

Analyses of Spherical antennas

K. W. Leung State Key Laboratory of Millimeter Waves & Department of Electronic Engineering, City University of Hong Kong Characteristics of spherical array

◆Array elements distributed on a sphere surface.

 Conformal, low profile, light weight, easy to install on aircraft surfaces

 Wide angular coverage by activating different array elements

◆ Stable antenna gain and radiation pattern for scanning

Spherical array application

SYNCHRONOUS RELAY SATELLITE



Earth-orbiting scientific satellite system

The user satellite is using the spherical array

Stockton & Hockensmith, "Application of spherical arrays -- A simple approach," Antennas and Propagation Society International Symposium, 1977, vol.15, pp. 202- 205, June 1977.

Hemispherical patch array for satellite data link: Electronic Switching Spherical Array (ESSA)



A cluster of 12 patch elements out of a total 120 are activated at a time.
Beam steering was accomplished by shifting the active part in small steps.
Hemispherical coverage with a moderate gain for steered beams (~13 dBic).

R. Stockton, "Electronic switching spherical array antenna", NASA (Unspecified Center), Apr. 1, 1978.

Elements of the ESSA: RHCP Microstrip Patch Antenna



Spherical helical antenna



➢Proposed by Mei

► Radiate circularly polarized fields over a wide beamwidth.

Suitable for systems requiring wide-beamwidths (e.g., GPS).

K. K. Mei and M. Meyer, "Solutions to spherical anisotropic antennas," *IEEE Trans. Antennas Progagat.*, vol. 12, pp.459-463, 1964.

Radiation patterns of spherical helical antenna



Radiation pattern of a 7-turn spherical helical antenna

> 3dB beamwidths: 60°

A. Safaai-Jazi and J. C. Cardoso, "Radiation characteristics of a spherical helical antenna," *IEE Proc. Microw. Antennas Propag.*, vol. 143, no. I, Feb. 1996

Comparison:radiation pattern of the cylindrical helical antenna



Measured electric field patterns of the 6-turn cylindrical helical antenna

> 3dB Beamwidth: ~ 40°

J. D. Kraus,"Helical Beam Antennas for Wide-Band Applications," *Proceedings of the IRE*,vol. 36, no.10, pp1236-1242,1948.

Spherical Solutions

- No edge-shaped boundaries as found in cylindrical and rectangular structures
- Closed-form Green's functions obtainable
- Exact solutions used as references for checking the accuracy of numerical or approximation techniques

Helmholtz Equation in Spherical Cooridnates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} + k^2\Psi = 0$$

Solution

 $\Psi_{m,n} = b_n(kr)P_m^n(\cos\theta)e^{jm\phi}$

where $b_n(kr)$ is the spherical Bessel function

 $P_n^m(\cos\theta)$ is the Associated Legendre function of the first kind

 $e^{jm\phi}$ is sinusoidal function.

Remark: Since the associated Legendre function of the second kind, $Q_n^m(\cos\theta)$ is singular at $\theta = 0$ or π , it is generally not used for engineering EM problems.

Spherical Bessel Functions

$$b_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+1/2}(kr)$$

where $B_{n+1/2}(kr)$ is the ordinary (cylindrical) Bessel function

 $b_n(kr) \sim j_n(kr), y_n(kr)$



Riccati-Bessel Functions (Schelkunoff-type Spherical Bessel Function)

- All EM fields can be found from 2 potential functions
- Define $\vec{A} = A_r \hat{r}$ and $\vec{F} = F_r \hat{r}$
- But A_r , F_r , are not solutions of Helmholtz equation
- Instead, A_r/r , F_r/r are solutions of Helmholtz equation
- Define Riccati-Bessel function

$$\hat{B}_{n}(kr) = krb_{n}(kr) = \sqrt{\frac{\pi kr}{2}}B_{n+1/2}(kr)$$

General Solutions of Sphericl Potential Functions

 $A_r, F_r, \sim \sum \hat{B}_n(kr) P_n^m(\cos\theta) e^{j^{m\phi}}$



Legendre Functions



E & H fields in Electromagnetics

$$E_r = \frac{1}{j\omega\varepsilon u_0} \left(\frac{\partial^2}{\partial r^2} + k^2\right) A_r$$

$$E_{\theta} = \frac{-1}{\varepsilon r \sin\theta} \frac{\partial F_r}{\partial \Phi} + \frac{1}{j\omega \varepsilon u_0 r} \frac{\partial A_r}{\partial r \partial \theta}$$
$$E_{\phi} = \frac{-1}{\varepsilon r} \frac{\partial F_r}{\partial \theta} + \frac{1}{j\omega \varepsilon u_0 r \sin\theta} \frac{\partial A_r}{\partial r \partial \phi}$$

$$H_r = \frac{1}{j\omega\varepsilon u_0} (\frac{\partial^2}{\partial r^2} + k^2) F_r$$

$$H_{\theta} = \frac{-1}{u_{0}r\sin\theta} \frac{\partial A_{r}}{\partial \Phi} + \frac{1}{j\omega\varepsilon u_{0}r} \frac{\partial F_{r}}{\partial r\partial\theta}$$
$$H_{\phi} = \frac{-1}{u_{0}r} \frac{\partial A_{r}}{\partial \theta} + \frac{1}{j\omega\varepsilon u_{0}r\sin\theta} \frac{\partial F_{r}}{\partial r\partial\phi}$$

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Grounded Spherical Hemispherical Dielectric Resonator Antenna: Embedded Magnetic Source







Electric & Magnetic Green's Functions Due to M_{ϕ}

$$G_{M_{\phi}}^{F_{rp}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} P_{n}^{m}(\cos\theta) e^{jm\phi} \cdot \begin{cases} \hat{J}_{n}'(kr') \hat{H}_{n}^{(2)}(kr) & r > r' \\ \hat{H}_{n}^{(2)}'(kr') \hat{J}_{n}(kr) & r < r' \end{cases}$$

$$G_{M_{\phi}}^{F_{rh}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{n}^{m}(\cos\theta) e^{jm\phi} \cdot \begin{cases} B_{nm} \hat{J}_{n}(kr) & r \le a \\ C_{nm} \hat{H}_{n}^{(2)}(k_{0}r) & r \ge a \end{cases}$$

$$G_{M_{\phi}}^{A_{rp}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} D_{nm} P_{n}^{m} (\cos \theta) e^{jm\phi} \cdot \begin{cases} \hat{J}_{n}(kr') \hat{H}_{n}^{(2)}(kr) & r > n \\ \hat{H}_{n}^{(2)}(kr') \hat{J}_{n}(kr) & r < n \end{cases}$$

$$G_{M_{\phi}}^{A_{rh}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m (\cos \theta) e^{jm\phi} \cdot \begin{cases} E_{nm} \hat{J}_n(kr) & r \le a \\ F_{nm} \hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

 $k = \sqrt{\varepsilon_r} \, k_0$

 $P_n^m(x)$: Associated Legendre function of the first kind (order *m*,degree *n*) $\hat{J}_n(x)$: Spherical Bessel function of the first kind (Schelkunoff type) $\hat{H}_n^{(2)}(x)$: Spherical Hankel function of the second kind (Schelkunoff type)

Particular solutions obtained by matching the boundary conditions at the source point (r = r')

$$E_{\theta}^{+} - E_{\theta}^{-} = -M_{\phi}$$

$$E_{\phi}^{+}-E_{\phi}^{-}=0$$

$$H_{\theta}^{+} - H_{\theta}^{-} = 0$$

$$H_{\phi}^+ - H_{\phi}^- = 0$$

$$A_{nm} = \frac{-\varepsilon r'}{4\pi} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!} \cdot m \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} M_{\phi s} P_n^m(\cos\theta) e^{-jm\phi} d\phi d\theta$$

$$D_{nm} = \frac{\omega\mu_o\varepsilon r'}{4\pi k} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} M_{\phi s} \sin\theta \frac{d}{d\theta} P_n^m(\cos\theta) e^{-jm\phi} d\phi d\theta$$

For a point source
$$M_{\phi s} = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{r'^2 \sin \theta}$$

$$G_{M_{\phi}}^{F_{rp}} = \frac{1}{r'\sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} a_{nm} P_{n}^{m}(\cos\theta') P_{n}^{m}(\cos\theta) \sin m(\phi - \phi') \cdot \begin{cases} \hat{J}_{n}'(kr') \hat{H}_{n}^{(2)}(kr) & r > r' \\ \hat{H}_{n}^{(2)}'(kr') \hat{J}_{n}(kr) & r < r' \end{cases}$$

$$G_{M_{\phi}}^{A_{rp}} = \frac{1}{r'} \sum_{n=1}^{\infty} \sum_{m=0}^{n} d_{nm} \frac{d}{d\theta'} P_{n}^{m}(\cos\theta') P_{n}^{m}(\cos\theta) \cos m(\phi - \phi') \cdot \begin{cases} \hat{J}_{n}(kr') \hat{H}_{n}^{(2)}(kr) & r > r' \\ \hat{H}_{n}^{(2)}(kr') \hat{J}_{n}(kr) & r < r' \end{cases}$$

where

$$a_{nm} = \frac{-j\varepsilon}{2\pi} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!} \cdot m$$

$$d_{nm} = \frac{\omega\mu_o\varepsilon}{\Delta_m 2\pi k} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{(n-m)!}{(n+m)!}$$

$$\Delta_m = \begin{cases} 1 & \text{for } m > 0 \\ 2 & \text{for } m = 0 \end{cases}$$

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Homogenous solutions obtained by matching the boundary conditions at the dielectric surface (r = a)

$$E_{\theta}^{+} - E_{\theta}^{-} = 0, \quad E_{\phi}^{+} - E_{\phi}^{-} = 0, \quad H_{\theta}^{+} - H_{\theta}^{-} = 0, \quad H_{\phi}^{+} - H_{\phi}^{-} = 0$$

$$G_{M_{\phi}}^{F_{rh}} = \frac{1}{r'\sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} P_n^m(\cos\theta') P_n^m(\cos\theta) \sin m(\phi - \phi') \hat{J}_n'(kr') \cdot \begin{cases} b_{nm} \hat{J}_n(kr) & r \le a \\ c_{nm} \hat{H}_n^{(2)}(k_0 r) & r \ge a \end{cases}$$

$$G_{M_{\phi}}^{A_{rh}} = \frac{1}{r'} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{d}{d\theta'} P_{n}^{m}(\cos\theta') P_{n}^{m}(\cos\theta) \cos m(\phi - \phi') \hat{J}_{n}(kr') \cdot \begin{cases} e_{nm} \hat{J}_{n}(kr) & r \le a \\ f_{nm} \hat{H}_{n}^{(2)}(k_{0}r) & r \ge a \end{cases}$$

where
$$b_{nm} = \frac{-a_{nm}}{\Delta_n^{TE}} \left[\hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_o a) - \frac{k}{k_o} \hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_o a) \right]$$

 $e_{nm} = \frac{-d_{nm}}{\Delta_n^{TM}} \left[\hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_o a) - \frac{k}{k_o} \hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_o a) \right]$
 $c_{nm} = -j \frac{k_o}{k} \cdot \frac{a_{nm}}{\Delta_n^{TE}} , \quad f_{nm} = j \frac{d_{nm}}{\Delta_n^{TM}}$

Magnetic Green's Function Due to M_r

$$(\nabla^2 + k^2) \frac{G_{M_r}^{F_{rp}}}{r} = \frac{-\varepsilon}{r} \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r^2 \sin \theta}$$

Particular solution

$$G_{M_r}^{F_{rp}} = \frac{1}{r'^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} g_{nm} P_n^m(\cos\theta') P_n^m(\cos\theta) \cos m(\phi - \phi') \cdot \begin{cases} \hat{J}_n(kr') \hat{H}_n^{(2)}(kr) & r > r' \\ \hat{H}_n^{(2)}(kr') \hat{J}_n(kr) & r < r' \end{cases}$$

where
$$g_{nm} = \frac{-j\varepsilon}{2\pi\Delta_m k} \frac{(n-m)!}{(n+m)!} (2n+1)$$

Homogenous solution

$$G_{M_{r}}^{F_{rh}} = \frac{1}{r'^{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_{n}^{m}(\cos\theta') P_{n}^{m}(\cos\theta) \cos m(\phi - \phi') \hat{J}_{n}(kr') \cdot \begin{cases} h_{nm} \hat{J}_{n}(kr) & r \le a \\ i_{nm} \hat{H}_{n}^{(2)}(k_{0}r) & r \ge a \end{cases}$$

where

$$h_{nm} = \frac{-g_{nm}}{\Delta_n^{TE}} \left[\hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_0 a) - \frac{k}{k_0} \hat{H}_n^{(2)}(ka) \hat{H}_n^{(2)}(k_0 a) \right]_{nm}^{i} = -j \frac{k}{k_0} \cdot \frac{g_{nm}}{\Delta_n^{TE}}$$
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Define 3 Dyadic Green's Function $G_{M_r}^{H_r} \hat{r} \hat{r}$, $G_{M_r}^{H_\theta} \hat{\theta} \hat{r}$, $G_{M_r}^{H_\theta} \hat{\phi} \hat{r}$

$$\overline{\overline{G}}_{M_r}^{H} = G_{M_r}^{H_r} \hat{r} \hat{r} + G_{M_r}^{H_\theta} \hat{\theta} \hat{r} + G_{M_r}^{H_\phi} \hat{\phi} \hat{r}$$

$$\overline{\overline{G}}_{M_{\phi}}^{H} = G_{M_{\phi}}^{H_{r}} \hat{r} \hat{\phi} + G_{M_{\theta}}^{H_{\phi}} \hat{\theta} \hat{\phi} + G_{M_{\phi}}^{H_{\phi}} \hat{\phi} \hat{\phi}$$

Total H-field due to $M_r \& M_{\phi}$ is given by

$$\overrightarrow{H} = \iint_{S_o} [\overrightarrow{\overline{G}}_{M_r}^{\overrightarrow{H}} \cdot (M_r' \hat{r}) + \overrightarrow{\overline{G}}_{M_\phi}^{\overrightarrow{H}} \cdot (M_{\phi}' \hat{\phi})] dS'$$

 $\vec{H} = \iint_{S_o} (G_{M_r}^{H_r} M_r' \hat{r} + G_{M_r}^{H_\theta} M_r' \hat{\theta} + G_{M_r}^{H_\theta} M_r' \hat{\phi}) + (G_{M_\phi}^{H_r} M_\phi' \hat{r} + G_{M_\phi}^{H_\theta} M_\phi' \hat{\theta} + G_{M_\phi}^{H_\theta} M_\phi' \hat{\phi}) dS'$ $= \iint_{S_o} (G_{M_r}^{H_r} M_r' + G_{M_\phi}^{H_r} M_\phi') \hat{r} + (G_{M_r}^{H_\theta} M_r' + G_{M_\phi}^{H_\theta} M_\phi') \hat{\theta} + (G_{M_r}^{H_\theta} M_r' + G_{M_\phi}^{H_\theta} M_\phi') \hat{\phi} dS'$

Since

$$H_{v} = H_{r}\sin\phi + H_{\phi}\cos\phi$$

$$M_r' = M_v' \sin \phi'$$

$$M_{\phi}' = M_{y}' \cos \phi'$$

we have

 $\vec{H} = \iint_{S_o} \left[\left(G_{M_r}^{H_r} \sin \phi' + G_{M_\phi}^{H_r} \cos \phi' \right) \sin \phi + \left(G_{M_r}^{H_\phi} \sin \phi' + G_{M_\phi}^{H_\phi} \cos \phi' \right) \cos \phi \right] M_y' dS'$ $= \iint_{S_o} G_{M_y}^{H_y} M_y' dS'$

Therefore, the required Green's function is given by $G_{M_{y}}^{H_{y}} = (G_{M_{r}}^{H_{r}} \sin \phi' + G_{M_{\phi}}^{H_{r}} \cos \phi') \sin \phi + (G_{M_{r}}^{H_{\phi}} \sin \phi' + G_{M_{\phi}}^{H_{\phi}} \cos \phi') \cos \phi$

Expressing the Green's function in $G_P \& G_H$:

$$G_{M_y}^{H_y} = G_P + G_H$$

where

$$G_{P} = \frac{1}{j\omega\mu_{0}\varepsilon} \cdot \frac{\sin\phi'\sin\phi}{r^{2}r'^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} n(n+1)g_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\cos m(\phi-\phi')\Phi_{n}(kr')\Psi_{n}(kr)$$

$$+\frac{1}{j\omega\mu_0\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{r^2r'}\sum_{n=1}^{\infty}\sum_{m=0}^{n}n(n+1)a_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\Phi_n'(kr')\Psi_n(kr)$$

$$\frac{k}{j\omega\mu_0\varepsilon}\cdot\frac{\sin\phi'\cos\phi}{r^2r'^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}mg_{nm}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\Phi_n(kr')\Psi_n'(kr)$$

$$\frac{1}{\mu_0} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \sum_{m=1}^n d_{nm} \frac{d}{d\theta'} P_n^m(\cos\theta') \frac{d}{d\theta} P_n^m(\cos\theta)\cos m(\phi - \phi') \Phi_n(kr') \Psi_n(kr)$$

$$+\frac{k}{j\omega\mu_{0}\varepsilon}\cdot\frac{\cos\phi'\cos\phi}{rr'}\sum_{n=1}^{\infty}\sum_{m=0}^{n}ma_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\cos m(\phi-\phi')\Phi_{n}'(kr')\Psi_{n}'(kr)$$

n which
$$\Phi_{n}(kr') = \begin{cases} \hat{J}_{n}(kr'), & r > r'\\ \hat{H}_{n}^{(2)}(kr'), & r < r' \end{cases} \Psi_{n}(kr) = \begin{cases} \hat{H}_{n}^{(2)}(kr), & r > r'\\ \hat{J}_{n}(kr), & r < r' \end{cases}$$

$$\begin{split} G_{H} &= \frac{1}{j\omega\mu_{0}\varepsilon} \cdot \frac{\sin\phi'\sin\phi}{r^{2}r'^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} n(n+1)h_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\cos m(\phi-\phi')\hat{J}_{n}(kr')\hat{J}_{n}(kr')\\ &+ \frac{1}{j\omega\mu_{0}\varepsilon} \cdot \frac{\cos\phi'\cos\phi}{r^{2}r'} \sum_{n=1}^{\infty} \sum_{m=0}^{n} n(n+1)b_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\sin m(\phi-\phi')\hat{J}_{n}(kr')\hat{J}_{n}(kr)\\ &- \frac{k}{j\omega\mu_{0}\varepsilon} \cdot \frac{\sin\phi'\cos\phi}{rr'^{2}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} mh_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\sin m(\phi-\phi')\hat{J}_{n}(kr')\hat{J}_{n}(kr)\\ &- \frac{1}{\mu_{0}} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} e_{nm}\frac{d}{d\theta'}P_{n}^{m}(\cos\theta')\frac{d}{d\theta}P_{n}^{m}(\cos\theta)\cos m(\phi-\phi')\hat{J}_{n}(kr')\hat{J}_{n}(kr)\\ &+ \frac{k}{j\omega\mu_{0}\varepsilon} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \sum_{m=0}^{n} mb_{nm}P_{n}^{m}(\cos\theta')P_{n}^{m}(\cos\theta)\cos m(\phi-\phi')\hat{J}_{n}(kr')\hat{J}_{n}(kr) \end{split}$$

Converting double summations to single summation

By applying the addition theorem for Legendre polynomials

$$P_n(\cos\xi) = \sum_{m=0}^{\infty} \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi')$$

where

 $\cos \xi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$

$$\begin{aligned} G_{M_y}^{H_y} &= G_p + G_H \\ G_p &= \frac{-1}{4\pi\omega\mu_0 k} \cdot \frac{\sin\phi'\sin\phi}{r^2r'^2} \sum_{n=1}^{\infty} n(n+1)(2n+1)P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr) \\ &- \frac{1}{4\pi\omega\mu_0} \cdot \frac{\cos\phi'\sin\phi}{r^2r'} \sum_{n=1}^{\infty} (2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n(kr) \\ &- \frac{1}{4\pi\omega\mu_0} \cdot \frac{\sin\phi'\cos\phi}{rr'^2} \sum_{n=1}^{\infty} (2n+1)\frac{\partial}{\partial\phi}P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n'(kr) \\ &- \frac{\omega\varepsilon}{4\pi k} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}P_n'(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr) \\ &- \frac{k}{4\pi\omega\mu_0} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n'(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n'(kr) \end{aligned}$$
where we have used the fact that $\theta = \theta' = \pi/2.$

$$\begin{split} G_{H} &= \frac{-1}{4\pi\omega\mu_{0}k} \cdot \frac{\sin\phi'\sin\phi}{r^{2}r'^{2}} \sum_{n=1}^{\infty} b_{n}n(n+1)(2n+1)P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{1}{4\pi\omega\mu_{0}} \cdot \frac{\cos\phi'\sin\phi}{r^{2}r'} \sum_{n=1}^{\infty} b_{n}(2n+1)\frac{\partial}{\partial\phi'}P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{1}{4\pi\omega\mu_{0}} \cdot \frac{\sin\phi'\cos\phi}{rr'^{2}} \sum_{n=1}^{\infty} b_{n}(2n+1)\frac{\partial}{\partial\phi'}P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{\omega\varepsilon}{4\pi k} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} e_{n}\frac{2n+1}{n(n+1)}P_{n}'(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{k}{4\pi\omega\mu_{0}} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} b_{n}\frac{2n+1}{n(n+1)}\frac{\partial^{2}}{\partial\phi\partial\phi'}P_{n}'(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ \end{split}$$

Numerical Problem for G_P

- G_P is an infinite summation over *n*
- Hankel functions of very high orders have very large ampliudes
- Difficult to handle numerically

Solution

Recall that physically G_p represents a z-directed electric field excited by a z-directed point current, therefore

$$G_P = \frac{-j}{\omega\mu_o} \left[\frac{\partial^2}{\partial y^2} + k^2 \right] \frac{e^{-jkR}}{4\pi R}$$

where

$$R = \sqrt{(r\cos\phi - r'\cos\phi')^2 + (y - r'\sin\phi')^2}$$

Mathematical Identity

Based on this fact, a mathematical identity can be established:

$$\begin{split} & \frac{-j}{\omega\mu_o} \left[\frac{\partial^2}{\partial y^2} + k^2 \right] \frac{e^{-jkR}}{4\pi R} \qquad (R = \sqrt{(r\cos\phi - r'\cos\phi')^2 + (y - r'\sin\phi')^2}) \\ & = \frac{-1}{4\pi\omega\mu_0 k} \cdot \frac{\sin\phi'\sin\phi}{r^2 r'^2} \sum_{n=1}^{\infty} n(n+1)(2n+1)P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr) \\ & -\frac{1}{4\pi\omega\mu_0} \cdot \frac{\cos\phi'\sin\phi}{r^2 r'} \sum_{n=1}^{\infty} (2n+1)\frac{\partial}{\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n(kr) \\ & -\frac{1}{4\pi\omega\mu_0} \cdot \frac{\sin\phi'\cos\phi}{rr'^2} \sum_{n=1}^{\infty} (2n+1)\frac{\partial}{\partial\phi}P_n(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n'(kr) \\ & -\frac{\omega\varepsilon}{4\pi k} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}P_n'(\cos(\phi-\phi'))\Phi_n(kr')\Psi_n(kr) \\ & -\frac{k}{4\pi\omega\mu_o} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n'(kr) \\ & -\frac{k}{4\pi\omega\mu_o} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\frac{\partial^2}{\partial\phi\partial\phi'}P_n(\cos(\phi-\phi'))\Phi_n'(kr')\Psi_n'(kr) \\ & -\frac{k}{32} \end{split}$$

Finally, the required Green's function is given as follows:

$$\begin{split} G_{M_{y}}^{H_{y}} &= \frac{-j}{\omega\mu_{o}} \left[\frac{\partial^{2}}{\partial y^{2}} + k^{2} \right] \frac{e^{-jkR}}{4\pi R} \\ &= \frac{-1}{4\pi\omega\mu_{0}k} \cdot \frac{\sin\phi'\sin\phi}{r^{2}r'^{2}} \sum_{n=1}^{\infty} b_{n}n(n+1)(2n+1)P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{1}{4\pi\omega\mu_{0}} \cdot \frac{\cos\phi'\sin\phi}{r^{2}r'} \sum_{n=1}^{\infty} b_{n}(2n+1)\frac{\partial}{\partial\phi'}P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{1}{4\pi\omega\mu_{0}} \cdot \frac{\sin\phi'\cos\phi}{rr'^{2}} \sum_{n=1}^{\infty} b_{n}(2n+1)\frac{\partial}{\partial\phi}P_{n}(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{\omega\varepsilon}{4\pi k} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} e_{n}\frac{2n+1}{n(n+1)}P_{n}'(\cos(\phi-\phi'))\hat{J}_{n}(kr')\hat{J}_{n}(kr) \\ &- \frac{k}{4\pi\omega\varepsilon} \cdot \frac{\cos\phi'\cos\phi}{rr'} \sum_{n=1}^{\infty} b_{n}\frac{2n+1}{n(n+1)}\frac{\partial^{2}}{\partial\phi\partial\phi'}P_{n}'(\cos(\phi-\phi'))\hat{J}_{n}'(kr')\hat{J}_{n}'(kr) \end{split}$$

Method-of-Moments Solution

Expand the equivalent magnetic current of the slot:

$$M(y) = \sum_{n=1}^{N} V_n f_n(y)$$

where

$f_n(x, y) = f_u$	$(x)f_p(y -$	$-y_n$)
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$f(x) = \begin{cases} \frac{1}{N} \\ \frac{1}{N} \end{cases}$	$\left(\frac{1}{W}\right)$	$\left x\right < W / 2$
) u (**)	0	x > W / 2

$$f_{p}(y) = \begin{cases} \frac{\sin k_{e}(h - |y|)}{\sin k_{e}h} & |y| < h \\ 0 & |y| > h \end{cases}$$

MoM Admittance of the Grounded Hemispherical Dielectric Resonator Antenna

$$Y^a_{mn} = Y^p_{mn} + Y^H_{mn}$$

where

$$Y_{mn}^{P} = -2 \iiint_{S_{0}} \iint_{S_{0}} f_{m}(x, y) G_{p} f_{n}(x', y') dS' dS$$

$$Y_{mn}^{H} = -2 \iiint_{S_0} \iint_{S_0} f_m(x, y) G_H f_n(x', y') dS' dS$$

However, G_P is singular

 \Rightarrow Difficult to integrate Y_{mn}^{p} numerically

Solution: Use the reduced kernel (Richmond form)

$$Y_{mn}^{p} = \frac{-2}{\eta^{2}} \left[\frac{-j\eta}{4\pi k} \int_{\frac{-L}{2}}^{\frac{L}{2}} \int_{\frac{-L}{2}}^{\frac{L}{2}} f_{p}(y - y_{m}) \frac{e^{-jk\zeta_{e}}}{\zeta_{e}^{5}} \left[(1 + jk\zeta_{e})(2\zeta_{e}^{2} - 3a_{e}^{2}) + a_{e}^{2}k^{2}\zeta_{e}^{2} \right] f_{p}(y' - y_{n}) dy' dy$$

where

$$\zeta_e = \sqrt{(y - y')^2 + {a_e}^2}$$

with $a_e = W/4$ being the equivalent radius of the slot.

Formulation of the Microstrip Feed network

- Apply the reciprocity approach as done by David Pozar*
 Not repeated here.
- ★ D. M. Pozar, "A reciprocity method of analysis for printed slot and slot-coupled microstrip antennas," *IEEE Trans. Antennas Progagat.*, vol.34, pp. 1439-1446, Dec. 1986.


At the reference plane (slot position): a = 12.5 mm, $\varepsilon_{ra} = 9.5$, $x_d = y_d = 0.0$, W = 0.9 mm, $W_f = 1.45 \text{ mm}$, d = 0.635 mm, $\varepsilon_{rs} = 2.96$, $L_s = 13.6 \text{ mm}$.



At the reference plane (slot position): a = 12.5 mm, $\varepsilon_{ra} = 9.5$, L = 13.5 mm, $y_d = 0.0$, W = 1.3 mm, $W_f = 1.45 \text{ mm}$, d = 0.635 mm, $\varepsilon_{rs} = 2.96$, $\mathcal{I}_s = 13.6 \text{ mm}$.



At the reference plane (slot position): a = 12.5 mm, $\varepsilon_{ra} = 9.5$, L = 13.5 mm, $x_d = 0.0$, W = 1.3 mm, $W_f = 1.45 \text{ mm}$, d = 0.635 mm, $\varepsilon_{rs} = 2.96$, $A_s = 13.6 \text{ mm}$.

Grounded Spherical Hemispherical Dielectric Resonator Antenna: Surface Electric Source



Advantages of Conformal-Strip Method

- Compatible with the probe-feed methodDrilling hole not required
- Very convenient post manufacturing trimmings
 - Cut shorter without leaving an air hole
 - Extended longer without the need for deepening the hole
- It is conformal

Hemispherical DRA for Demonstration

- Analytical closed-form Green function can be obtained
- * Efficient in numerical implementation
- Excited in the fundamental broadside TE₁₁₁
 mode

Co-ordinate system in the analysis



DRA Green's Function

Inside the DRA (r < a)

$$F_{r1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} \hat{J}_n(kr) P_n^m(\cos\theta) e^{jm\phi}$$

$$A_{r1} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{nm} \hat{J}_n(kr) P_n^m(\cos\theta) e^{jm\phi}$$

Outside the DRA (r > a)

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$$F_{r2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{nm} \hat{H}_{n}^{(2)}(k_{0}r) P_{n}^{m}(\cos\theta) e^{jm\phi}$$

$$A_{r2} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} D_{nm} \hat{H}_{n}^{(2)}(k_{0}r) P_{n}^{m}(\cos\theta) e^{jm\phi}$$

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(1)

(2)

(3)

(4)

Boundary Conditions

At the DRA-air interface: $\hat{r} \times (\vec{E}_2 - \vec{E}_1) = 0$ (5) $\hat{r} \times (\vec{H}_2 - \vec{H}_1) = J_{\phi s} \hat{\phi}$ (6)

where $J_{\phi s}$ is the conformal strip current.

On the DRA surface, we have r = r' = a and $G_1 = G_2 = G$, which is given by:

$$G = \frac{j\eta_0}{4\pi a^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{2n+1}{n(n+1)\Delta_n^{\text{TE}}} \hat{J}_n(ka) \hat{H}_n^{(2)}(k_0 a)$$

$$\cdot \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_n^m(\cos\theta)}{d\theta} \cdot \frac{dP_n^m(\cos\theta')}{d\theta'} \cos m(\phi - \phi')$$

$$- \frac{j\eta_0}{4\pi a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{2n+1}{n(n+1)\Delta_n^{\text{TM}}} \hat{J}_n'(ka) \hat{H}_n^{(2)}(k_0 a)$$

$$\cdot 2m^2 \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{P_n^m(\cos\theta)}{\sin\theta} \cdot \frac{P_n^m(\cos\theta')}{\sin\theta'} \cos m(\phi - \phi')$$
where $\Delta_n^{\text{TE}} = \hat{J}_n(ka) \hat{H}_n^{(2)}(k_0 a) - \frac{k}{k_0} \hat{J}_n'(ka) \hat{H}_n^{(2)}(k_0 a)$
(8)
$$\Delta_n^{\text{TM}} = \hat{J}_n'(ka) \hat{H}_n^{(2)}(k_0 a) - \frac{k}{k_0} \hat{J}_n(ka) \hat{H}_n^{(2)}(k_0 a)$$

$$\Delta_m = \begin{cases} 1, \ m > 0 \\ 2, \ m = 0 \end{cases}$$
(10) and η_0 is the wave impedance in vacuum.

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MoM Solution for the Strip Current

Using the MoM:

$$I(\phi) = \sum_{q=1}^{N} I_q f_q(\phi)$$

where

$$f_{q}(\phi) = \begin{cases} \frac{\sin k_{e}(h - a | \phi - \phi_{q} |)}{\sin k_{e} h}, & a | \phi - \phi_{q} | < h \\ 0, & a | \phi - \phi_{q} | \ge h \end{cases}$$
(15)

in which

$$h = \frac{L}{N+1}, \quad \phi_q = \frac{1}{a} \left(\frac{-L}{2} + qh \right), \quad k_e = \sqrt{(\varepsilon_r + 1)/2} k_0 \quad (16-18)$$

(14)

MoM Solution for the Strip Current (Cont')

By Galerkin's procedure, the following matrix equation is obtained:

$$[Z_{pq}][I_q] = [f_p(0)]$$
⁽¹⁹⁾

where

$$Z_{pq} = \frac{-1}{W^2} \iiint_{S_0} f_p(\phi) \ G(\theta, \phi; \theta', \phi') f_q(\phi') \ dS' dS$$
(20)

The input impedance is given by

$$Z_{\rm in} = \frac{1}{\sum_{q=1}^{N} I_q f_q(0)}$$

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(21)

Problems in Evaluating Z_{pq}

- The Green function G is singular as r → r'
 Excessive no. of modal terms required
 Higher-order Hankel functions difficult to handle numerically
- * Considerable computation time required

Solution

Integral Z_{pq} evaluated using novel recurrence formulas.

First express Z_{pq} in the following form:

$$Z_{pq} = \frac{-ja^2\eta_0}{4\pi W^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ \frac{\hat{J}_n(ka)\hat{H}_n^{(2)}(k_0a)}{\Delta_n^{TE}} \sum_{m=0}^n \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} [\Theta_1(n,m)]^2 \Phi_1(p,q,m) - \frac{\hat{J}_n'(ka)\hat{H}_n^{(2)'}(k_0a)}{\Delta_n^{TM}} \sum_{m=1}^n 2m^2 \cdot \frac{(n-m)!}{(n+m)!} [\Theta_2(n,m)]^2 \Phi_1(p,q,m) \right\}$$

where

$$\Theta_1(n,m) = \int_{\theta_1}^{\theta_2} \frac{dP_n^m(\cos\theta)}{d\theta} \sin\theta \,d\theta, \qquad \Theta_2(n,m) = \int_{\theta_1}^{\theta_2} P_n^m(\cos\theta) \,d\theta$$

$$\Phi_1(p,q,m) = \int_{\phi=\phi_p-\phi_h}^{\phi_p+\phi_h} \int_{\phi'=\phi_q-\phi_h}^{\phi_q+\phi_h} f_p(\phi) \cos m(\phi-\phi') f_q(\phi') d\phi' d\phi, \quad \phi_h = h/a$$

Recurrence formulas for $\Theta_2(n,m)$

(A) Recurrence formula recursive in *n*

$$\Theta_2(n+1,m) = \frac{1}{(n+1)(n-m+1)} \left\{ -(2n+1) \left[1 - (-1)^{n+m} \right] \sqrt{1 - x_1^2} P_n^m(x_1) + n(n+m) \Theta_2(n-1,m) \right\}$$

Initial values : $\Theta_2(0,0) = 2 \sin^{-1}x_1$ $\Theta_2(1,0) = 0$ $\Rightarrow \Theta_2(n,0)$ can be found for all *n*.

(B) Recurrence formula recursive in m

 $\Theta_2(n, m+2) = -2[1+(-1)^{n+m}]P_n^{m+1}(x_1) + (n+m+1)(n-m)\Theta_2(n,m)$

Initial values : $\Theta_2(n,0)$ [from (A) above] $\Theta_2(n,1) = [(-1)^n - 1] P_n(x_1)$ $\Rightarrow \Theta_2(n,m)$ can be found for all *m* and *n*.

Evaluation of $\Theta_2(n,m)$ and $\Phi_1(p,q,m)$

Evaluation of $\Theta_1(n,m)$

 $\Theta_1(n, m)$ are found in terms of $\Theta_2(n, m)$:

$$\Theta_1(n,m) = \left[(-1)^{n+m} - 1 \right] \sqrt{1 - x_1^2} P_n^m(x_1) - \frac{1}{2n+1} \left[(n+m)\Theta_2(n-1,m) - (m-n-1)\Theta_2(n+1,m) \right]$$

Analytical evaluation of $\Phi_1(p,q,m)$

$$\Phi_{1}(p,q,m) = \int_{\phi=\phi_{p}-\phi_{h}}^{\phi_{p}+\phi_{h}} \int_{\phi'=\phi_{q}-\phi_{h}}^{\phi_{q}+\phi_{h}} f_{p}(\phi) \cos m(\phi-\phi') f_{q}(\phi') d\phi' d\phi$$
$$= \left(\frac{2k_{e}a(\cos k_{e}h - \cos m\phi_{h})}{(\sin k_{e}h)(m - k_{e}a)(m + k_{e}a)}\right)^{2} \cos[m(\phi_{p} - \phi_{q})]$$

Radiation Fields

$$E_{\theta}(r,\theta,\phi) = \frac{j\eta_0 a}{4\pi W} \cdot \frac{e^{-jk_0 r}}{r} \sum_{q=1}^N I_q E_{\theta q}(\theta,\phi)$$
$$E_{\phi}(r,\theta,\phi) = \frac{-\eta_0 a}{4\pi W} \cdot \frac{e^{-jk_0 r}}{r} \sum_{q=1}^N I_q E_{\phi q}(\theta,\phi)$$

where $E_{\phi q}(\theta, \phi)$ and $E_{\phi q}(\theta, \phi)$ are found in a similar fashion.

Convergence Check

Input impedance (Ω)



Measured and Calculated Input Impedances



 $a = 12.5 \text{ mm}, \epsilon_r = 9.5, l = 12.0 \text{ mm}, \text{ and } W = 1.2 \text{ mm}$

Results



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Modified Configuration with a Parasitic Patch



Modified Configuration with a Parasitic Patch

Superscript A : excitation strip Superscript B : parasitic patch

E-field vanishes on the excitation strip

$${}^{A}E^{\theta}_{J_{\theta}} + {}^{B}E^{\theta}_{J_{\theta}} + {}^{B}E^{\theta}_{J_{\phi}} + {}^{A}E^{i} = 0$$

In terms of Green functions

$$\iint_{S_A} G_{J_\theta}^{E_\theta} J_\theta^A dS' + \iint_{S_B} G_{J_\theta}^{E_\theta} J_\theta^B dS' + \iint_{S_B} G_{J_\phi}^{E_\theta} J_\phi^B dS' + {}^A E^i = 0$$

Current expansions of Excitation & Parasitic Patches

$$I_{\theta}^{A}(\theta) = \sum_{p=1}^{N_{1}} I_{\theta p}^{A} f_{\theta p}^{A}(\theta)$$

$$I_{\theta}^{B}(\theta,\phi) = \sum_{l=1}^{N_{2}} \sum_{m=1}^{N_{4}} I_{lm}^{B\theta} f_{\theta l}^{B}(\theta) g_{\phi m}(\phi)$$

$$I_{\phi}^{B}(\theta,\phi) = \sum_{\nu=1}^{N_{3}} \sum_{n=1}^{N_{5}} I_{\nu n}^{B\phi} g_{\theta n}(\theta) f_{\phi \nu}^{B}(\phi)$$

where $f_{\theta l}^{B}(\theta)$ and $g_{\theta n}(\theta)$ are of the following form:

$$f_{\theta l}^{B}(\theta), g_{\theta n}(\theta) \sim \begin{cases} \frac{\sin(\theta_{h} - |\theta - \theta_{p}|)}{\sin \theta_{h}}, & a |\theta - \theta_{p}| < h \\ 0, & \text{elsewhere} \end{cases}$$

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New Integrals of $P_n^m(\cos \theta)$

$$\Theta_{\theta\theta}(p,n,m) = \int_{\theta_p - \theta_h}^{\theta_p + \theta_h} P_n^m(\cos\theta) f_{\theta p}(\theta) d\theta$$

$$\Theta_{\theta\theta^2}(p,n,m) = \int_{\theta_p - \theta_h}^{\theta_p + \theta_h} \frac{dP_n^m(\cos\theta)}{d\theta} \sin\theta f_{\theta^p}(\theta) d\theta$$

Their recurrence formulas have also been found [A] but are not included here for brevity.

[A] K. W. Leung, and H. K. Ng, "Theory and experiment of circularly polarized dielectric resonator antenna with a parasitic patch," *IEEE Trans. Antennas Propagat.*, vol. 51, pp.405-412, Mar. 2003.

Measured and Calculated Results



a=12.5mm, \mathcal{E}_r =9.5, l_1 =14mm, l_2 =7.9mm, W_1 =1.2mm, W_2 =2.2mm, and ϕ_0 =157.4°.

Measured and Calculated Results (Cont'd)



a=12.5mm, \mathcal{E}_r =9.5, l_1 =14mm, l_2 =7.9mm, W_1 =1.2mm, W_2 =2.2mm, and ϕ_0 =157.4°.

Measured and Calculated Results (Cont'd)



The method can be used to formulate microstrip antenna problems.



Analysis of Spherical Slot Antenna



Green's Function of the Antenna

 $G^{e,c} = G_P^{e,c} + G_H^{e,c}$

$$\begin{split} G_{P}^{e,c} &= \frac{-1}{4\pi a^{2} \eta_{e,c}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{2n+1}{n(n+1)} \cdot \hat{J}_{n}(k_{e,c}a) \hat{H}_{n}^{(2)}(k_{e,c}a) \cdot \frac{2}{\Delta_{m}} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_{n}^{m}(\cos\theta)}{d\theta} \cdot \frac{dP_{n}^{m}(\cos\theta)}{d\theta'} \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e,c}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{2n+1}{n(n+1)} \hat{J}_{n}'(k_{e,c}a) \hat{H}_{n}^{(2)'}(k_{e,c}a) \cdot 2m^{2} \frac{(n-m)!}{(n+m)!} P_{n}^{m}(\cos\theta) P_{n}^{m}(\cos\theta') \cos m(\phi-\phi') \\ &G_{H}^{e} = \frac{-1}{4\pi a^{2} \eta_{e}} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \beta_{n}^{TM} \frac{2n+1}{n(n+1)} [\hat{H}_{n}^{(2)}(k_{e}a)]^{2} \cdot \frac{2}{\Delta_{m}} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_{n}^{m}(\cos\theta)}{d\theta} \cdot \frac{dP_{n}^{m}(\cos\theta')}{d\theta'} \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \beta_{n}^{TE} \frac{2n+1}{n(n+1)} [\hat{H}_{n}^{(2)}(k_{e}a)]^{2} \cdot 2m^{2} \frac{(n-m)!}{(n+m)!} \frac{dP_{n}^{m}(\cos\theta)}{d\theta} \cdot \frac{dP_{n}^{m}(\cos\theta')}{d\theta'} \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \beta_{n}^{TE} \frac{2n+1}{n(n+1)} [\hat{H}_{n}^{(2)'}(k_{e}a)]^{2} \cdot 2m^{2} \frac{(n-m)!}{(n+m)!} P_{n}^{m}(\cos\theta) P_{n}^{m}(\cos\theta') \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{n} \sum_{m=1}^{n} \beta_{n}^{TE} \frac{2n+1}{n(n+1)} [\hat{H}_{n}^{(2)'}(k_{e}a)]^{2} \cdot 2m^{2} \frac{(n-m)!}{(n+m)!} P_{n}^{m}(\cos\theta) P_{n}^{m}(\cos\theta') \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{n} \sum_{m=1}^{n} \beta_{n}^{TE} \frac{2n+1}{n(n+1)} [\hat{H}_{n}^{(2)'}(k_{e}a)]^{2} \cdot 2m^{2} \frac{(n-m)!}{(n+m)!} P_{n}^{m}(\cos\theta) P_{n}^{m}(\cos\theta') \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{n} \sum_{m=1}^{n} \beta_{n}^{TE} \frac{2n+1}{n(n+1)!} [\hat{H}_{n}^{(2)'}(k_{e}a) \hat{H}_{n}^{(2)}(k_{e}a)] \frac{2n+1}{d\theta'} \cdot \frac{1}{n(n+1)!} \cdot \frac{2}{\Delta_{m}} \cdot \frac{(n-m)!}{(n+m)!} \cdot \frac{dP_{n}^{m}(\cos\theta)}{d\theta'} \cos m(\phi-\phi') \\ &- \frac{1}{4\pi a^{2} \eta_{e}} \cdot \frac{1}{\sin\theta \sin\theta'} \sum_{n=1}^{n} \sum_{m=1}^{n} \frac{1}{m} \left\{ \left[b_{n} \hat{H}_{n}^{(2)}(k_{e}a) \hat{J}_{n}^{*}(k_{e}a) + c_{n} \hat{J}_{n}(k_{e}a) \hat{H}_{n}^{(2)'}(k_{e}a) \right\} \right\}$$

MoM Admittances

$$Y_{pq} = \frac{-1}{W^2} \iiint_{S_0} f_p(\phi) \left[G^e + G^c \right] f_q(\phi') \, dS' dS$$

where

$$f_{q}(\phi) = \begin{cases} \frac{\sin k_{e}'(h - a|\phi - \phi_{q}|)}{\sin k_{e}'h}, & a|\phi - \phi_{q}| < h \\ 0, & a|\phi - \phi_{q}| \ge h \end{cases}$$
$$h = \frac{L}{N+1}, \qquad \phi_{q} = \frac{1}{a} \left(\frac{-L}{2} + qh\right), \qquad k_{e}' = \sqrt{(\varepsilon_{r} + 1)/2} k_{0}$$

Let

$$Y_{pq} = \left(Y_{pqP}^e + Y_{pqP}^c\right) + \left(Y_{pqH}^e + Y_{pqH}^c\right)$$

where

$$Y_{pqP,H}^{e,c} = \frac{-1}{W^2} \iiint_{S_0} f_p(\phi) G_{P,H}^{e,c} f_q(\phi') dS' dS$$

Method A

Numerical integration

$$Y_{pqP}^{e,c} = \frac{-1}{\eta_{e,c}^2} \int_{-\phi_1}^{\phi_1} \int_{-\phi_1}^{\phi_1} f_p(\phi) \left\{ \frac{-j\eta_{e,c}}{4\pi} \cdot \frac{e^{-jk_{e,c}R}}{k_{e,c}R^5} [R^2(k_{e,c}^2R^2 - jk_{e,c}R - 1)\cos(\phi - \phi')] \right\} f_q(\phi') a^2 d\phi' d\phi$$
$$-a^2(k_{e,c}^2R^2 - 3jk_{e,c}R - 3)\sin^2(\phi - \phi')] \right\}$$

Recurrence formulas

$$Y_{pqH} = \frac{a^2}{4\pi W^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ A(n) \sum_{m=0}^{n} \frac{2}{\Delta_m} \cdot \frac{(n-m)!}{(n+m)!} \left[\Theta_1(n,m) \right]^2 \Phi(p,q,m) \right\}$$

$$+ B(n) \sum_{m=1}^{n} 2m^2 \cdot \frac{(n-m)!}{(n+m)!} [\Theta_2(n,m)]^2 \Phi(p,q,m)$$

where

$$R = \sqrt{4a^2 \sin^2[(\phi - \phi')/2] + r_1^2}$$

$$A(n) = \frac{\beta_n^{TM}}{\eta_e} \left[\hat{H}_n^{(2)}(k_e a) \right]^2 + \frac{1}{\eta_c} \left[e_n \hat{H}_n^{(2)}(k_c a) \hat{J}_n(k_c a) + f_n \hat{J}_n'(k_c b) \hat{H}_n^{(2)}(k_c a) \right]$$

$$B(n) = \frac{\beta_n^{TE}}{\eta_e} \left[\hat{H}_n^{(2)}(k_e a) \right]^2 + \frac{1}{\eta_c} \left[b_n \hat{H}_n^{(2)}(k_c a) \hat{J}_n(k_c a) + c_n \hat{J}_n(k_c b) \hat{H}_n^{(2)}(k_c a) \right]_{68}$$

Method B

- Combine the particular and homogeneous solutions
- Only recurrence formulas are used
- Advantage: Easy to implement and fast (no numerical integration required)
- Disadvantage: ??

Convergence Check for Method A



Convergence Check for Method B



Comparison between Methods A & B

Advantages of Method A

- Method A converges much faster than Method B
- More suitable for problems of larger spherical sizes

Advantages of Method B

- Numerical integrations not required
- Easier to implement
- Much faster than Method A

Method A	Method B	Numerical integration for both G_p, G_H
32 sec.	112 sec.	77 sec.
Measured & Calculated (Method B) Results



Measured and calculated input impedances of the rectangular slot for a = 6.25 cm, b = 4.0 cm, L = 12.46 cm, W = 2.4 mm, and $\varepsilon_r = 1$.

Recurrence Formulas for Integral of $\hat{J}_n(kr)$



MoM Admittance

Define $Y^D \& Y^C$ as the dielectric & cavity admittances, respectively.

$$Y_{pqH}^{D,C} = \frac{-1}{2\pi\omega\mu_0 k} \sum_{n=1}^{\infty} n(n+1)(2n+1)\gamma_n^{D,C} A_n(p) A_n(q)$$

where

$$A_{n}(i) = \int_{y_{i-1}}^{y_{i+1}} \frac{\hat{J}_{n}(ky)}{y^{2}} \cdot \frac{\sin k_{e}(h - |y - y_{i}|)}{\sin k_{e}h} dy$$

Recurrence formula for $A_n(i)$

$$\frac{(n+2)(n+3)}{(2n+1)(2n+3)}A_{n+2}(i) + \left[\frac{k_e^2}{k^2} - \frac{(2n^2+2n-3)}{(2n-1)(2n+3)}\right]A_n(i) + \frac{(n-1)(n-2)}{(2n+1)(2n-1)}A_{n-2}(i)$$
$$= \frac{k_e}{k^2 \sin k_e h} \sum_{j=-1}^1 B(j) \frac{\hat{J}_n(ky_{i+j})}{y_{i+j}^2}$$

Backward recurrence for $A_n(i)$, needless to know $A_0, A_1, A_2, \& A_3$.

Measured and Calculated Results



 $a = 12.5 \text{ mm}, \epsilon_{r1} = 9.5, L = 20.0 \text{ mm}, W = 1.0 \text{ mm}, = 0, \text{ and } N = 5$

Solving Planar annular problem using spherical solution



MoM Solution

- First obtain the spherical Green's function of H_{ϕ} due to a ϕ -directed magnetic point current
- Substitute $\theta = \theta' = \pi/2$ (both source & field one the ground plane)

MoM Solution (Cont'd)

$$Z_{\rm in} = \sum_{q=1}^{N} \frac{1}{Y_{qq}}$$

where

$$Y_{qq} = \frac{\pi \Delta_{q-1}^2}{W^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \cdot \frac{(n-q+1)!}{(n+q-1)!} \cdot \left\{ \frac{1}{\Delta_{q-1}} \left[P_n^q(0) \right]^2 \left[\frac{\alpha_1^{TM}(n)}{\eta_1} + \frac{\alpha_2^{TM}(n)}{\eta_2} \right] + (q-1)^2 \left[P_n^{q-1}(0) \right]^2 \left[\frac{\alpha_1^{TE}(n)}{\eta_1} + \frac{\alpha_2^{TE}(n)}{\eta_2} \right] \right\}$$

in which

$$P_n^m(0) = \begin{cases} (-1)^{(n+m)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n+m-1)}{2 \cdot 4 \cdot 6 \cdots (n-m)} & n+m \text{ even} \\ 0 & n+m \text{ odd} \end{cases}$$

$$\Delta_{q-1} = \begin{cases} 2 & q = 1 \\ 1 & \text{otherwise} \end{cases}$$

Measured and Calculated Results



 $a = 19.25 \text{ mm}, b = 10.75 \text{ mm}, W = 1.5 \text{ mm}, \varepsilon_{r1} = 4, \text{ and } \varepsilon_{r2} = 1$

