equation is obtained:

$$[Z_{pq}][I_q] = [f_p(0)] \tag{19}$$

where

$$Z_{pq} = \frac{-1}{W^2} \iint_{S_0} \iint_{S_0} f_p(\phi)\, G(\theta,\phi;\theta',\phi')\, f_q(\phi')\, dS'dS \tag{20}$$

The input impedance is given by

$$Z_{\text{in}} = \frac{1}{\sum_{q=1}^{N} I_q f_q(0)} \tag{21}$$

Problems in Evaluating Z_{pq}

- The Green function G is singular as $\vec{r} \to \vec{r}\,'$
- Excessive no. of modal terms required
- Higher-order Hankel functions difficult to handle numerically
- Considerable computation time required

Solution

Integral Z_{pq} evaluated using novel recurrence formulas.

First express Z_{pq} in the following form:

$$Z_{pq} = \frac{-ja^2\eta_0}{4\pi W^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{\frac{\hat{J}_n(ka)\hat{H}_n^{(2)}(k_0a)}{\Delta_n^{TE}}\sum_{m=0}^{n}\frac{2}{\Delta_m}\cdot\frac{(n-m)!}{(n+m)!}[\Theta_1(n,m)]^2\,\Phi_1(p,q,m)\right.$$

$$\left.-\frac{\hat{J}_n'(ka)\hat{H}_n^{(2)}{}'(k_0a)}{\Delta_n^{TM}}\sum_{m=1}^{n}2m^2\cdot\frac{(n-m)!}{(n+m)!}[\Theta_2(n,m)]^2\,\Phi_1(p,q,m)\right\}$$

where

$$\Theta_1(n,m) = \int_{\theta_1}^{\theta_2}\frac{dP_n^m(\cos\theta)}{d\theta}\sin\theta\,d\theta, \qquad \Theta_2(n,m) = \int_{\theta_1}^{\theta_2}P_n^m(\cos\theta)\,d\theta$$

$$\Phi_1(p,q,m) = \int_{\phi=\phi_p-\phi_h}^{\phi_p+\phi_h}\int_{\phi'=\phi_q-\phi_h}^{\phi_q+\phi_h}f_p(\phi)\cos m(\phi-\phi')\,f_q(\phi')\,d\phi'\,d\phi, \qquad \phi_h = h/a.$$

Recurrence formulas for $\Theta_2(n,m)$

(A) Recurrence formula recursive in n

$$\Theta_2(n+1,m) = \frac{1}{(n+1)(n-m+1)}\left\{-(2n+1)\left[1-(-1)^{n+m}\right]\sqrt{1-x_1^2}\,P_n^m(x_1) + n(n+m)\Theta_2(n-1,m)\right\}$$

Initial values :

$\Theta_2(0,0) = 2\sin^{-1}x_1$

$\Theta_2(1,0) = 0$

$\Rightarrow$ $\Theta_2(n,0)$ can be found for all n.

(B) Recurrence formula recursive in m

$$\Theta_2(n,m+2) = -2[1+(-1)^{n+m}]P_n^{m+1}(x_1) + (n+m+1)(n-m)\Theta_2(n,m)$$

Initial values :

$\Theta_2(n,0)$ [from (A) above]

$\Theta_2(n,1) = [(-1)^n - 1]\,P_n(x_1)$

$\Rightarrow$ $\Theta_2(n,m)$ can be found for all m and n.

Evaluation of $\Theta_2(n,m)$ and $\Phi_1(p,q,m)$

Evaluation of $\Theta_1(n,m)$

$\Theta_1(n, m)$ are found in terms of $\Theta_2(n, m)$:

$$\Theta_1(n,m) = \left[(-1)^{n+m} - 1\right]\sqrt{1-x_1^2}\, P_n^m(x_1) - \frac{1}{2n+1}\left[(n+m)\Theta_2(n-1,m) - (m-n-1)\Theta_2(n+1,m)\right]$$

Analytical evaluation of $\Phi_1(p,q,m)$

$$\Phi_1(p,q,m) = \int_{\phi=\phi_p-\phi_h}^{\phi_p+\phi_h} \int_{\phi'=\phi_q-\phi_h}^{\phi_q+\phi_h} f_p(\phi)\cos m(\phi-\phi')\, f_q(\phi')\, d\phi'\, d\phi$$

$$= \left(\frac{2k_e a\left(\cos k_e h - \cos m\phi_h\right)}{\left(\sin k_e h\right)\left(m - k_e a\right)\left(m + k_e a\right)} \right)^2 \cos[m(\phi_p - \phi_q)]$$

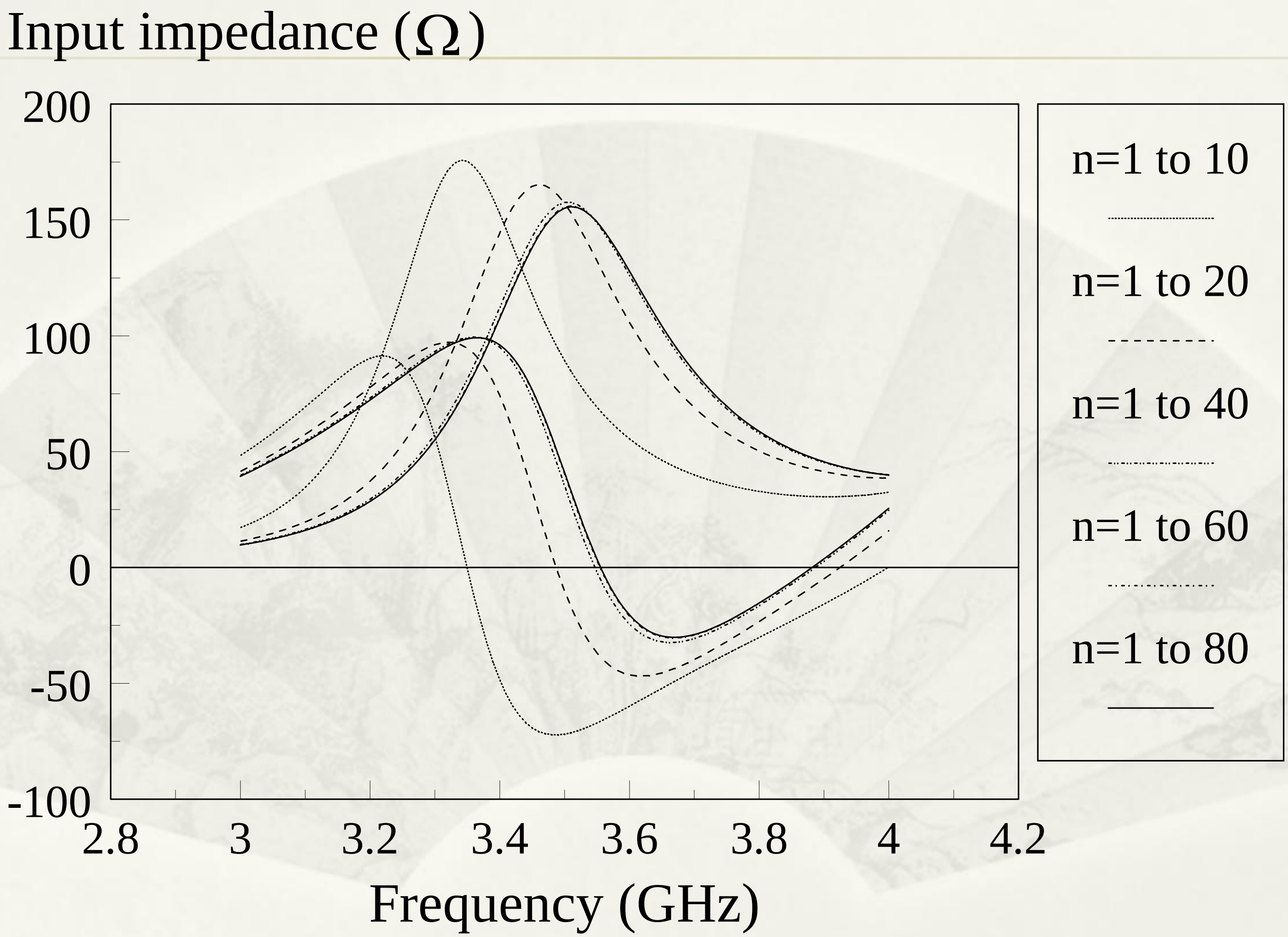
Radiation Fields

$$E_\theta(r,\theta,\phi) = \frac{j\eta_0 a}{4\pi W} \cdot \frac{e^{-jk_0 r}}{r} \sum_{q=1}^{N} I_q E_{\theta q}(\theta,\phi)$$

$$E_\phi(r,\theta,\phi) = \frac{-\eta_0 a}{4\pi W} \cdot \frac{e^{-jk_0 r}}{r} \sum_{q=1}^{N} I_q E_{\phi q}(\theta,\phi)$$

where $E_{\phi q}(\theta,\phi)$ and $E_{\phi q}(\theta,\phi)$ are found in a similar fashion.

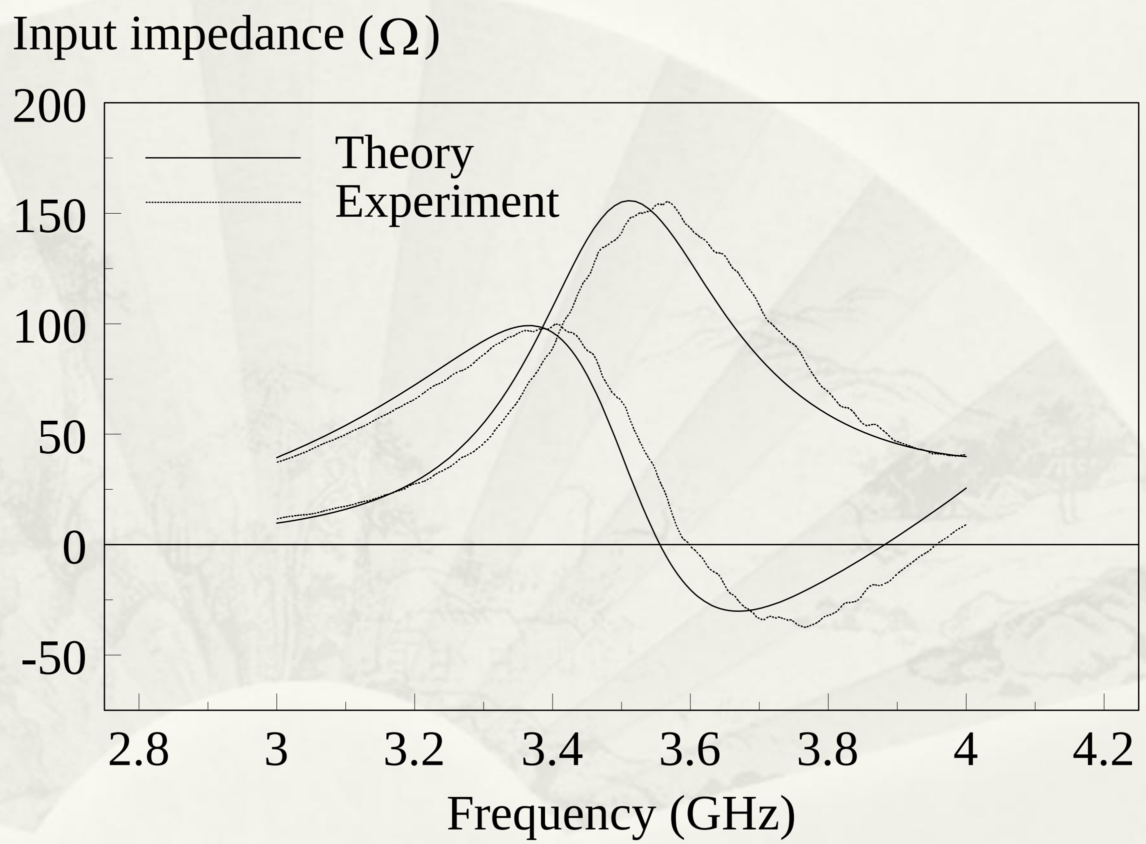
Convergence Check



a = 12.5 mm, ε_r = 9.5, l = 12.0 mm, and W = 1.2 mm.

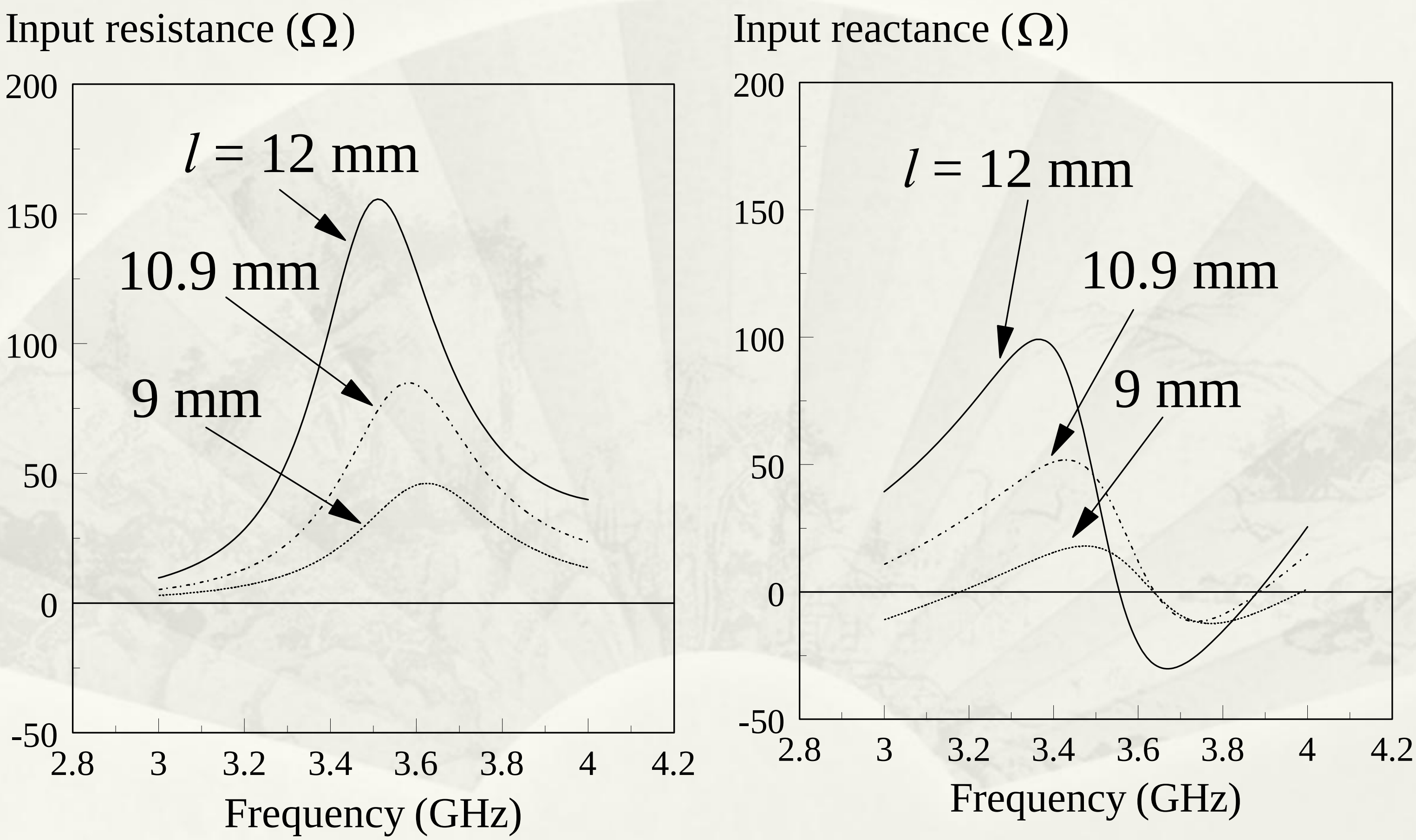
Measured and Calculated Input Impedances

- Measured resonant frequency (zero reactance):3.60 GHz
- Calculated resonant frequency (zero reactance):3.56 GHz
- Error :1.1 %



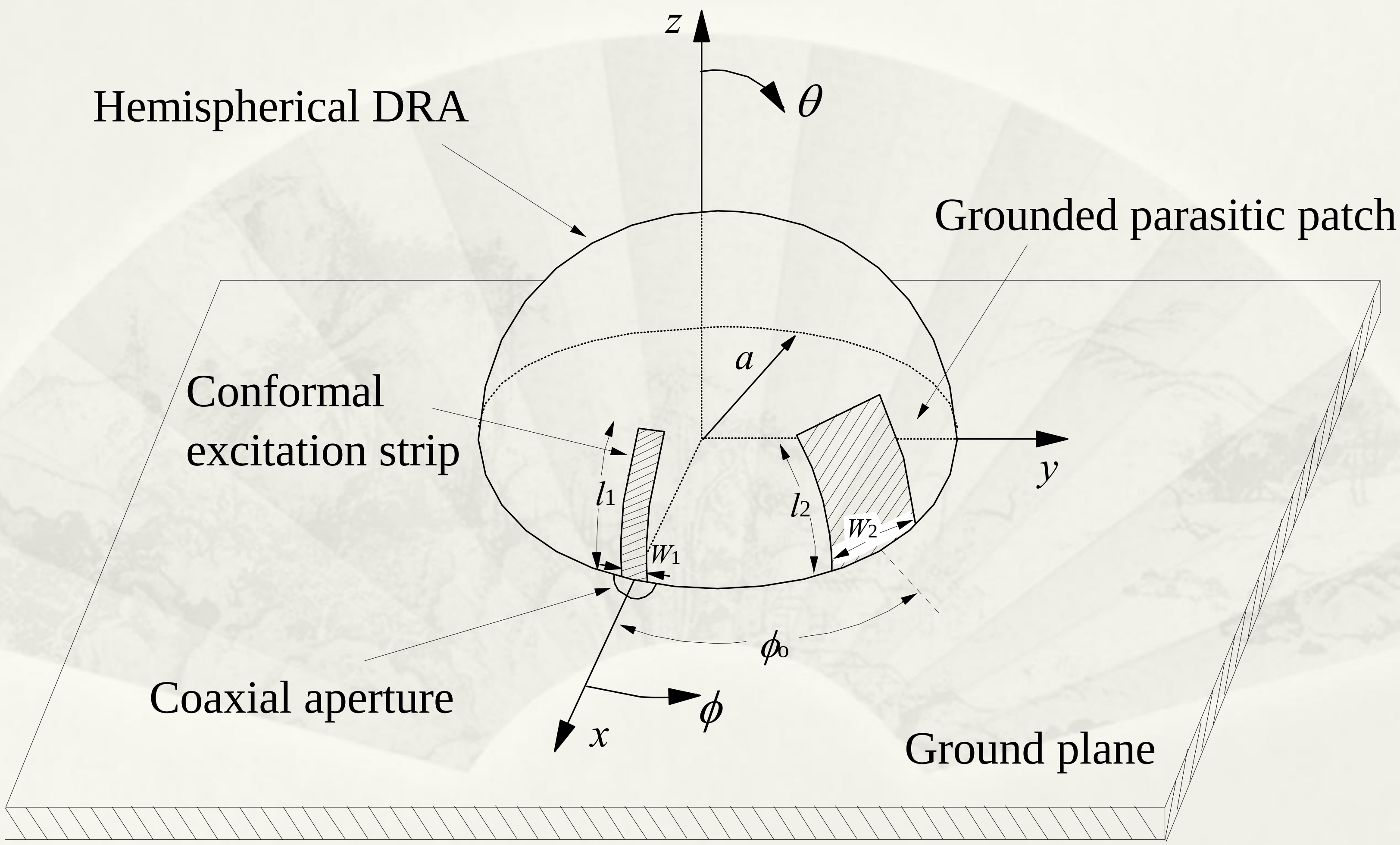
a = 12.5 mm, ε_r = 9.5, l = 12.0 mm, and W = 1.2 mm

Results



a = 12.5 mm, ε_r = 9.5, and W = 1.2 mm

Modified Configuration with a Parasitic Patch



Modified Configuration with a Parasitic Patch

Superscript A : excitation strip
Superscript B : parasitic patch

E-field vanishes on the excitation strip

$$^{A}E_{J_\theta}^{\theta} + {}^{B}E_{J_\theta}^{\theta} + {}^{B}E_{J_\phi}^{\theta} + {}^{A}E^{i} = 0$$

In terms of Green functions

$$\iint_{S_A} G_{J_\theta}^{E_\theta} J_\theta^A dS' + \iint_{S_B} G_{J_\theta}^{E_\theta} J_\theta^B dS' + \iint_{S_B} G_{J_\phi}^{E_\theta} J_\phi^B dS' + {}^{A}E^{i} = 0$$

Current expansions of Excitation & Parasitic Patches

$$I_\theta^A(\theta) = \sum_{p=1}^{N_1} I_{\theta p}^A f_{\theta p}^A(\theta)$$

$$I_\theta^B(\theta,\phi) = \sum_{l=1}^{N_2}\sum_{m=1}^{N_4} I_{lm}^{B\theta} f_{\theta l}^B(\theta) g_{\phi m}(\phi)$$

$$I_\phi^B(\theta,\phi) = \sum_{v=1}^{N_3}\sum_{n=1}^{N_5} I_{vn}^{B\phi} g_{\theta n}(\theta) f_{\phi v}^B(\phi)$$

where $f_{\theta l}^B(\theta)$ and $g_{\theta n}(\theta)$ are of the following form:

$$f_{\theta l}{}^B(\theta), g_{\theta n}(\theta) \sim \begin{cases} \dfrac{\sin(\theta_h - |\theta-\theta_p|)}{\sin\theta_h}, & a\,|\theta-\theta_p| < h \\ 0, & \text{elsewhere} \end{cases}$$

New Integrals of $P_n^m(\cos\theta)$

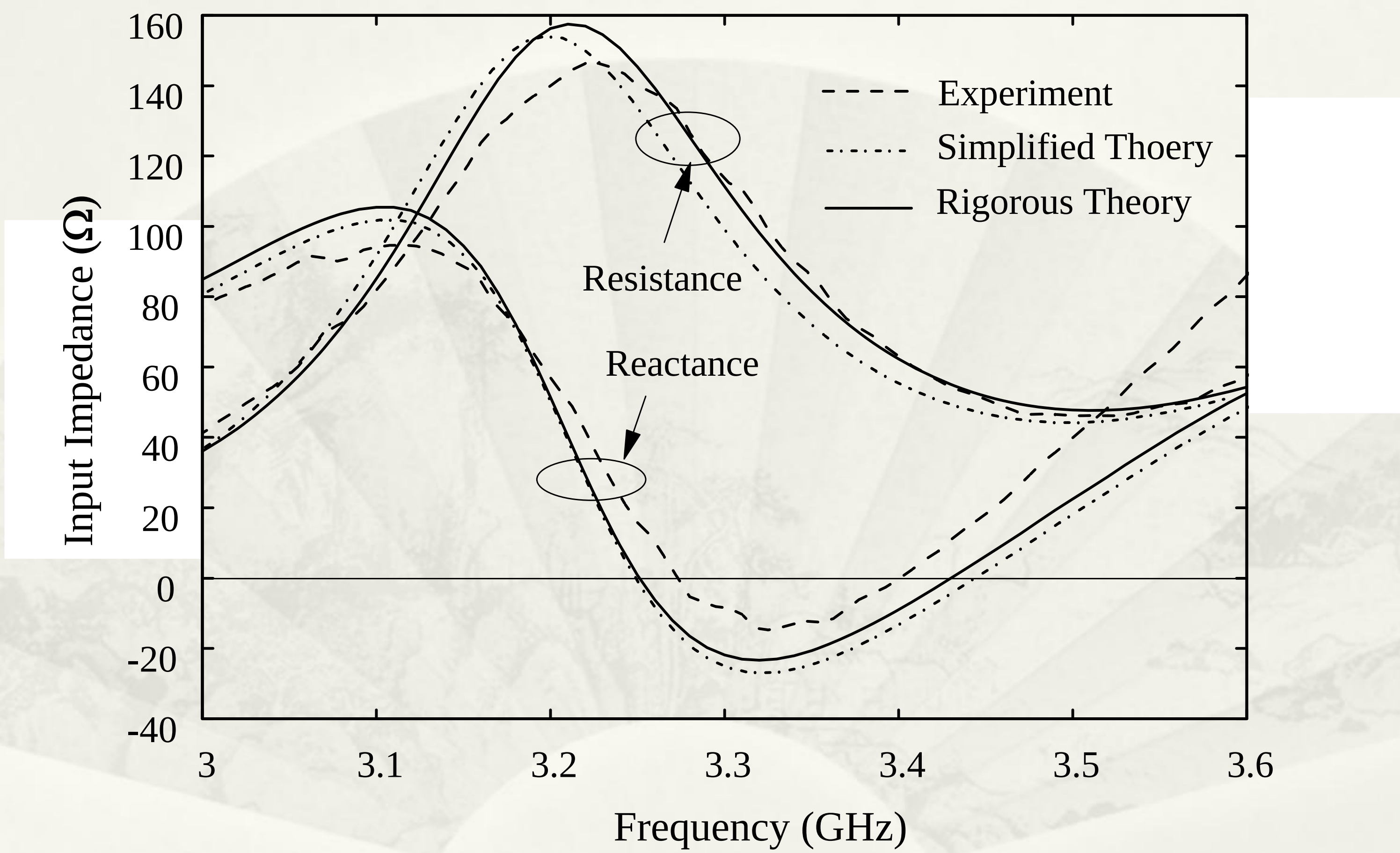
$$\Theta_{\theta\theta 1}(p,n,m)=\int_{\theta_p-\theta_h}^{\theta_p+\theta_h} P_n^m(\cos\theta) f_{\theta p}(\theta)d\theta$$

$$\Theta_{\theta\theta 2}(p,n,m)=\int_{\theta_p-\theta_h}^{\theta_p+\theta_h} \frac{dP_n^m(\cos\theta)}{d\theta}\sin\theta\, f_{\theta p}(\theta)d\theta$$

Their recurrence formulas have also been found [A] but are not included here for brevity.

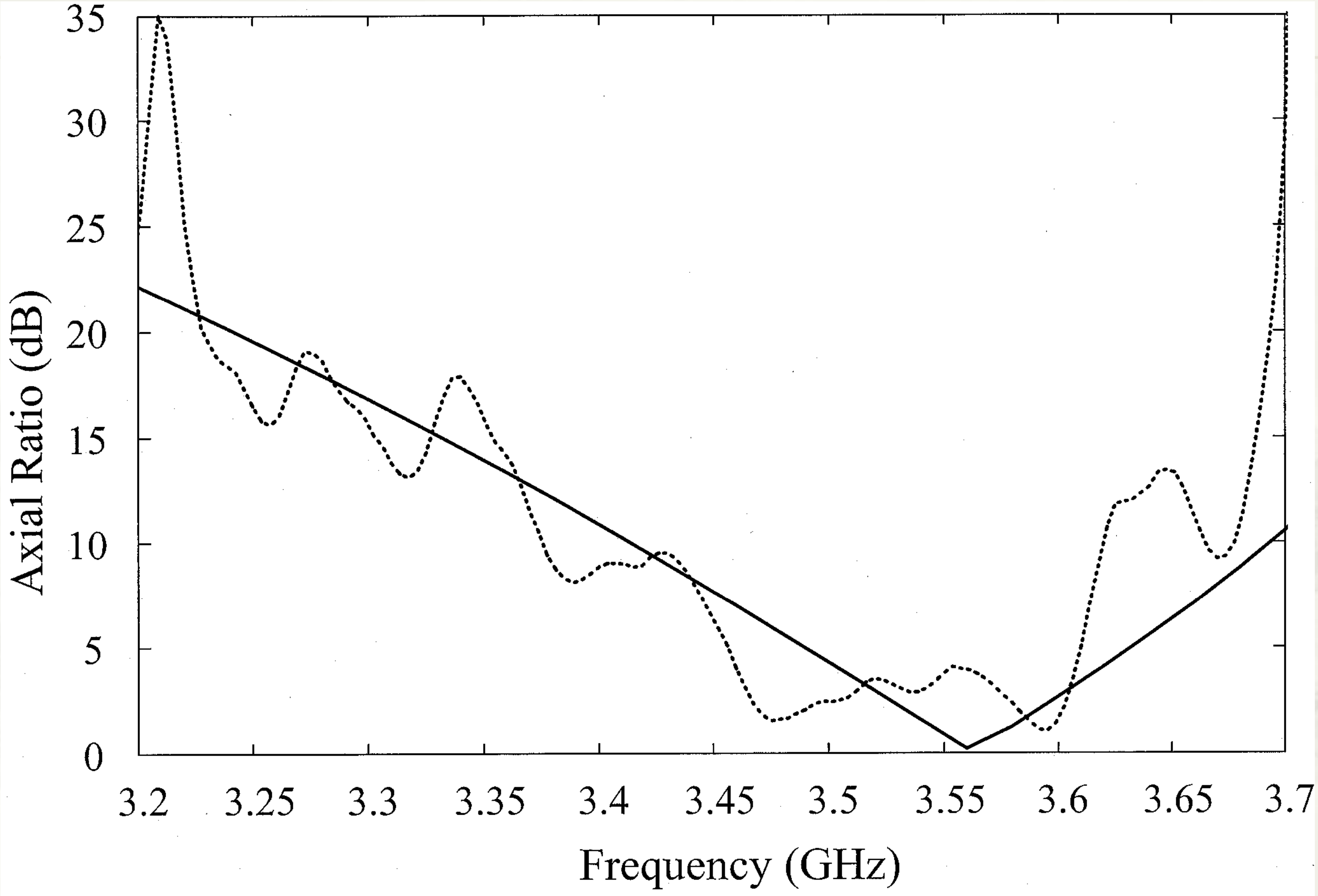
[A] K. W. Leung, and H. K. Ng, "Theory and experiment of circularly polarized dielectric resonator antenna with a parasitic patch," *IEEE Trans. Antennas Propagat.*, vol. 51, pp.405-412, Mar. 2003.

Measured and Calculated Results



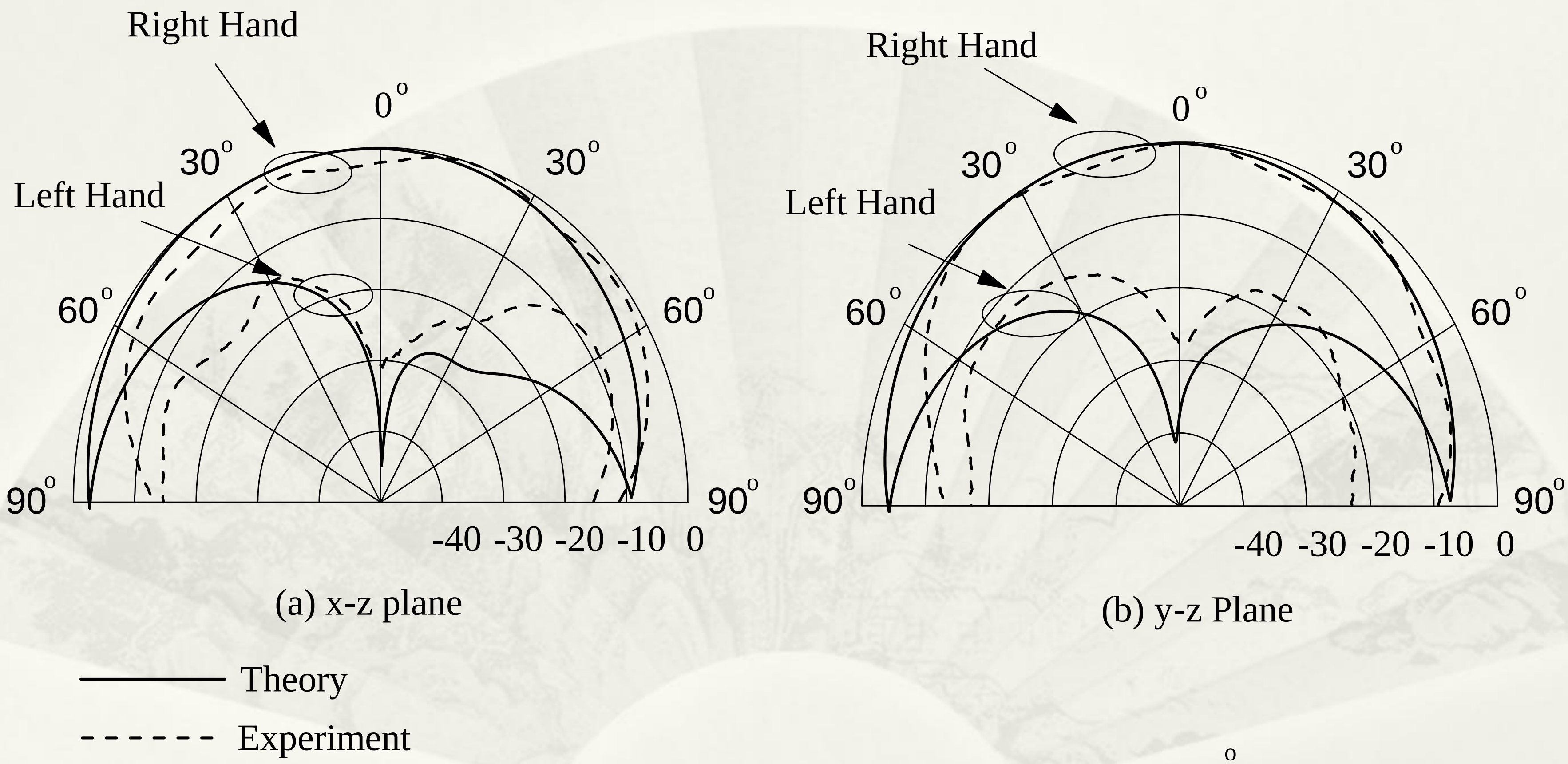
a=12.5mm, ε_r =9.5, l_1=14mm, l_2=7.9mm, W_1=1.2mm, W_2=2.2mm, and ϕ_o=157.4º.

Measured and Calculated Results (Cont'd)



a=12.5mm, ε_r =9.5, l_1=14mm, l_2=7.9mm, W_1=1.2mm, W_2=2.2mm, and ϕ_o=157.4°.

Measured and Calculated Results (Cont'd)



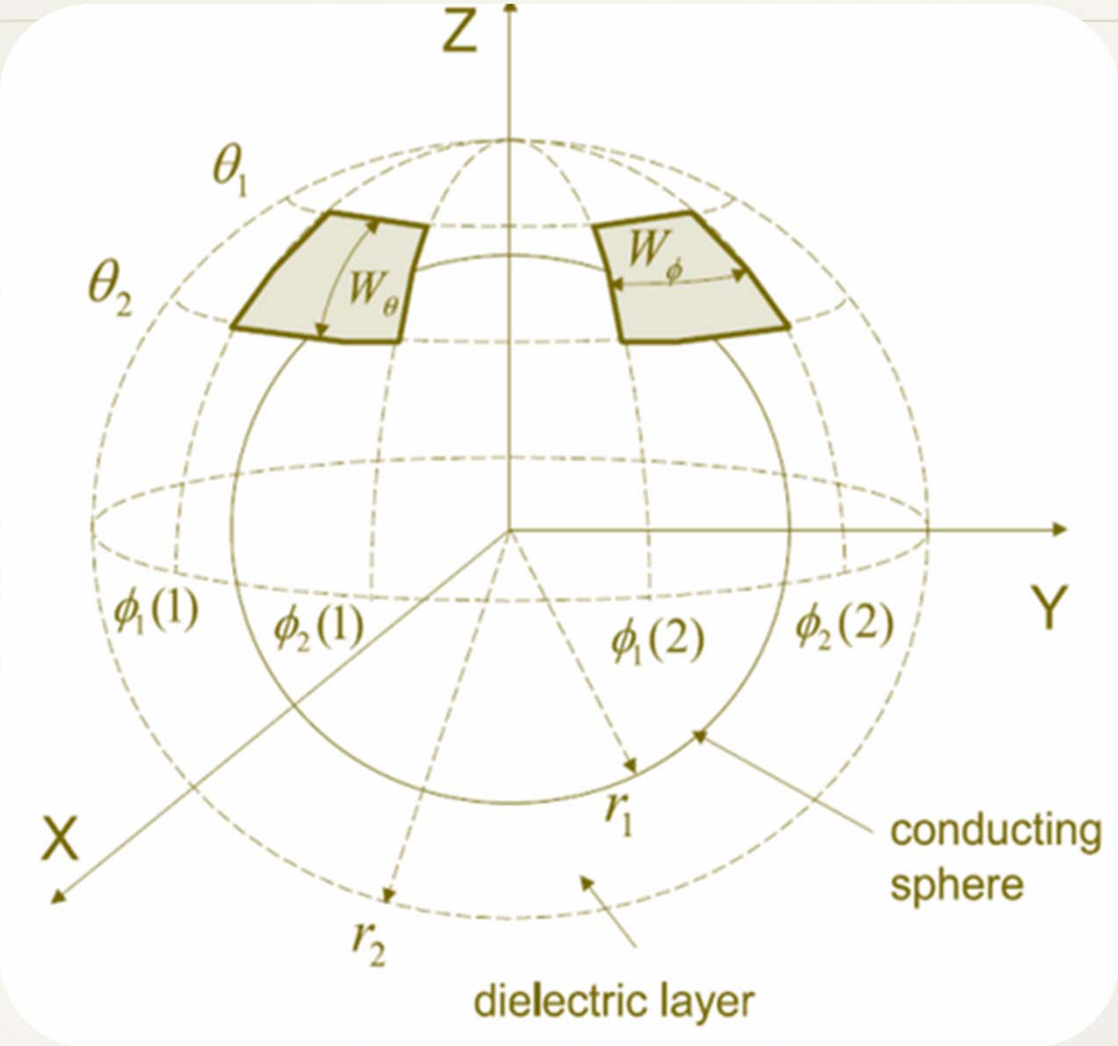
(a) x-z plane

(b) y-z Plane

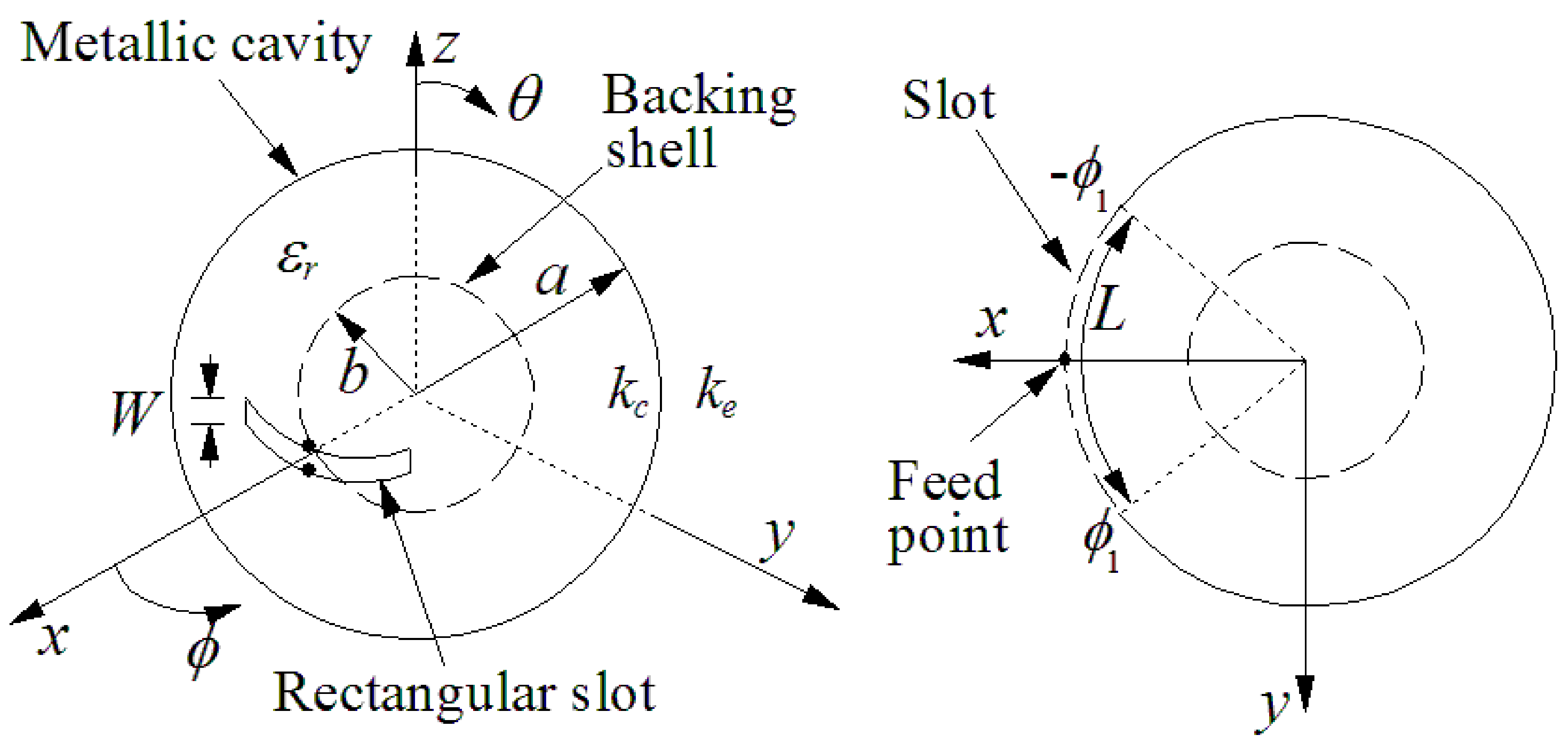
o

f = 3.52 GHz

The method can be used to formulate microstrip antenna problems.



Analysis of Spherical Slot Antenna



Perspective View

Top View

Green's Function of the Antenna

$$G^{e,c} = G_P^{e,c} + G_H^{e,c}$$

$$G_P^{e,c} = \frac{-1}{4\pi a^2 \eta_{e,c}} \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{2n+1}{n(n+1)} \cdot \hat{J}_n(k_{e,c}a)\hat{H}_n^{(2)}(k_{e,c}a) \cdot \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_n^m(\cos\theta)}{d\theta} \cdot \frac{dP_n^m(\cos\theta')}{d\theta'} \cos m(\phi-\phi')$$

$$- \frac{1}{4\pi a^2 \eta_{e,c}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{2n+1}{n(n+1)} \hat{J}_n'(k_{e,c}a)\hat{H}_n^{(2)}{}'(k_{e,c}a) \cdot 2m^2 \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi')$$

$$G_H^e = \frac{-1}{4\pi a^2 \eta_e} \sum_{n=1}^{\infty}\sum_{m=0}^{n} \beta_n^{TM} \frac{2n+1}{n(n+1)} \left[\hat{H}_n^{(2)}(k_e a)\right]^2 \cdot \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_n^m(\cos\theta)}{d\theta} \cdot \frac{dP_n^m(\cos\theta')}{d\theta'} \cos m(\phi-\phi')$$

$$- \frac{1}{4\pi a^2 \eta_e} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty}\sum_{m=1}^{n} \beta_n^{TE} \frac{2n+1}{n(n+1)} \left[\hat{H}_n^{(2)}{}'(k_e a)\right]^2 \cdot 2m^2 \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi')$$

$$G_H^c = \frac{-1}{4\pi a^2 \eta_c} \sum_{n=1}^{\infty}\sum_{m=0}^{n} \left\{ \left[e_n \hat{H}_n^{(2)}{}'(k_c a)\hat{J}_n(k_c a) + f_n \hat{J}_n'(k_c a)\hat{H}_n^{(2)}(k_c a) \right] \frac{2n+1}{n(n+1)} \cdot \right.$$

$$\left. \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_n^m(\cos\theta)}{d\theta} \cdot \frac{dP_n^m(\cos\theta')}{d\theta'} \cos m(\phi-\phi') \right\}$$

$$- \frac{1}{4\pi a^2 \eta_c} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty}\sum_{m=1}^{n} \left\{ \left[b_n \hat{H}_n^{(2)}(k_c a)\hat{J}_n'(k_c a) + c_n \hat{J}_n(k_c a)\hat{H}_n^{(2)}{}'(k_c a) \right] \frac{2n+1}{n(n+1)} \cdot \right.$$

$$\left. 2m^2 \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi') \right\}$$

MoM Admittances

$$Y_{pq} = \frac{-1}{W^2} \iint_{S_0} \iint_{S_0} f_p(\phi) \left[G^e + G^c \right] f_q(\phi') \, dS' dS$$

where

$$f_q(\phi) = \begin{cases} \dfrac{\sin k_e'(h - a|\phi - \phi_q|)}{\sin k_e' h}, & a|\phi - \phi_q| < h \\ 0, & a|\phi - \phi_q| \geq h \end{cases}$$

$$h = \frac{L}{N+1}, \qquad \phi_q = \frac{1}{a}\left(\frac{-L}{2} + qh \right), \qquad k_e' = \sqrt{(\varepsilon_r + 1)/2}\, k_0$$

Let

$$Y_{pq} = \left(Y^e_{pqP} + Y^c_{pqP} \right) + \left(Y^e_{pqH} + Y^c_{pqH} \right)$$

where

$$Y^{e,c}_{pqP,H} = \frac{-1}{W^2} \iint_{S_0} \iint_{S_0} f_p(\phi) \, G^{e,c}_{P,H} \, f_q(\phi') \, dS' dS$$

Method A

Numerical integration

$$Y_{pqP}^{e,c} = \frac{-1}{\eta_{e,c}^2} \int_{-\phi_1}^{\phi_1} \int_{-\phi_1}^{\phi_1} f_p(\phi) \left\{ \begin{aligned} &\frac{-j\eta_{e,c}}{4\pi} \cdot \frac{e^{-jk_{e,c}R}}{k_{e,c}R^5} [R^2(k_{e,c}^2 R^2 - jk_{e,c}R - 1)\cos(\phi-\phi') \\ &\qquad - a^2(k_{e,c}^2 R^2 - 3jk_{e,c}R - 3)\sin^2(\phi-\phi')] \end{aligned} \right\} f_q(\phi') a^2 \, d\phi' \, d\phi$$

Recurrence formulas

$$Y_{pqH} = \frac{a^2}{4\pi W^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ A(n) \sum_{m=0}^{n} \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} [\Theta_1(n,m)]^2 \Phi(p,q,m) + B(n) \sum_{m=1}^{n} 2m^2 \cdot \frac{(n-m)!}{(n+m)!} [\Theta_2(n,m)]^2 \Phi(p,q,m) \right\}$$

where

$$R = \sqrt{4a^2 \sin^2[(\phi-\phi')/2] + r_1^2}$$

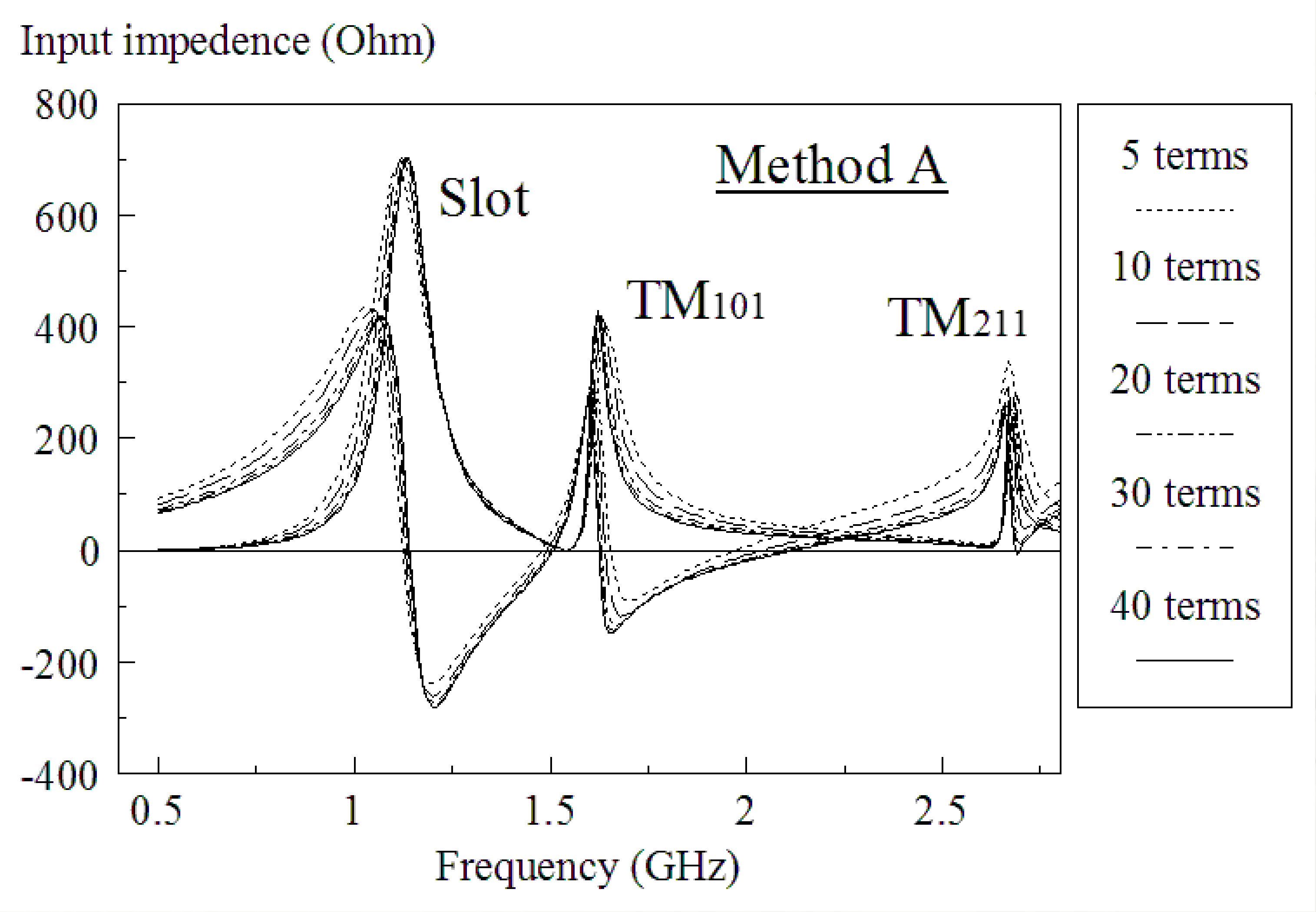
$$A(n) = \frac{\beta_n^{TM}}{\eta_e} \left[\hat{H}_n^{(2)}(k_e a)\right]^2 + \frac{1}{\eta_c} \left[e_n \hat{H}_n^{(2)}{}'(k_c a) \hat{J}_n(k_c a) + f_n \hat{J}_n{}'(k_c b) \hat{H}_n^{(2)}(k_c a)\right]$$

$$B(n) = \frac{\beta_n^{TE}}{\eta_e} \left[\hat{H}_n^{(2)}{}'(k_e a)\right]^2 + \frac{1}{\eta_c} \left[b_n \hat{H}_n^{(2)}(k_c a) \hat{J}_n{}'(k_c a) + c_n \hat{J}_n(k_c b) \hat{H}_n^{(2)}{}'(k_c a)\right]$$

Method B

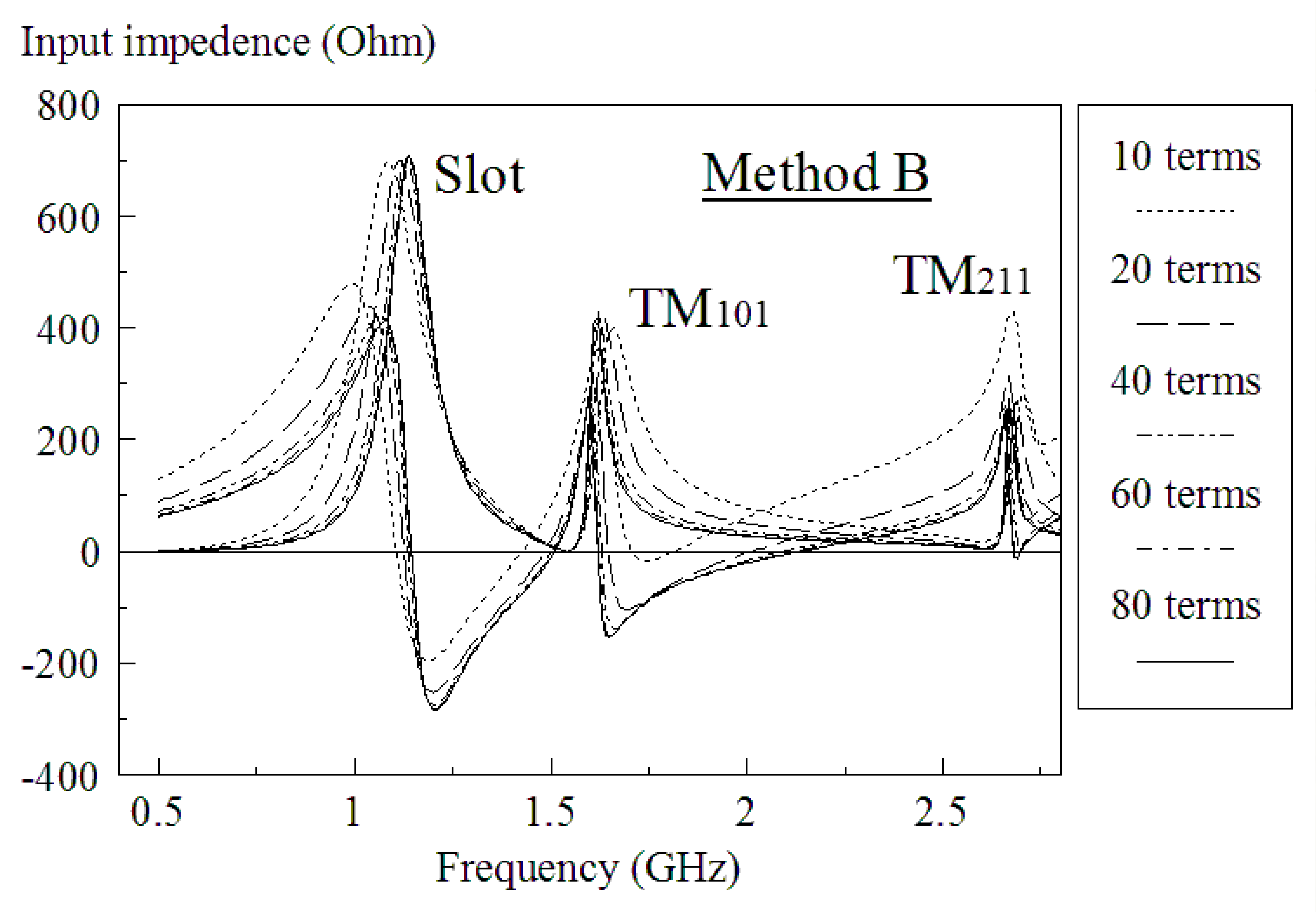
- Combine the particular and homogeneous solutions
- Only recurrence formulas are used
- Advantage: Easy to implement and fast (no numerical integration required)
- Disadvantage: ??

Convergence Check for Method A



a = 6.25 cm, b = 3 cm, L = 12.46 cm, W = 2.4 mm, and ε_r = 1.

Convergence Check for Method B



a = 6.25 cm, b = 3 cm, L = 12.46 cm, W = 2.4 mm, and ε_r = 1.

Comparison between Methods A & B

Advantages of Method A

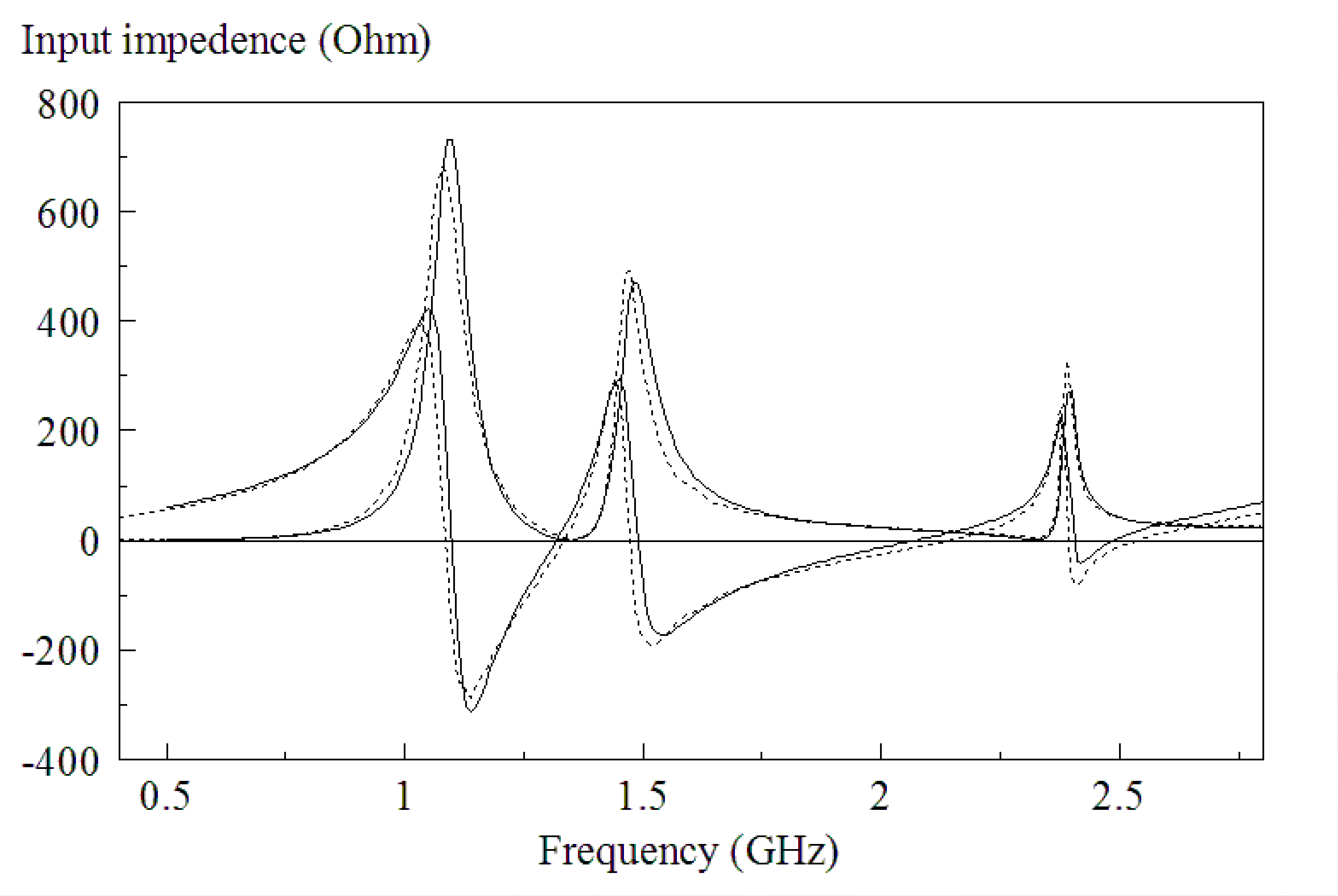
- Method A converges much faster than Method B
- More suitable for problems of larger spherical sizes

Advantages of Method B

- Numerical integrations not required
- Easier to implement
- Much faster than Method A

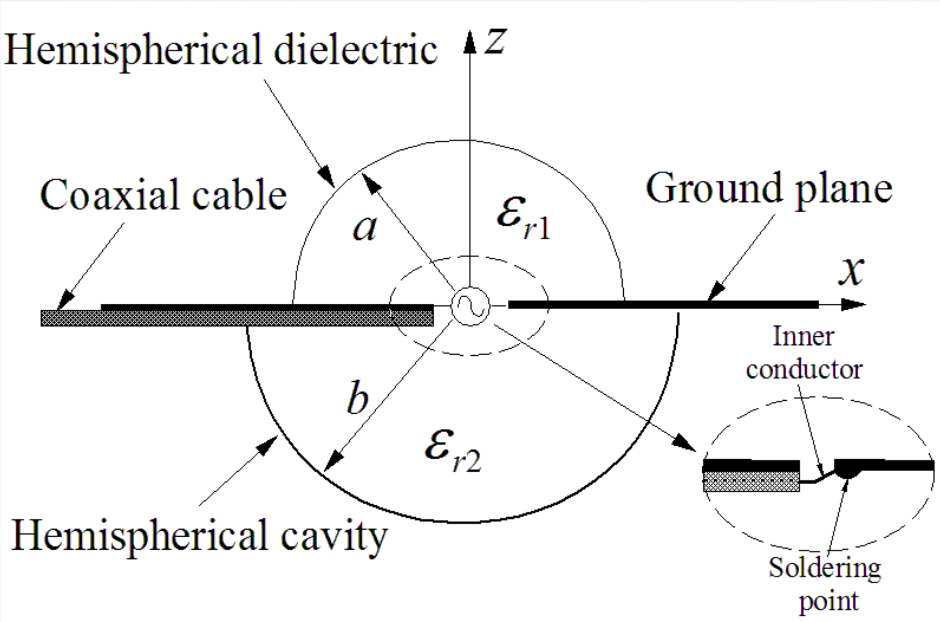
Method A	Method B	Numerical integration for both G_p, G_H
32 sec.	112 sec.	77 sec.

Measured & Calculated (Method B) Results

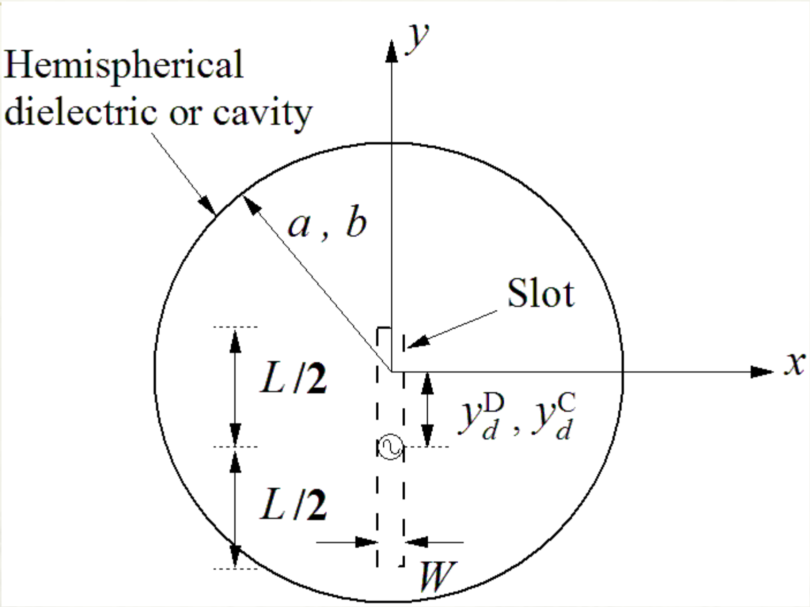


Measured and calculated input impedances of the rectangular slot for a = 6.25 cm, b = 4.0 cm, L = 12.46 cm, W = 2.4 mm, and $\varepsilon_r = 1$.

Recurrence Formulas for Integral of $\hat{J}_n(kr)$



Side View



Top View

MoM Admittance

Define Y^D & Y^C as the dielectric & cavity admittances, respectively.

$$Y_{pqH}^{D,C} = \frac{-1}{2\pi\omega\mu_0 k}\sum_{n=1}^{\infty} n(n+1)(2n+1)\gamma_n^{D,C} A_n(p)A_n(q)$$

where

$$A_n(i) = \int_{y_{i-1}}^{y_{i+1}} \frac{\hat{J}_n(ky)}{y^2}\cdot\frac{\sin k_e(h-|y-y_i|)}{\sin k_e h}dy$$

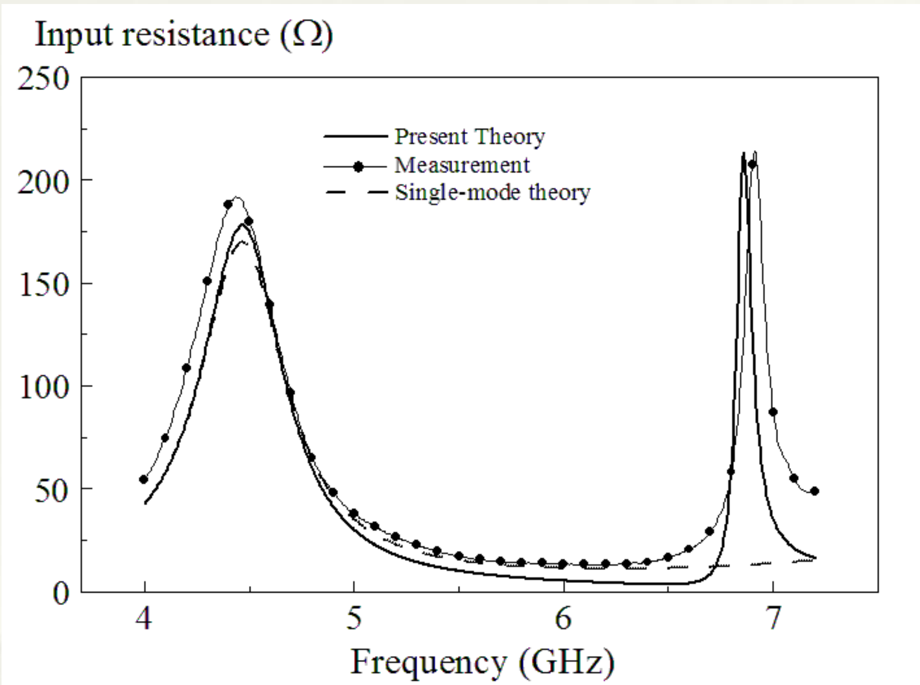
Recurrence formula for $A_n(i)$

$$\frac{(n+2)(n+3)}{(2n+1)(2n+3)}A_{n+2}(i)+[\frac{k_e^2}{k^2}-\frac{(2n^2+2n-3)}{(2n-1)(2n+3)}]A_n(i)+\frac{(n-1)(n-2)}{(2n+1)(2n-1)}A_{n-2}(i)$$

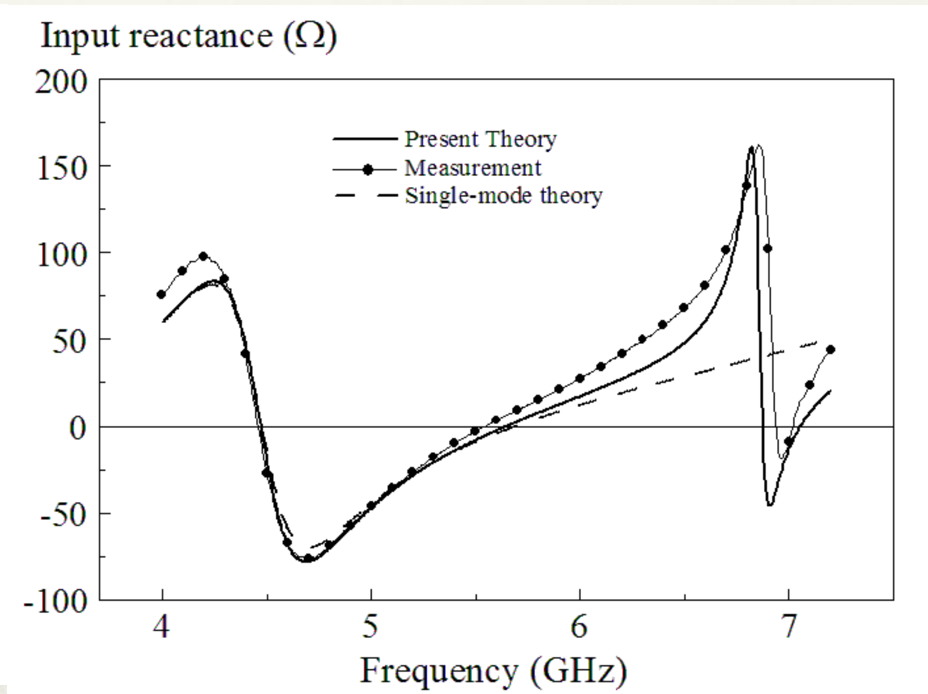
$$=\frac{k_e}{k^2\sin k_e h}\sum_{j=-1}^{1}B(j)\frac{\hat{J}_n(ky_{i+j})}{y_{i+j}^2}$$

Backward recurrence for $A_n(i)$, needless to know A_0, A_1, A_2, & A_3.

Measured and Calculated Results



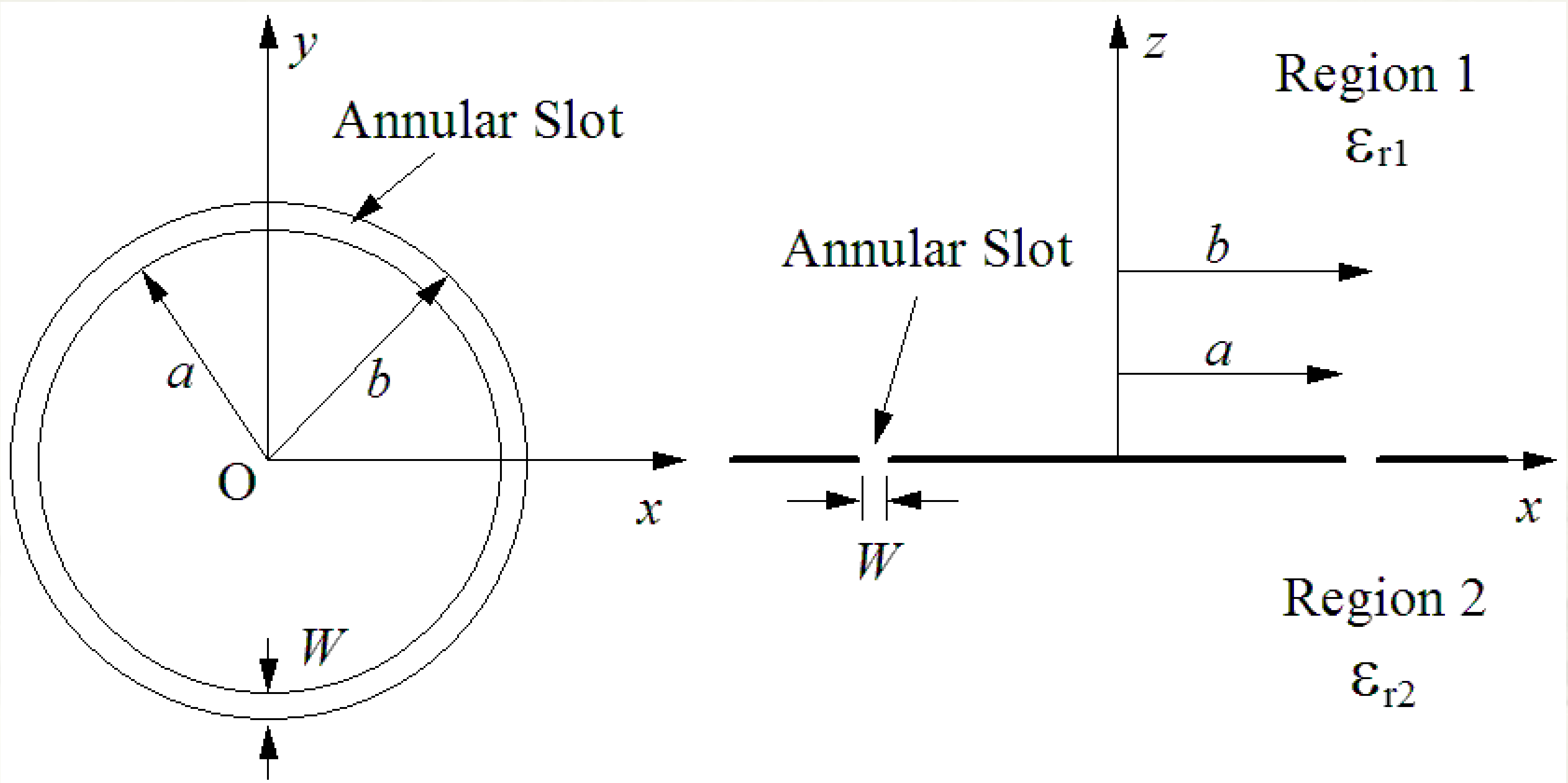
Input resistance



Input reactance

a = 12.5 mm, ε_{r1} = 9.5, L = 20.0 mm, W = 1.0 mm, = 0, and N = 5

Solving Planar annular problem using spherical solution



Top View

Side View

MoM Solution

- First obtain the spherical Green's function of H_ϕ due to a ϕ-directed magnetic point current

- Substitute $\theta = \theta' = \pi/2$ (both source & field one the ground plane)

MoM Solution (Cont'd)

$$Z_{\text{in}} = \sum_{q=1}^{N} \frac{1}{Y_{qq}}$$

where

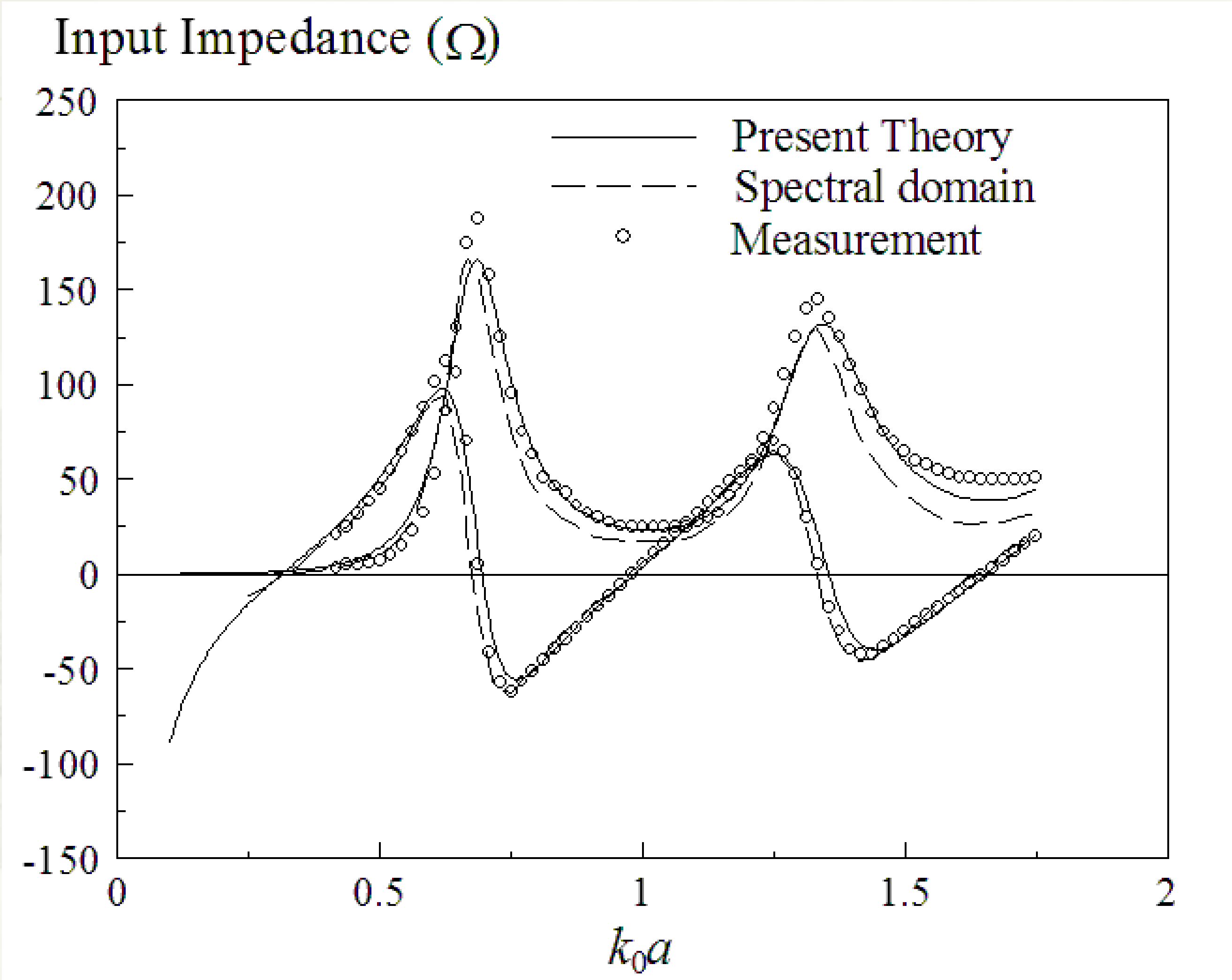
$$Y_{qq} = \frac{\pi \Delta_{q-1}^2}{W^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \cdot \frac{(n-q+1)!}{(n+q-1)!} \cdot \left\{ \frac{1}{\Delta_{q-1}} \left[P_n^q(0) \right]^2 \left[\frac{\alpha_1^{TM}(n)}{\eta_1} + \frac{\alpha_2^{TM}(n)}{\eta_2} \right] + (q-1)^2 \left[P_n^{q-1}(0) \right]^2 \left[\frac{\alpha_1^{TE}(n)}{\eta_1} + \frac{\alpha_2^{TE}(n)}{\eta_2} \right] \right\}$$

in which

$$P_n^m(0) = \begin{cases} (-1)^{(n+m)/2} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (n+m-1)}{2 \cdot 4 \cdot 6 \cdots (n-m)} & n+m \text{ even} \\ 0 & n+m \text{ odd} \end{cases}$$

$$\Delta_{q-1} = \begin{cases} 2 & q = 1 \\ 1 & \text{otherwise} \end{cases}$$

Measured and Calculated Results



a = 19.25 mm, b = 10.75 mm, W = 1.5 mm, ε_{r1} = 4, and ε_{r2} = 1

Q & A