The Particle Filtering Methodology in Signal Processing

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5. Handling constant parameters
6. Rao-Blackwellization
7. Prediction and smoothing
8. Multiple particle filters

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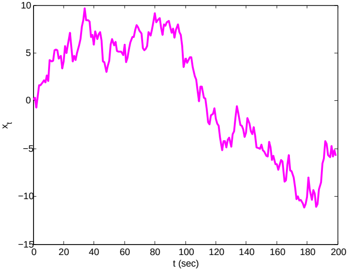
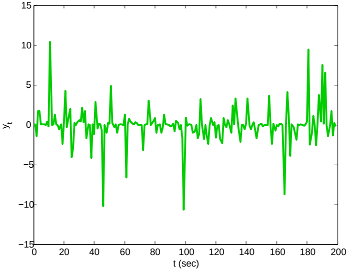
- The main objective of sequential signal processing is the recursive estimation of the state of the system based on the available measurements





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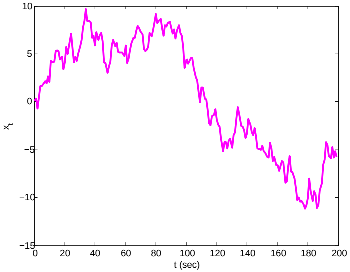

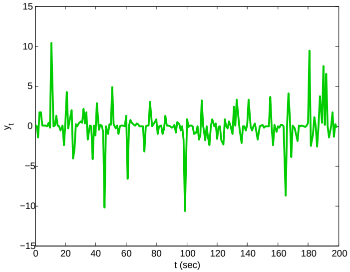

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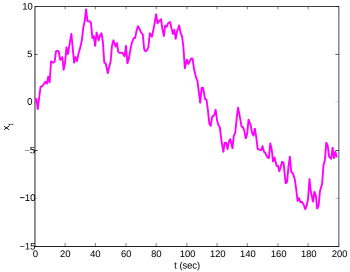

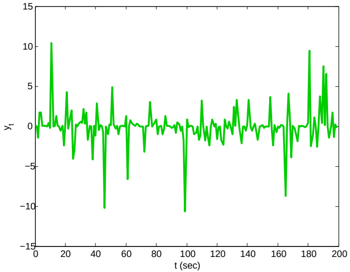

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- Optimal filtering methods for sequential signal processing are based on analytical updates of the densities of interest

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Representation of dynamic systems

Many problems can be stated in terms of *dynamic systems*

1. *State equation*: hidden random signal to be estimated

$$\mathbf{x}_t = f\left(\mathbf{x}_{t-1}, \mathbf{u}_t\right)$$

$\mathbf{x}_t$	signal (state) of interest
f	state transition function (possibly nonlinear)
$\mathbf{u}_t$	state perturbation noise

2. *Observation equation*: available information

$$\mathbf{y}_t = g\left(\mathbf{x}_t, \mathbf{v}_t\right)$$

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Alternative representation of dynamic systems

One can also represent the dynamic system in terms of *densities*

1. *State equation*: $p\left(\mathbf{x}_t|\mathbf{x}_{t-1},\boldsymbol{\theta}\right)$
2. *Observation equation*: $p\left(\mathbf{y}_t|\mathbf{x}_t,\boldsymbol{\psi}\right)$

The form of the density functions depends on:

- the functions $f(\cdot)$ and $g(\cdot)$
- the densities of $\mathbf{u}_t$ and $\mathbf{v}_t$

$$\left.\begin{array}{c}\boldsymbol{\theta}\in\Re^{L_\theta}\\ \boldsymbol{\psi}\in\Re^{L_\psi}\end{array}\right\}\longrightarrow\begin{array}{l}\text{Unknown fixed parameter vectors}\\ \text{They can be included in } \mathbf{x}_t \text{ as static parameters}\end{array}$$

Important densities

Three densities play a critical role in *sequential signal processing*

1. *Filtering density*: $p\left(\mathbf{x}_t|\mathbf{y}_{1:t}\right)$, where $\mathbf{y}_{1:t} = \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_t\}$
2. *Predictive density*: $p\left(\mathbf{x}_{t+l}|\mathbf{y}_{1:t}\right)$, where $l \geq 1$

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Tracking the important densities

Objective: Track the densities by exploiting *recursive relationships*

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Example: Bearings-only tracking

Consider the tracking of an object based on bearings-only measurements

$$\mathbf{x}_t = \mathbf{G}_x \mathbf{x}_{t-1} + \mathbf{G}_u \mathbf{u}_t$$

where $\mathbf{x}_t = [x_{1,t} \;\; x_{2,t} \;\; \dot{x}_{1,t} \;\; \dot{x}_{2,t}]^\top$

Bearings-only range measurements are obtained by J sensors placed at known locations in the sensor field

$$y_{j,t} = \arctan\left(\frac{x_{2,t} - l_{2,j}}{x_{1,t} - l_{1,j}}\right) + v_t(n)$$

- Given the measurements of J sensors, $\mathbf{y}_t = [y_{1,t} \; y_{2,t} \;\; \cdots \;\; y_{J,t}]^\top$, and the movement model of the object, the goal is to track the object in time, that is, estimate its position and velocity

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- In the literature this problem is known as *target tracking*

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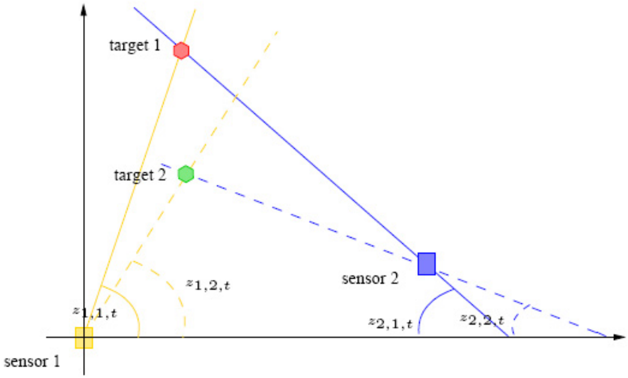
$$y_{j,t} = \arctan\left(\frac{x_{2,t} - l_{2,j}}{x_{1,t} - l_{1,j}}\right) + v_t(n)$$

- Given the measurements of J sensors, $\mathbf{y}_t = [y_{1,t} \; y_{2,t} \;\; \cdots \;\; y_{J,t}]^\top$, and the movement model of the object, the goal is to track the object in time, that is, estimate its position and velocity
- In the literature this problem is known as *target tracking*



A target in a two-dimensional space and two BOT sensors

Recursion for filtering

- Suppose that at time $t-1$, we know the observations $\mathbf{y}_{1:t-1}$ and the a posteriori PDF $p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1})$
- Once $\mathbf{y}_t$ becomes available, we update the PDF

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Outline

Approximation of densities by discrete random measures

Densities can be approximated by *discrete random measures*

$$\chi = \left\{ \mathbf{x}_t^{(m)}, w_t^{(m)} \right\}_{m=1}^{M}$$

- The discrete random measure approximates the density by

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Example of approximation of densities

Assume that independent particles, $\mathbf{x}_t^{(m)}$, can be drawn from $p(\mathbf{x}_t|\mathbf{y}_{1:t})$

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Strong law of large numbers	*Central limit theorem*
$\widehat{E}(h(\mathbf{X}_t)) \xrightarrow{a.s.} E(h(\mathbf{X}_t))$	$\widehat{E}(h(\mathbf{X}_t)) \xrightarrow{d} \mathcal{N}\left(E(h(\mathbf{X}_t)), \frac{\sigma_h^2}{M}\right)$

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The concept of importance sampling (IS)

- In practice, we often cannot draw samples directly from $p(\mathbf{x}_t|\mathbf{y}_{1:t})$
- Particles are drawn from $\pi(\mathbf{x}_t)$ and the estimate of $E\left(h(\mathbf{X}_t)\right)$ becomes

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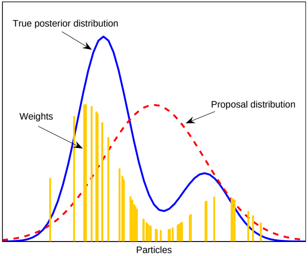
Summary

We use a random measure to approximate $p(\mathbf{x}_t|\mathbf{y}_{1:t})$ in two steps:

1. Drawing samples from a proposal function $\pi(\mathbf{x}_t)$, which needs to be known only up to a multiplicative constant, i.e.,

$$\mathbf{x}_t^{(m)} \sim \pi(\mathbf{x}_t), \quad m = 1, 2, \cdots, M$$

2. Computing the weights of the particles $w_t^{(m)}$



How do we obtain χ_t from χ_{t-1}?

We assume $\chi_{t-1} = \left\{ \mathbf{x}_{t-1}^{(m)}, w_{t-1}^{(m)} \right\}_{m=1}^{M}$ is given.

- *Step one: Generation of new particles from a proposal function, i.e.,*

$$\mathbf{x}_t^{(m)} \sim \pi(\mathbf{x}_t), \quad m = 1, \cdots, M$$

  and augmentation of the particle stream $\mathbf{x}_{0:t-1}^{(m)}$ with $\mathbf{x}_t^{(m)}$

- *Step two: Computation of the particle weights of* $\mathbf{x}_t^{(m)}$, $m = 1, \cdots, M$, *i.e.*

$$w_t^{(m)} \propto w_{t-1}^{(m)} \times \text{update factor}, \quad m = 1, \cdots, M$$

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Mathematical formulation of the algorithm

Particle generation

Basis

$$\pi\left(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}\right) = \pi\left(\mathbf{x}_t|\mathbf{x}_{0:t-1},\mathbf{y}_{1:t}\right)\pi\left(\mathbf{x}_{0:t-1}|\mathbf{y}_{1:t-1}\right)$$

$$\mathbf{x}_{0:t-1}^{(m)} \sim \pi\left(\mathbf{x}_{0:t-1}|\mathbf{y}_{1:t-1}\right)$$

$$w_{t-1}^{(m)} \propto \frac{p(\mathbf{x}_{0:t-1}^{(m)}|\mathbf{y}_{t-1})}{\pi(\mathbf{x}_{0:t-1}^{(m)}|\mathbf{y}_{1:t-1})}$$

Augmentation of the trajectory $\mathbf{x}_{0:t-1}^{(m)}$ with $\mathbf{x}_t^{(m)}$

$$\mathbf{x}_t^{(m)} \sim \pi\left(\mathbf{x}_t|\mathbf{x}_{0:t-1}^{(m)},\mathbf{y}_{1:t}\right), \quad m = 1,\cdots,M$$

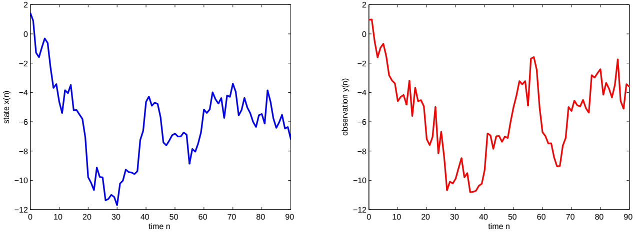
Weight update

$$w_t^{(m)} \propto \frac{p\left(\mathbf{y}_t|\mathbf{x}_t^{(m)}\right)p\left(\mathbf{x}_t^{(m)}|\mathbf{x}_{t-1}^{(m)}\right)}{\pi\left(\mathbf{x}_t^{(m)}|\mathbf{x}_{0:t-1}^{(m)},\mathbf{y}_{1:t}\right)}w_{t-1}^{(m)}, \quad m = 1,\cdots,M$$

Example: random walk

Consider the state-space model

$$
\begin{aligned}
x_t &= x_{t-1} + u_t \\
y_t &= x_t + v_t
\end{aligned}
$$



	$t = 0$	$t = 1$	$t = 2$
x_t	1.44	0.89	-1.28
y_t	0.96	0.98	-0.51

Example: SIR for random walk

- We choose as a proposal function

$$\pi(x_t) = p(x_t|x_{t-1})$$

- Therefore, we generate the particles according to

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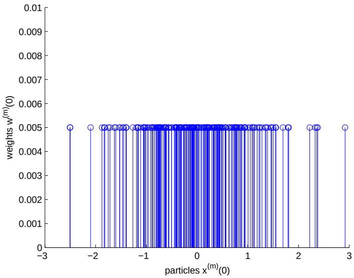
Example: Step by step SIR for random walk

Initialization: $t = 0$

$$x_0^{(m)} \sim \mathcal{N}(0,1) \qquad m = 1, 2, \ldots, M.$$

Note that all the weights are equal



Example: Step by step SIR for random walk (cont.)

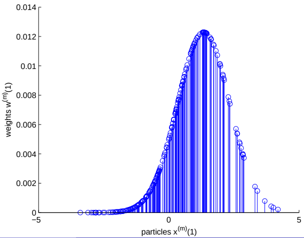
Time instant: $t = 1$

- Generation of particles using the proposal function:

$$x_1^{(m)} \sim p\left(x_1 | x_0^{(m)}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_1 - x_0^{(m)})^2}{2}\right)$$

- Weight update:

$$w_1^{(m)} \propto \exp\left(-\frac{(y_1 - x_1^{(m)})^2}{2}\right)$$



Example: Step by step SIR for random walk (cont.)

Time instant: $t = 2$

- Generation of particles using the proposal function:

$$x_2^{(m)} \sim p\left(x_2 | x_1^{(m)}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_2 - x_1^{(m)})^2}{2}\right)$$

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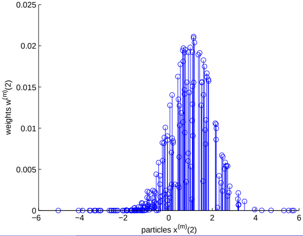

Choice of proposal function

- The proposal (importance) function plays a crucial role in the performance of PF
- It is desirable to use easy-to-sample proposal functions that produce particles with a large enough variance in order to avoid exploration of the state space in too narrow regions and thereby contributing to losing the tracks of the state, but not too large to alleviate generation of too dispersed particles

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The optimal importance function

The optimal choice for the importance function is the true posterior distribution, i.e.,

$$\pi\left(\mathbf{x}_t|\mathbf{x}_{0:t-1}^{(m)}, \mathbf{y}_{1:t}\right) = p\left(\mathbf{x}_t|\mathbf{x}_{t-1}^{(m)}, \mathbf{y}_t\right)$$

This choice corresponds to the weight calculation

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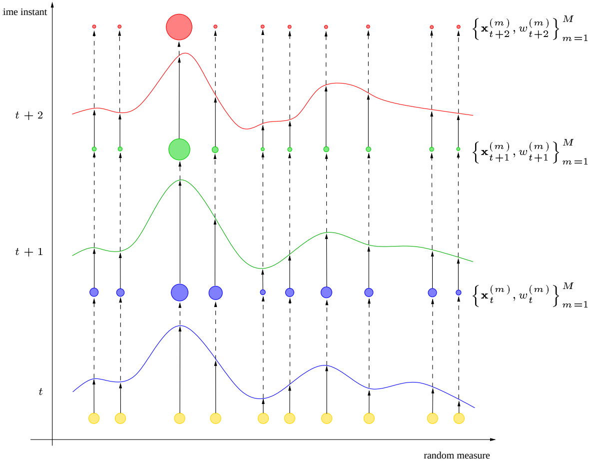
Degeneracy of the random measure: Resampling

In particle filtering the discrete random measure degenerates quickly and only few particles are assigned meaningful weights

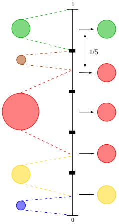


Resampling

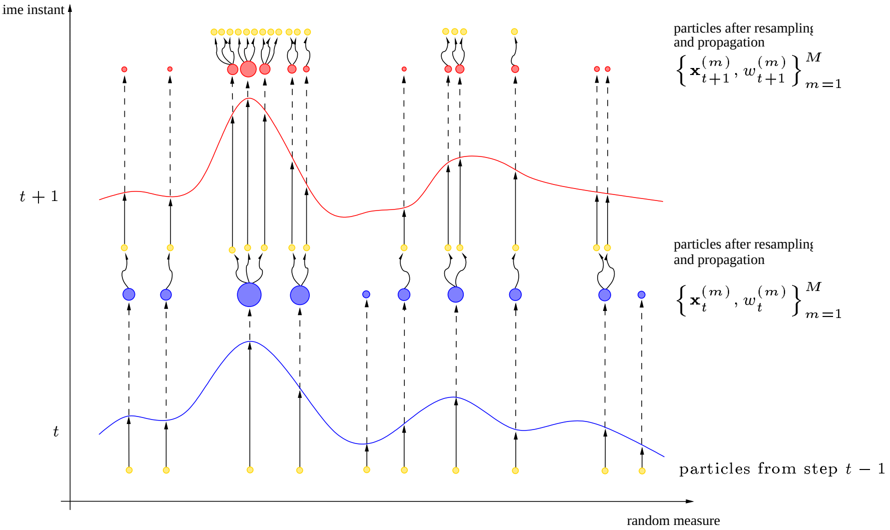
- A measure of this degeneracy is the *effective particle size*

$$M_{\text{eff}} = \frac{M}{1 + \mathrm{Var}\left(w_t^{*(m)}\right)} \quad \rightarrow \quad \widehat{M}_{\text{eff}} = \frac{1}{\sum_{m=1}^{M}\left(w_t^{(m)}\right)^2}$$

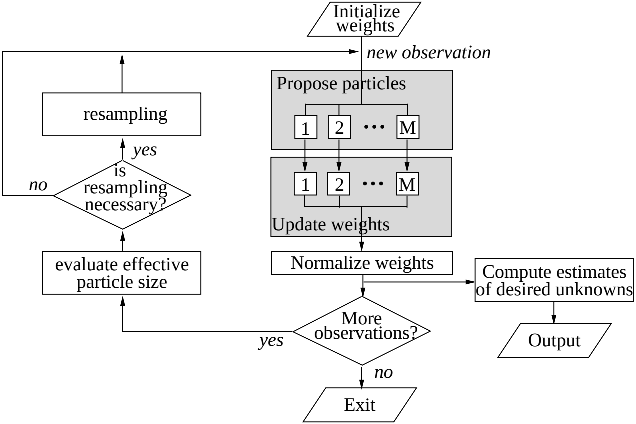
- Degeneracy is reduced by using good importance sampling functions and resampling
- Resampling eliminates particles with small weights and replicates the ones with large weights



Pictorial description of resampling



Flowchart of the SIR algorithm



Outline

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It explores the state space by using the latest measurements.

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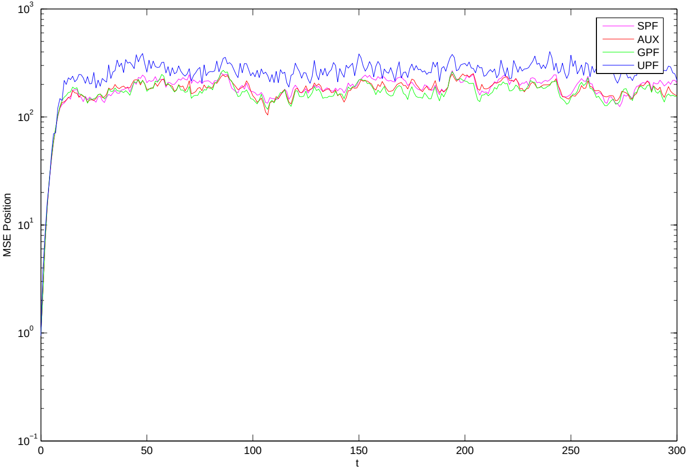
Comparison of methods – Bearings only tracking



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Kernel-based auxiliary particle filter

- The inclusion of fixed parameters in the model implies extending the random measure to the form

$$\chi_t = \left\{ \mathbf{x}_t^{(m)}, \boldsymbol{\theta}_t^{(m)}, w_t^{(m)} \right\}_{m=1}^{M}$$

- The random measure approximates the density of interest $p(\mathbf{x}_t, \theta \mid \mathbf{y}_t)$, which can be decomposed as

$$p(\mathbf{x}_t, \theta \mid \mathbf{y}_{1:t}) \propto p(\mathbf{y}_t \mid \mathbf{x}_t, \theta) p(\mathbf{x}_t \mid \theta, \mathbf{y}_{1:t}) p(\theta \mid \mathbf{y}_{1:t})$$

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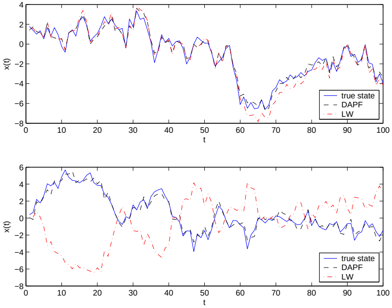
Density assisted particle filter

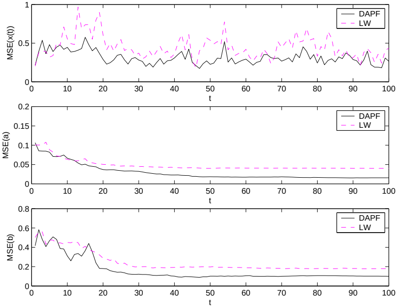
The approximating densities can be other than Gaussians or mixtures of Gaussians

1. Draw particles of $\boldsymbol{\theta}$ from $p(\boldsymbol{\theta}|\mathbf{y}_{0:t-1})$, i.e.,
$$\boldsymbol{\theta}_{t-1}^{(m)} \sim p(\boldsymbol{\theta}|\mathbf{y}_{0:t-1})$$
2. Draw particles according to
$$\mathbf{x}_t^{(m)} \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}^{(m)}, \boldsymbol{\theta}_{t-1}^{(m)})$$
3. Set $\boldsymbol{\theta}_t^{(m)} = \boldsymbol{\theta}_{t-1}^{(m)}$
4. Update and normalize the weights
$$w_t^{(m)} \propto p(\mathbf{y}_t|\mathbf{x}_t^{(m)}, \boldsymbol{\theta}_t^{(m)})$$
5. Estimate the parameters of the density $p(\boldsymbol{\theta}|\mathbf{y}_{0:t})$ from
$$\chi_t = \left\{\mathbf{x}_t^{(m)}, \boldsymbol{\theta}_t^{(m)}, w_t^{(m)}\right\}_{m=1}^{M}$$

Handling constant parameters – example

$$
\begin{aligned}
x_t &= a x_{t-1} + u_t \\
y_t &= b x_t + v_t
\end{aligned}
$$





Outline

An efficient approach: Rao-Blackwellized PF

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Example: Bearings-only tracking with biased measurements

Consider K targets moving along a 2D sensor field of N sensors

$$\mathbf{x}_t = \mathbf{G}_x \mathbf{x}_{t-1} + \mathbf{G}_u \mathbf{u}_t$$

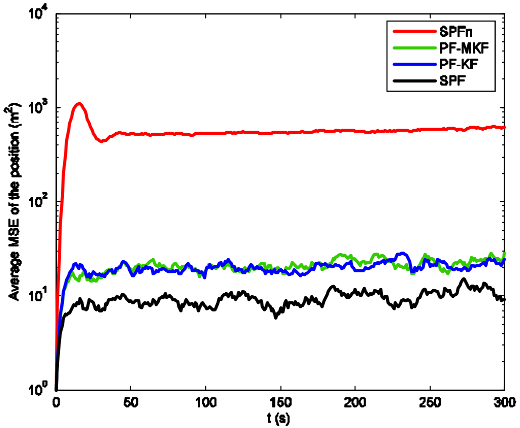
The $n-$th sensor at time instant t measures the bearing information of the targets sensed in the field

$$\begin{aligned} \mathbf{y}_{n,t} &= \mathbf{g}_n(\mathbf{x}_t) + \mathbf{b}_n + \mathbf{v}_{n,t} \qquad\qquad n = 1, \cdots, N \\ \mathbf{g}_n(\mathbf{x}_t) &= \left[\arctan\left(\frac{x_{2,1,t} - x_{2,n}}{x_{1,1,t} - x_{1,n}} \right), \cdots, \arctan\left(\frac{x_{2,K,t} - x_{2,n}}{x_{1,K,t} - x_{1,n}} \right) \right]^\top \end{aligned}$$

where $\mathbf{b}_n$ represents the unknown bias of the $n-$th sensor

Comparison of algorithms

Averaged MSE in m^2 of two targets obtained by different methods using $M = 500$ particles and $J = 50$ independent runs



Outline

Prediction

The prediction problem revolves around the estimation of the predictive density $p(\mathbf{x}_{t+l}|\mathbf{y}_{1:t})$, where $l > 1$

The approximation of the predictive density $p(\mathbf{x}_{t+l}|\mathbf{y}_{1:t})$ is obtained by

1. drawing particles $\mathbf{x}_{t+1}^{(m)}$ from $p(\mathbf{x}_{t+1}|\mathbf{x}_{t}^{(m)})$
2. drawing particles $\mathbf{x}_{t+2}^{(m)}$ from $p(\mathbf{x}_{t+2}|\mathbf{x}_{t+1}^{(m)})$
3. $\cdots$
4. drawing particles $\mathbf{x}_{t+l}^{(m)}$ from $p(\mathbf{x}_{t+l}|\mathbf{x}_{t+l-1}^{(m)})$
5. and associating with the samples $\mathbf{x}_{t+l}^{(m)}$ the weights $w_t^{(m)}$ and thereby forming the random measure $\{\mathbf{x}_{t+l}^{(m)}, w_t^{(m)}\}_{m=1}^{M}$.

Smoothing

All the information about $\mathbf{x}_t$ in this case is in the PDF $p(\mathbf{x}_t|\mathbf{y}_{1:T})$

Forward PF

Run PF in the forward direction and store all the random measures

$\chi_t = \{\mathbf{x}_t^{(m)}, w_t^{(m)}\}_{m=1}^M, \quad n = 1, 2, \cdots, N$

Backward recursions

Set the smoothing weights

$w_{s,T}^{(m)} = w_T^{(m)}$ and $\chi_{s,T} = \{\mathbf{x}_T^{(m)}, w_{s,T}^{(m)}\}_{m=1}^M$

For $n = N-1, \cdots, 1, 0$

Computation of the smoothing weights $w_{s,t}$

Compute the smoothing weights of $\mathbf{x}_t^{(m)}$ by

$$w_{s,t}^{(m)} = \sum_{j=1}^M w_{s,t+1}^{(j)} \frac{w_t^{(m)} p(\mathbf{x}_{t+1}^{(j)}|\mathbf{x}_t^{(m)})}{\sum_{l=1}^M w_t^{(l)} p(\mathbf{x}_{t+1}^{(j)}|\mathbf{x}_t^{(l)})}$$

Construction of the smoothing random measure $\chi_{s,t}$

Set $\chi_{s,t} = \{\mathbf{x}_t^{(m)}, w_{s,t}^{(m)}\}_{m=1}^M$

Outline

Multiple particle filters

We are interested in particle filtering methods for complex systems that can be represented by the following state-space model:

$$
\begin{aligned}
\mathbf{x}_t &= f_x(\mathbf{x}_{t-1}, \mathbf{u}_t) \\
\mathbf{y}_t &= f_y(\mathbf{x}_t, \mathbf{v}_t)
\end{aligned}
$$

We assume that the dimension of $\mathbf{x}_t$ is high

From practice we know that high-dimensional states would, in general, require a very large number of particles for accurate tracking of the posterior pdf of $\mathbf{x}_t$

Multiple particle filters

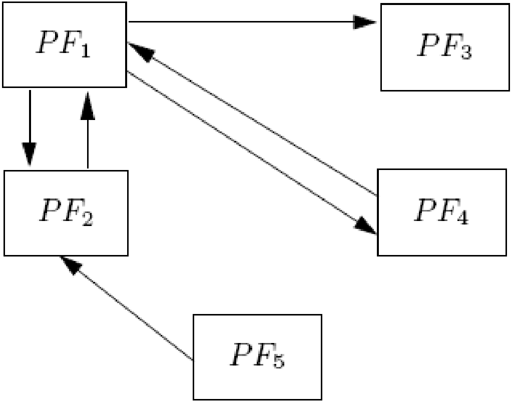
The underlying idea is to use multiple particle filters that communicate information about their posterior pdfs

The particle filters operate on partitioned subspaces of the complete state space

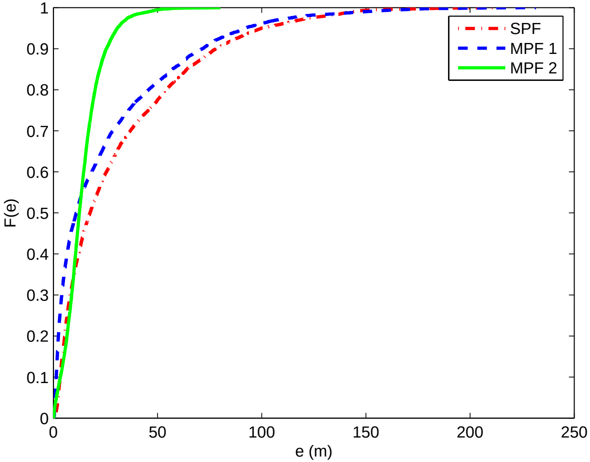
The state space is decomposed into separate subspaces of lower dimensionality which form a partition of the original space

We assume that the interest is in finding the marginal posterior densities of the state vectors that span these subspaces

Multiple particle filters



A system of multiple particle filters.



CDFs of RMSEs ($M = 1000,\ M_k = 500$)

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- We expect that the approximations deteriorate as the dimension of the state space increases but hope that as the number of particles $M \rightarrow \infty$, the approximation of the density will improve

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Convergence theorems

Theorem 1: If $p(\mathbf{x}_t|\mathbf{x}_{t-1})$ satisfies some mild conditions and the likelihood function $p(\mathbf{y}_t|\mathbf{x}_t)$ is bounded, continuous and strictly positive, then

$$\lim_{M\to\infty} \chi_t^M = p(\mathbf{x}_t|\mathbf{y}_{1:t})$$

almost surely

Theorem 2: If the likelihood function $p(\mathbf{y}_t|\mathbf{x}_t)$ is bounded, for all $t \geq 1$ there exists a constant c_t independent of M such that for any bounded function

$$E(e_t^2) \leq c_t \frac{\|h(\mathbf{x}_t)\|^2}{M}$$

where $E(\cdot)$ is an expectation operator, $h(\mathbf{x}_t)$ is a function of $\mathbf{x}_t$, and $\|h(\mathbf{x}_t)\|$ denotes the supremum norm, i.e.,

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