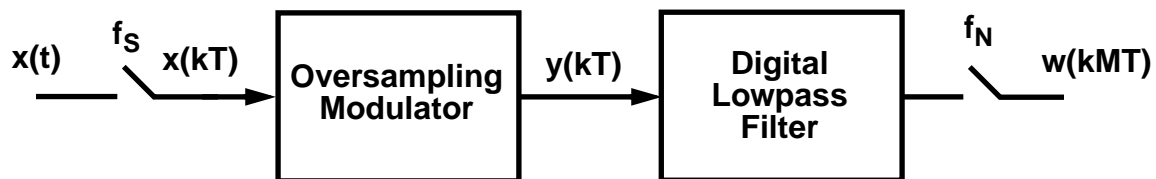


Oversampled A/D Conversion

Basic concept:

Exchange resolution in time for that in amplitude through the use of oversampling, feedback and digital filtering.



$$f_s = 1/T = \text{sampling rate}$$

$$f_N = 1/MT = \text{Nyquist rate}$$

$$M = \text{oversampling ratio}$$

Oversampling Modulators

Predictive

- Δ Modulation
- DPCM (Differential PCM)

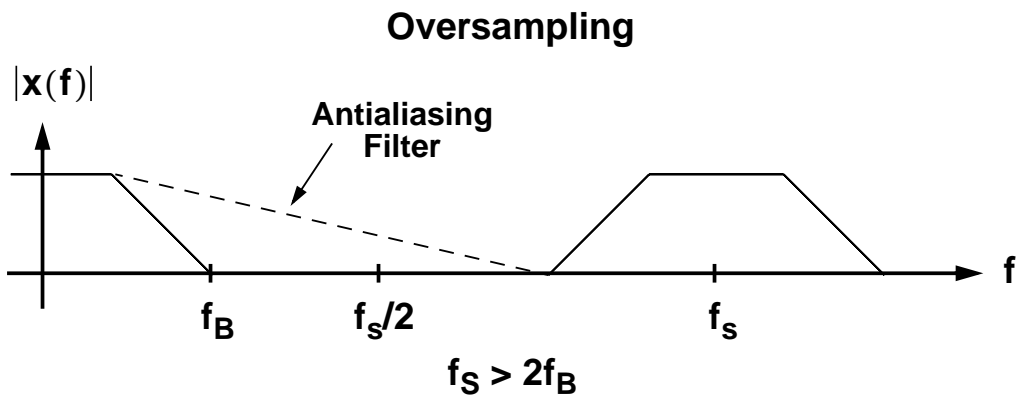
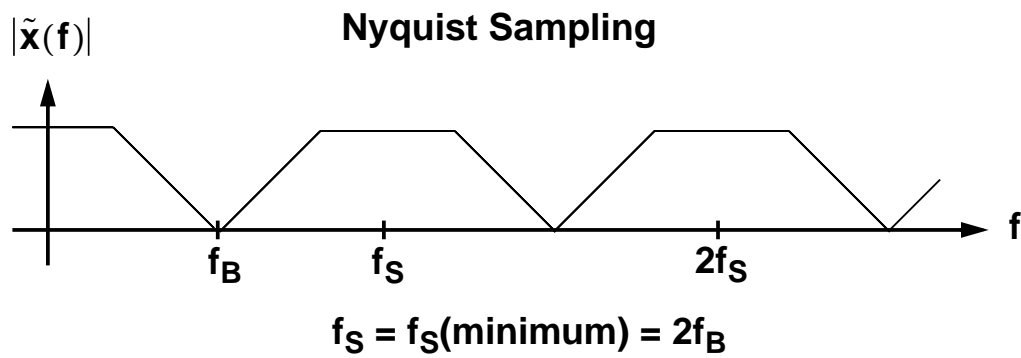
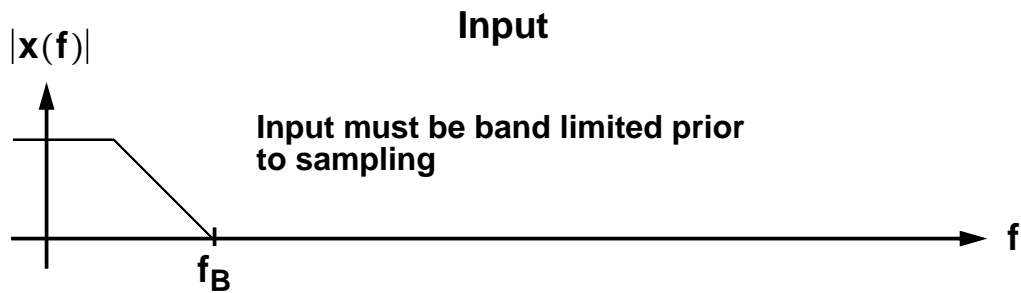
Noise-Shaping

- $\Sigma\Delta$ Modulation
- Cascaded (Multistage) $\Sigma\Delta$ Modulation
- Multilevel $\Sigma\Delta$ Modulation
- Interpolation

Benefits of oversampling:

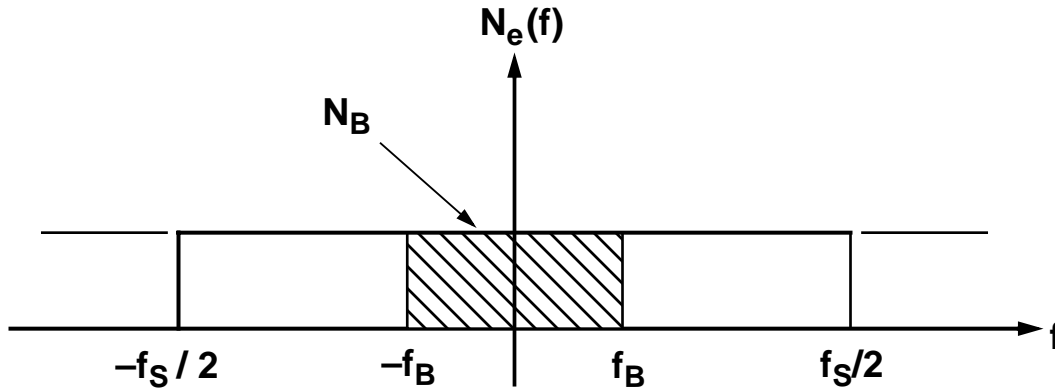
- Relaxed transition band requirements for analog antialiasing (and reconstruction) filters
- Reduced baseband quantization noise power

Antialiasing



Baseband Noise

For an active discrete-time quantizer with step size Δ and sampling rate f_s (which is not in overload), the quantization noise power is distributed uniformly across the Nyquist bandwidth.



The power spectral density of the quantization error, e , is

$$N_e(f) = \frac{\overline{e^2}}{f_s} = \left(\frac{\Delta^2}{12}\right) \frac{1}{f_s}$$

and all of the quantization noise is aliased into the Nyquist band, $-f_s/2$ to $f_s/2$.

When $f_B = f_s/2$, then the baseband ($-f_B < f < f_B$) quantization noise power is

$$S_{B0} = \int_{-f_B}^{f_B} N_e(f) df = \frac{\Delta^2}{12}$$

When $f_B < f_S/2$, the baseband quantization noise power is

$$\begin{aligned} S_B &= \int_{-f_B}^{f_B} N_e(f) df = \left(\frac{\Delta^2}{12}\right) \left(\frac{1}{f_S}\right) \int_{-f_B}^{f_B} N_e(f) df \\ &= S_{B0} \left(\frac{2f_B}{f_S}\right) = \frac{S_{B0}}{M} \end{aligned}$$

where

$$M \equiv \frac{f_S}{2f_B} = \text{OVERSAMPLING RATIO}$$

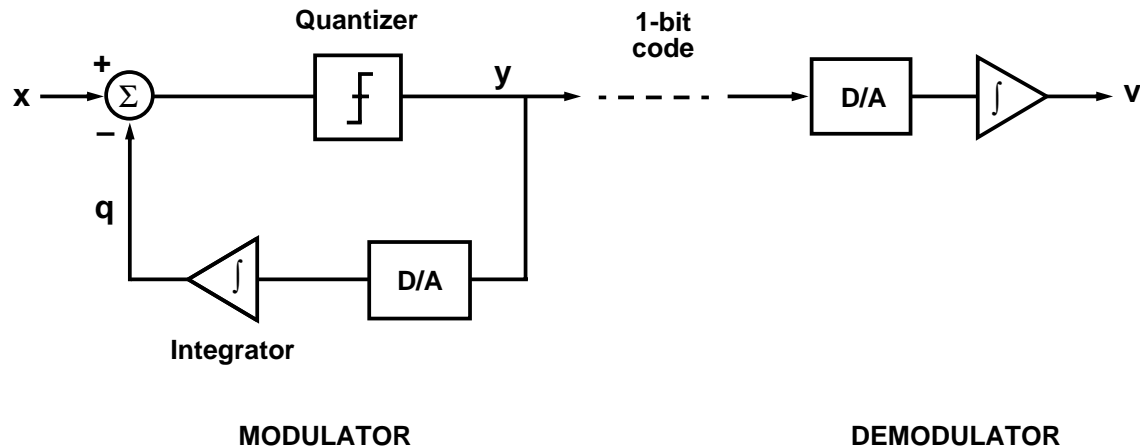
2× increase in M ⇒ 3dB reduction in S_B
⇒ 1/2 bit increase in resolution

A much greater improvement in resolution with increasing M can be obtained by embedding the quantizer in a feedback loop.

FEEDBACK can be use for PREDICTION (Δ modulation) or NOISE SHAPING ($\Sigma\Delta$ modulation)

In general, noise shaping modulators are more robust and easier to implement than predictive modulators

Delta Modulation



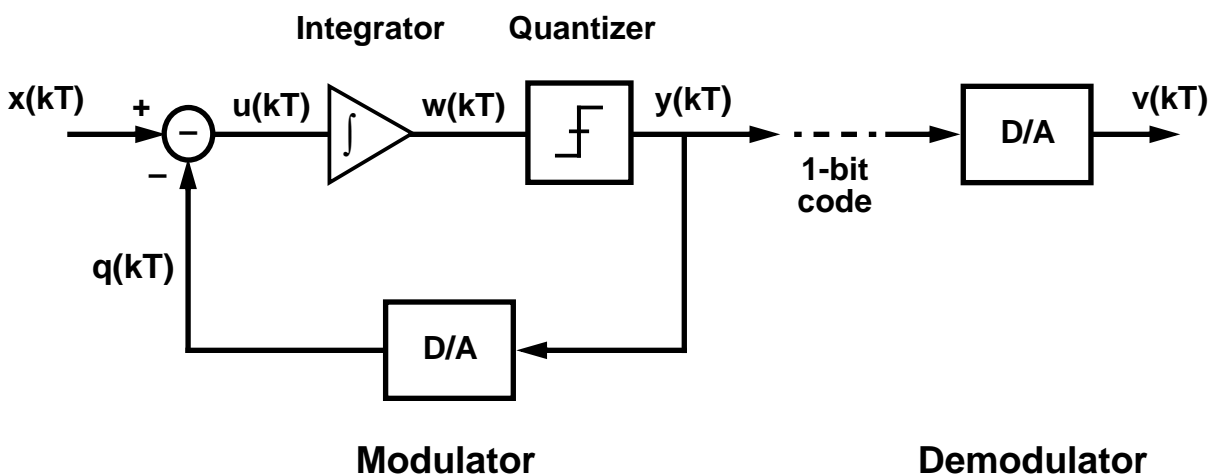
- Quantizes the difference between the input x , and the quantization signal, q
- q is generated by accumulating the quantized differences
- Typically a 1-bit quantizer with a small step size; step size can be adapted to accommodate “slope overload” (signal changing too fast).
- Fundamental practical problem is the accumulation of D/A mismatch error in the demodulator

Noise Shaping Modulators

- Sample and coarsely quantize the input at a rate well above the Nyquist rate
- Shape the spectrum of the quantization noise so as to push most of its energy outside the signal baseband
- Out-of-band noise, including quantization noise, is suppressed by a subsequent digital lowpass filter (DECIMATION FILTER)
- Output of the digital filter can be resampled at a lower sampling rate if the filter provides adequate antialiasing, as well as noise suppression

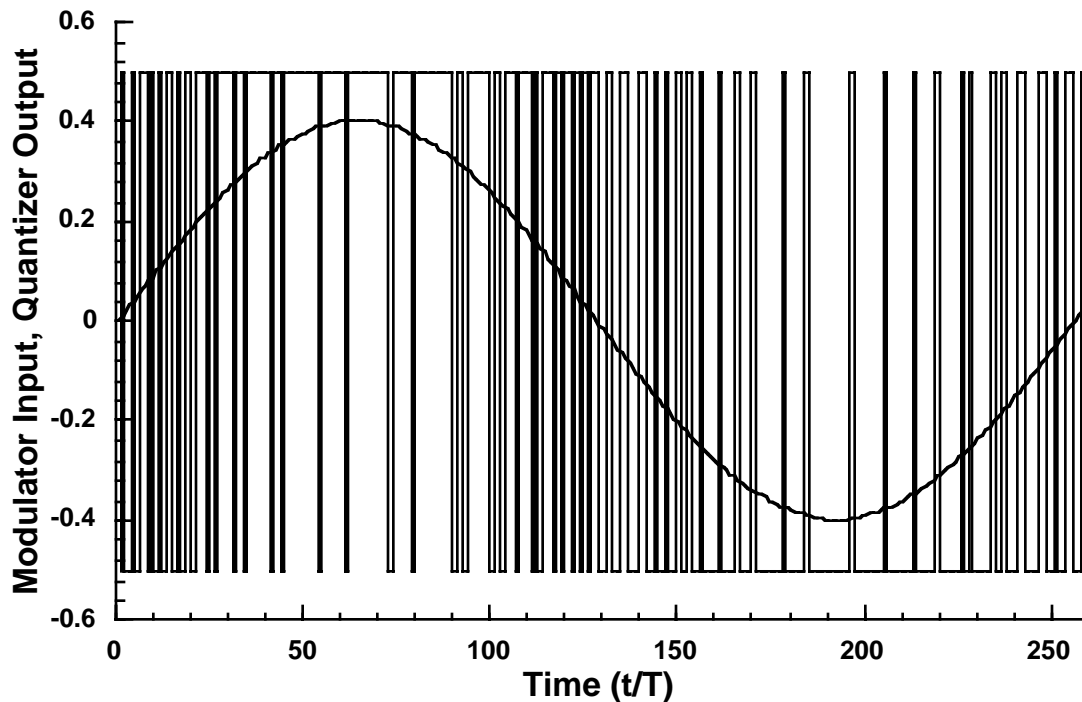
$\Sigma\Delta$ Modulation

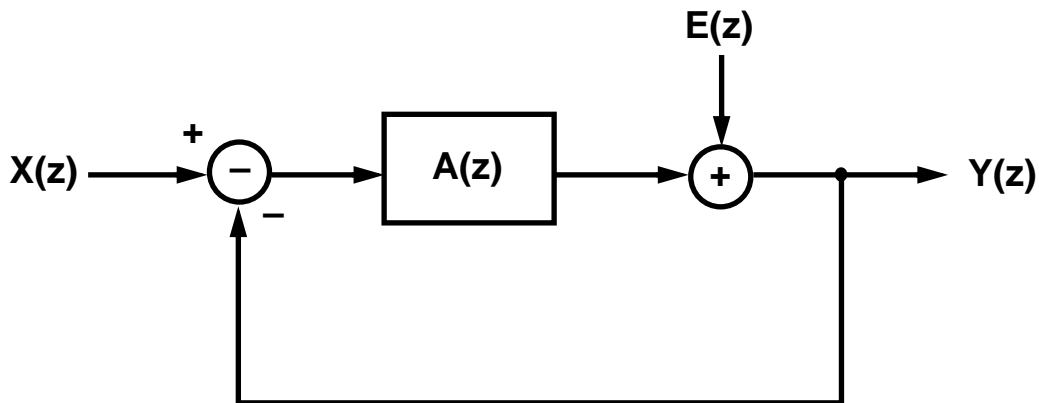
Simplest noise-shaping modulator is a first-order $\Sigma\Delta$ (or $\Delta\Sigma$) modulator with 1-bit quantization



- Integrator accumulates the difference between the input, $x(kT)$, and the quantization signal, $q(kT)$
- Feedback keeps the integrator output, $w(kT)$, near zero, thus minimizing the low-frequency difference between x and q
- For 1-bit quantization:
 - No D/A nonlinearity
 - Quantizer just a comparator
 - 2-level D/A converter can be an analog switch network toggling between + and – full scale

$\Sigma\Delta$ Modulator Response



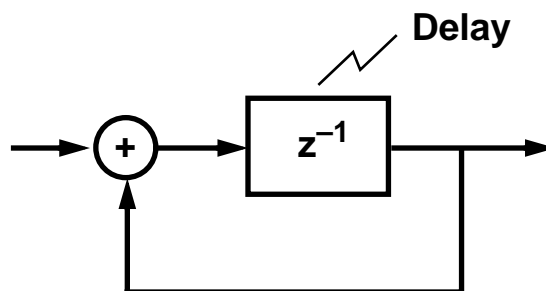
Linearized Discrete-Time Model:

where
$$A(z) = \frac{z^{-1}}{1 - z^{-1}}$$

In this representation of a “first-order” $\Sigma\Delta$ modulator, the quantization error is modeled as an additive error sequence, $e(kT)$, with the z-transform $E(z)$.

It is not strictly valid to assume the error sequence is random and uncorrelated with the input, especially when a 2-level quantizer is used. For a 2-level quantizer, the quantization error is highly correlated with the modulator input. Nonetheless, the model does illustrate the shaping of the quantization noise spectrum. It does not account for the appearance of strong discrete noise tones in that spectrum.

$A(z)$ as specified above is simply a delaying discrete-time integrator that can be implemented as:



From the above model it follows that

$$Y(z) = A(z)[X(z) - Y(z)] + E(z)$$

$$\therefore Y(z) = \left[\frac{A(z)}{1 + A(z)} \right] X(z) + \left[\frac{1}{1 + A(z)} \right] E(z)$$

$$1 + A(z) = 1 + \frac{z^{-1}}{1 - z^{-1}} = \frac{1}{1 - z^{-1}}$$

$$\frac{A(z)}{1 + A(z)} = \left(\frac{z^{-1}}{1 - z^{-1}} \right) (1 - z^{-1}) = z^{-1}$$

$$\therefore Y(z) = z^{-1}X(z) + H_E(z)E(z)$$

where

$$H_E(z) = 1 - z^{-1} \quad (\text{first-order difference})$$

Thus, in the output $Y(z)$, the quantization error is filtered by the first-order difference $H_E(z)$

In the frequency domain

$$\begin{aligned} H_E(j\omega) &= (1 - e^{-j\omega T}) = 2e^{-j\omega T/2} \left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2} \right) \\ &= 2e^{-j\omega T/2} [j \sin(\omega T/2)] \\ &= (2e^{-j\omega T/2})(e^{-j\pi/2}) [\sin(\omega T/2)] \\ &= [2 \sin(\omega T/2)] e^{-j(\omega T - \pi)/2} \end{aligned}$$

where

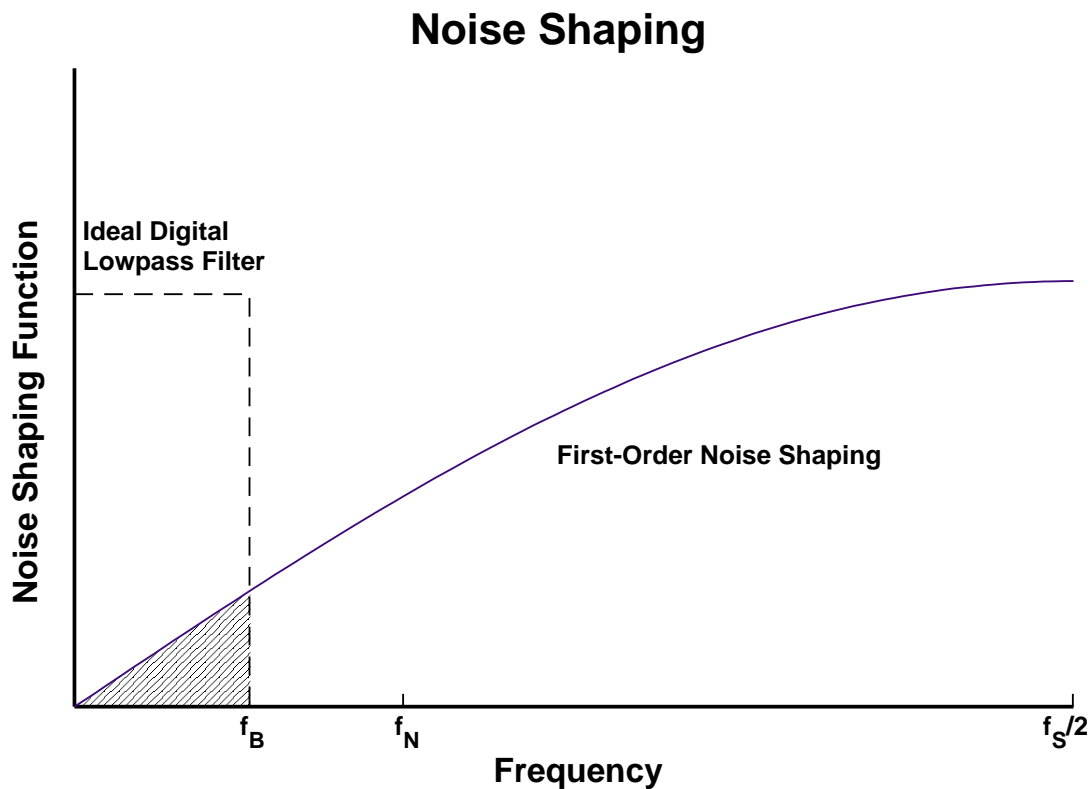
$$T = 1/f_s$$

Thus,

$$\begin{aligned} |H_E(f)| &= 2 \left| \sin\left(\frac{\omega T}{2}\right) \right| = 2 \left| \sin\left(\frac{2\pi f T}{2}\right) \right| \\ &= 2 |\sin(\pi f T)| = 2 |\sin(\pi f / f_S)| \end{aligned}$$

If $N_e(f)$ is the power spectral density of the quantization error $e(kT)$, the spectral power density of the quantization noise in the modulator output is:

$$N_y(f) = |H_E(f)|^2 N_e(f)$$



If it is assumed that the spectrum of the quantization error is white, which is not actually the case, and if S_Q denotes the quantization error power,

$$S_Q \equiv \overline{e^2} = \frac{\Delta^2}{12}$$

then

$$N_e(f) = \frac{S_Q}{f_S} = \left(\frac{\Delta^2}{12}\right) \frac{1}{f_S}$$

and the baseband quantization noise power in the modulator output is

$$\begin{aligned} S_B &= \int_{-f_B}^{f_B} N_Y(f) df = \int_{-f_B}^{f_B} |H_E(f)|^2 N_e(f) df \\ &= \frac{S_Q}{f_S} \int_{-f_B}^{f_B} [2\sin(\pi f T)]^2 df \end{aligned}$$

If $f \ll f_S = 1/T$, then

$$\sin(\pi f T) \cong \pi f T = \pi(f/f_S)$$

and

$$\begin{aligned} S_B &\cong 4 \left(\frac{S_Q}{f_S}\right) \int_{-f_B}^{f_B} \left[\pi \left(\frac{f}{f_S}\right)\right]^2 df = 4\pi^2 \left(\frac{S_Q}{f_S^3}\right) \left[\frac{f^3}{3}\right]_{-f_B}^{f_B} \\ &= 4\pi^2 \left(\frac{S_Q}{f_S^3}\right) \left(\frac{2f_B^3}{3}\right) = \frac{\pi^2}{3} \left(\frac{2f_B}{f_S}\right)^3 S_Q \end{aligned}$$

$$\therefore \mathbf{S_B} \cong \frac{\pi^2}{3} \left(\frac{1}{\mathbf{M}}\right)^3 \mathbf{S_Q} = \frac{\pi^2}{3} \left(\frac{1}{\mathbf{M}}\right)^3 \left(\frac{\Delta^2}{12}\right)$$

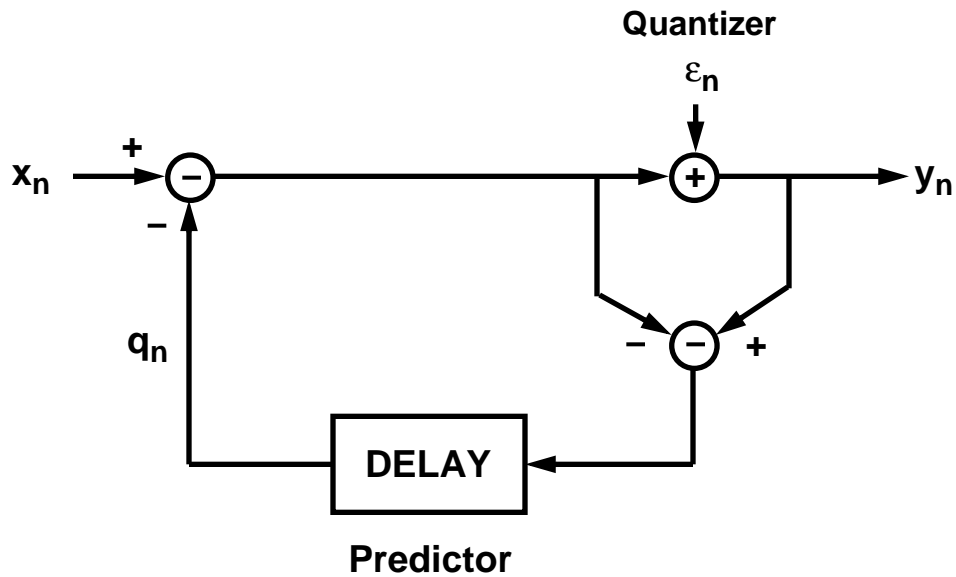
Since a full-scale sinusoid has an amplitude of at most $\Delta/2$, the maximum dynamic range of the modulator is

$$\begin{aligned} \text{DR} &= 10\log \left[\frac{(\Delta/2)^2/2}{\mathbf{S_B}} \right] \\ &= 10\log \left[\frac{\Delta^2/8}{\frac{\pi^2}{3} \left(\frac{1}{\mathbf{M}}\right)^3 \left(\frac{\Delta^2}{12}\right)} \right] = 10\log \left[\frac{9}{2\pi^2} \mathbf{M}^3 \right] \\ &= 10\log \left[\frac{9}{2\pi^2} \right] + 30\log(\mathbf{M}) \end{aligned}$$

Each $2\times$ increase in \mathbf{M} results in a 9dB in dynamic range, which corresponds to 1.5 bits of resolution

Because of the spectral tones that result from the correlation of the quantization error with the input, the dynamic range of a first-order $\Sigma\Delta$ modulator with 1-bit quantization is not as large as this result indicates

Error Compensation Model of a $\Sigma\Delta$ Modulator



Configuration of original patent on $\Sigma\Delta$ modulation (Cutler, U.S. Patent 2,927,962, 3/8/60); referred to as a transmitting terminal using error compensation

$$y_n = x_n - q_n + \epsilon_n$$

$$q_n = \epsilon_{n-1}$$

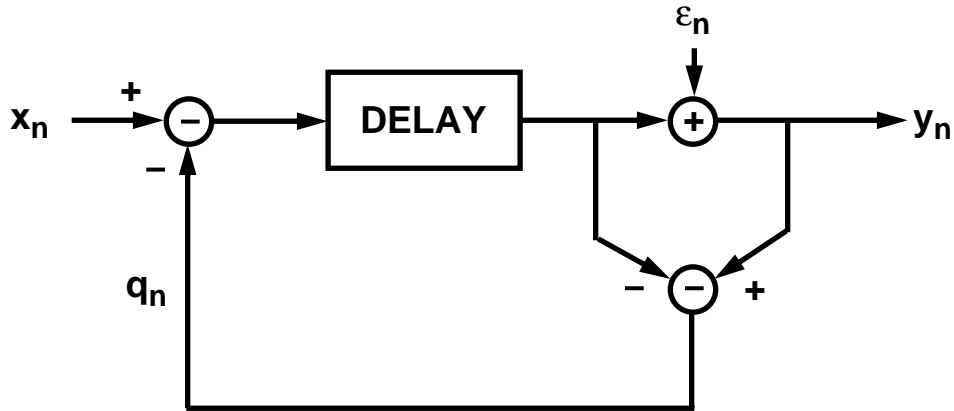
$$\therefore y_n = x_n + \epsilon_n - \epsilon_{n-1}$$

$$Y(z) = X(z) + (1 - z^{-1})E(z)$$

Thus, this topology provides first-order noise shaping. However, it is difficult to implement.

The error compensation topology can be rearranged as follows to obtain the conventional first-order $\Sigma\Delta$ modulator.

First, the delay is moved in to the forward path:

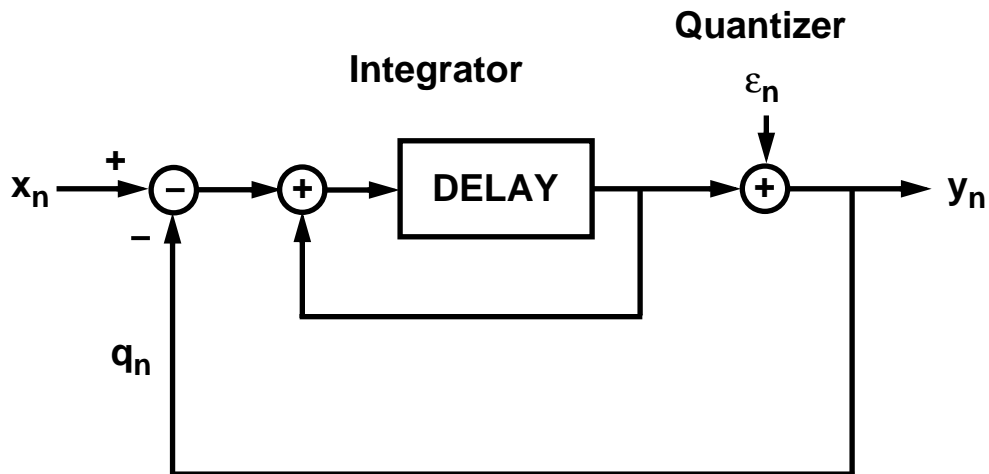


This configuration implements the noise-differencing relationship with a delay of the input:

$$y_n = x_{n-1} + \epsilon_n - \epsilon_{n-1}$$

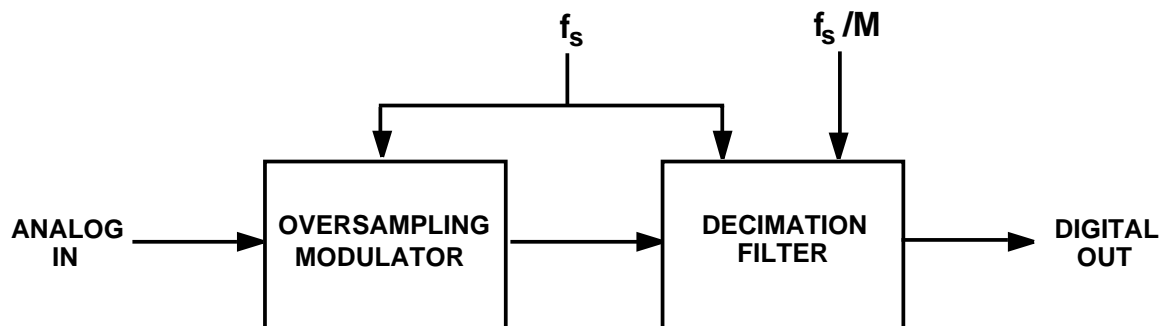
$$Y(z) = z^{-1}X(z) + (1 - z^{-1})E(z)$$

Next, simply rearrange the paths at the input and output of the quantizer to obtain the conventional first-order $\Sigma\Delta$ modulator



Oversampling A/D Conversion

Combine an oversampling noise-shaping modulator with a low-pass digital filter that removes the out-of-band quantization noise. The digital filter also provides the antialiasing need to allow resampling at a lower sampling rate (“decimation”).



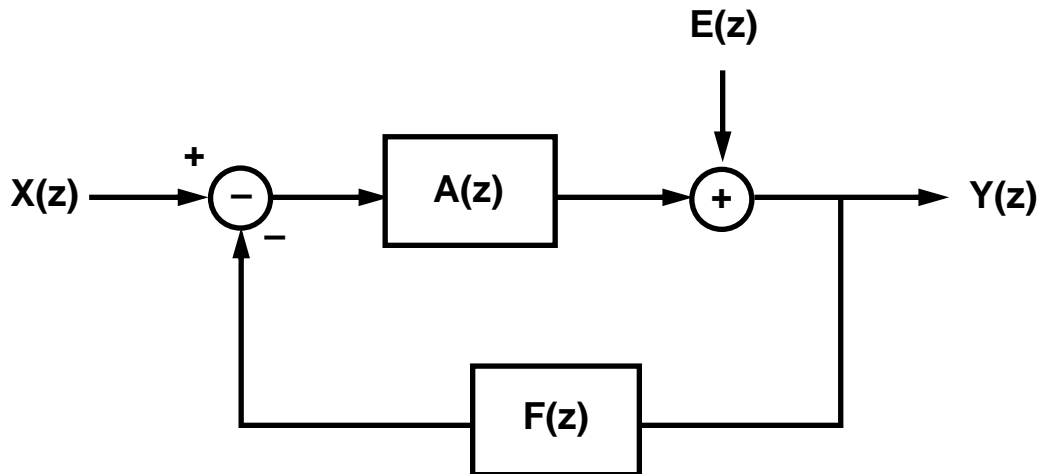
f_s = Sampling Rate

M = Oversampling Ratio

Higher-Order $\Sigma\Delta$ Modulators

- Single quantizer
 - Multi-loop noise differencing
 - Single-loop with multi-order filter
- Cascaded (multistage)

Single-Quantizer Modulator



$$Y(z) = H_X(z)X(z) + H_E(z)E(z)$$

$$H_X(z) = \frac{A(z)}{1 + A(z)F(z)}$$

$$H_E(z) = \frac{1}{1 + A(z)F(z)}$$

$$A(z) = \frac{H_X(z)}{H_E(z)}$$

$$F(z) = \frac{1 - H_E(z)}{H_X(z)}$$

Noise Differencing Modulators

Class of modulators for which

$$Y(z) = z^{-n}X(z) + (1 - z^{-1})^L E(z)$$

That is

$$H_X(z) = z^{-n} \quad \text{and} \quad H_E(z) = (1 - z^{-1})^L$$

Noise differencing modulators can be implemented with a single quantizer and L nested loops. However, limit cycle instability occurs for $L > 2$.

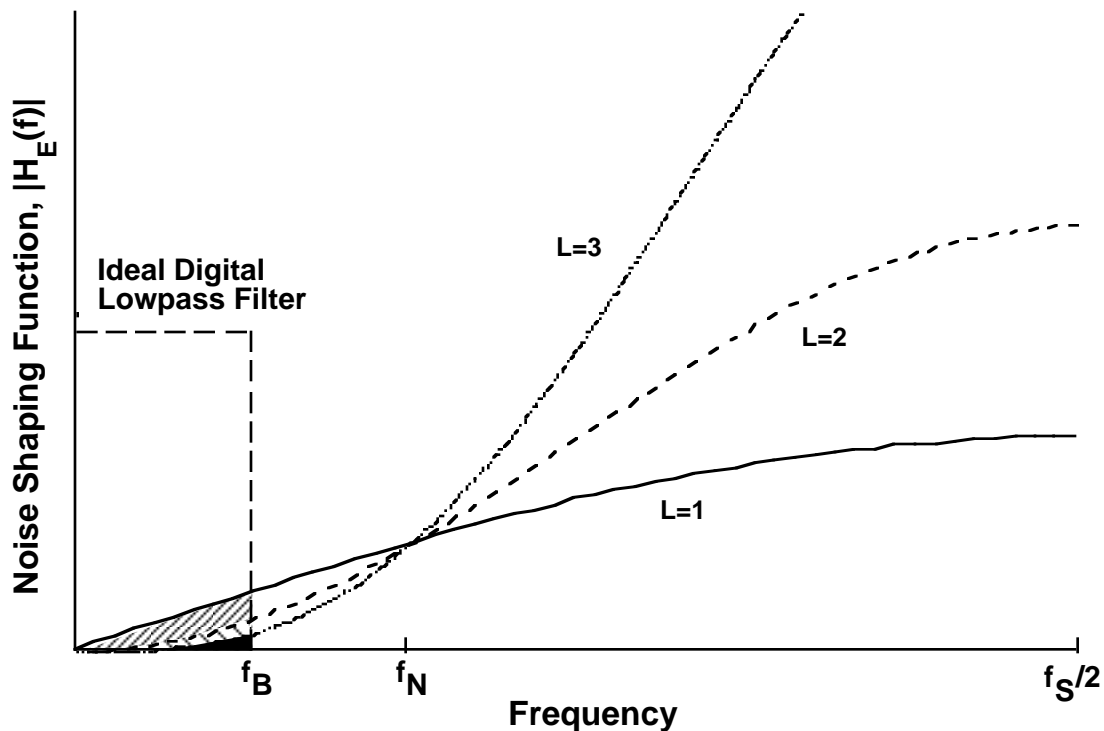
For an L^{th} order noise differencing modulator

$$H_E(z) = (1 - z^{-1})^L$$

$$|H_E(f)| = |2 \sin(\pi f / f_S)|^L$$

The quantization noise is thus shaped as

$$N_y(f) = |H_E(f)|^2 N_e(f) = [2 \sin(\pi f / f_S)]^{2L} N_e(f)$$



Baseband Quantization Noise

As in the case of the first-order modulator, if it is assumed that the quantization noise is white with a uniform spectral density S_Q/f_S , then the quantization noise remaining in the baseband of the output is

$$\begin{aligned} S_B &= \int_{-f_B}^{f_B} N_Y(f) df = \int_{-f_B}^{f_B} |H_E(f)|^2 N_e(f) df \\ &= \frac{S_Q}{f_S} \int_{-f_B}^{f_B} [2 \sin(\pi f T)]^{2L} df \end{aligned}$$

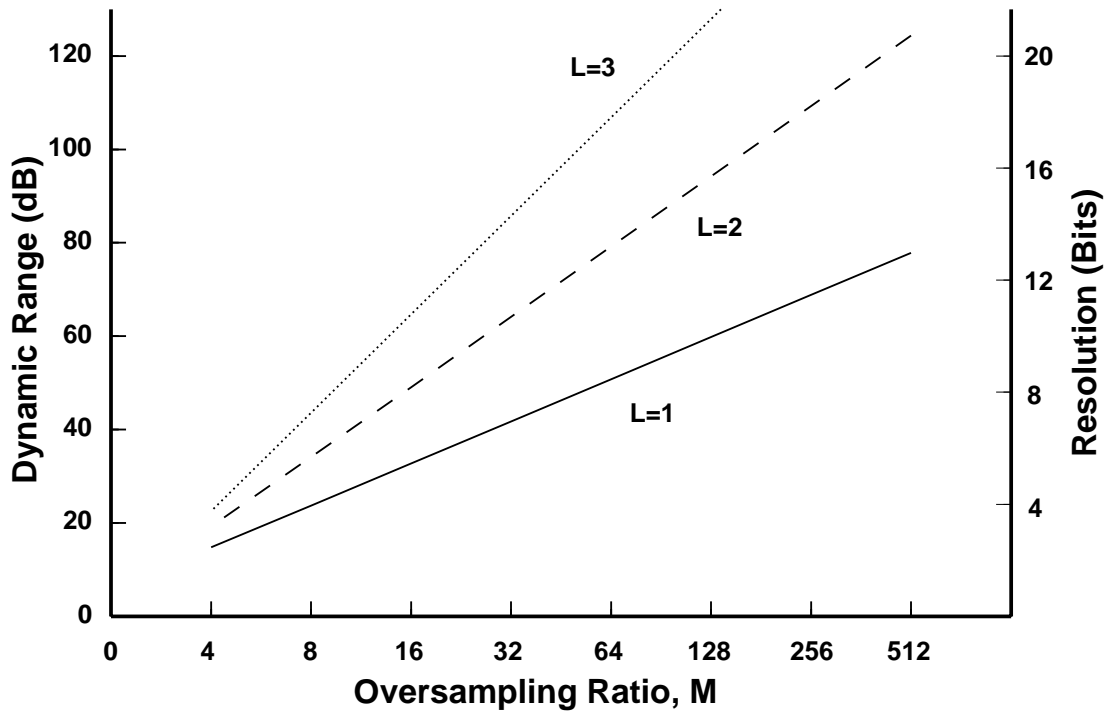
If $f \ll f_S = 1/T$, then

$$\sin(\pi f T) \cong \pi f T = \pi(f/f_S)$$

and

$$\begin{aligned} S_B &= 2^{2L} \left(\frac{S_Q}{f_S} \right) \int_{-f_B}^{f_B} \left[\pi \left(\frac{f}{f_S} \right) \right]^{2L} df = 2^{2L} \pi^{2L} \left(\frac{S_Q}{f_S^{2L+1}} \right) \left[\frac{f^{2L+1}}{2L+1} \right] \Bigg|_{-f_B}^{f_B} \\ &= 2^{2L} \pi^{2L} \left(\frac{S_Q}{f_S^{2L+1}} \right) \left(\frac{2f_B^{2L+1}}{2L+1} \right) = \left(\frac{\pi^{2L}}{2L+1} \right) \left(\frac{2f_B}{f_S} \right)^{2L+1} S_Q \\ &= \left(\frac{\pi^{2L}}{2L+1} \right) \left(\frac{1}{M} \right)^{2L+1} S_Q \end{aligned}$$

Thus, for an L^{th} order modulator, every doubling of M results in an increase in dynamic range of $6L+3$ dB ($L+0.5$ bits)



Noise differencing modulators can be implemented with a single quantizer and L nested loops. However, limit cycle instability occurs for $L > 2$. Thus, we consider the case where a single quantizer is used with $L = 1$ and $L = 2$:

For $H_X(z) = 1$

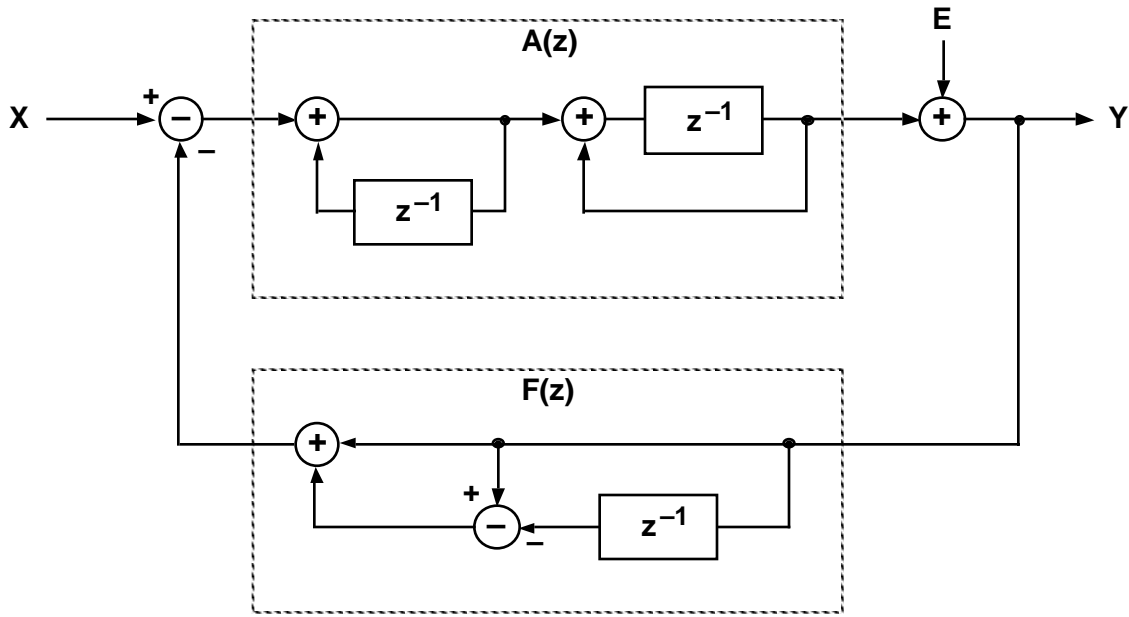
$$L = 1: \quad H_E(z) = (1 - z^{-1}) \Rightarrow A(z) = \frac{z^{-1}}{1 - z^{-1}}$$

$$F(z) = 1$$

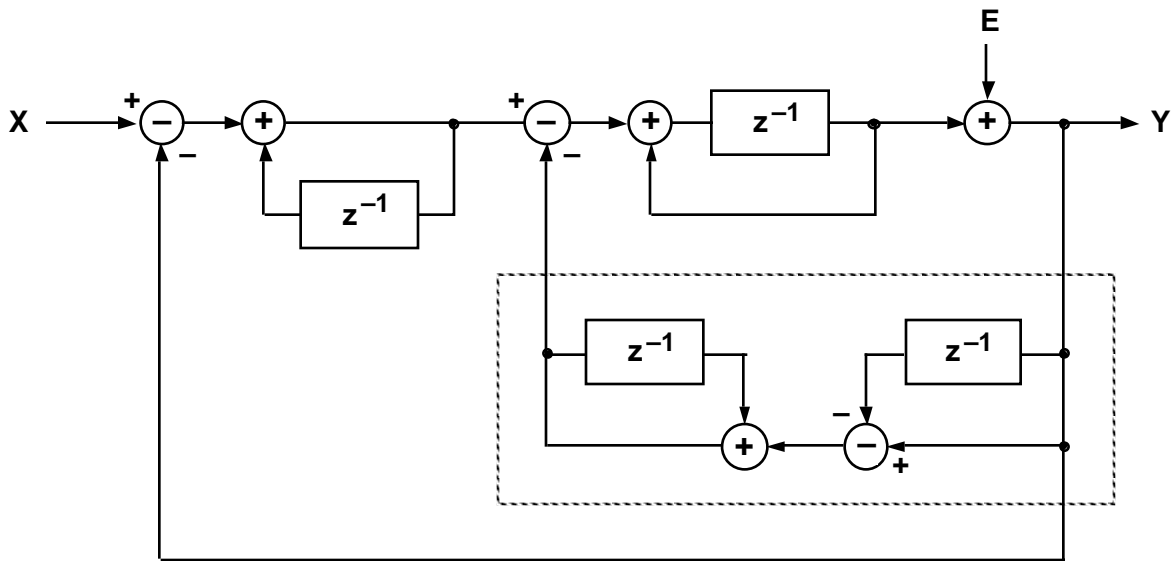
$$L = 2: \quad H_E(z) = (1 - z^{-1})^2 \Rightarrow A(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$$

$$F(z) = 2 - z^{-1} \neq 1$$

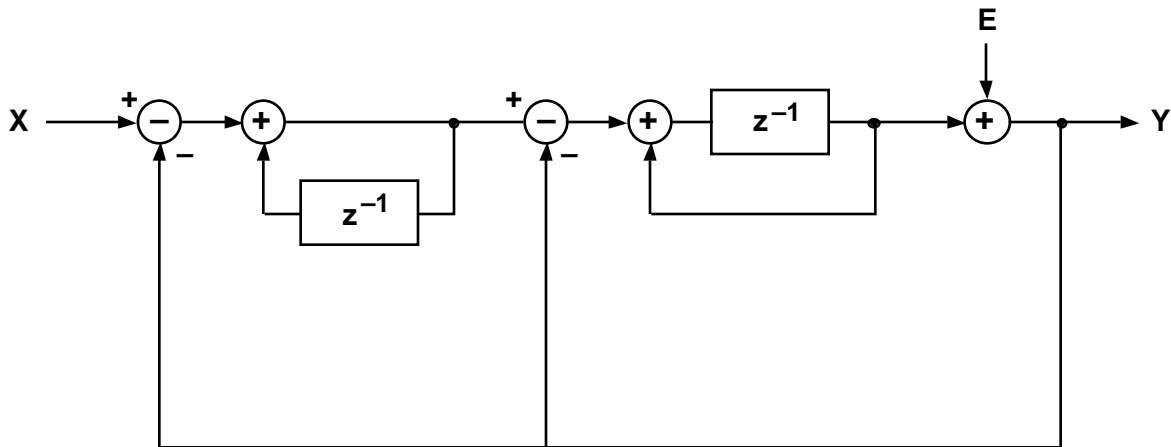
A canonical realization of a 2nd-order noise-differencing modulator is thus:



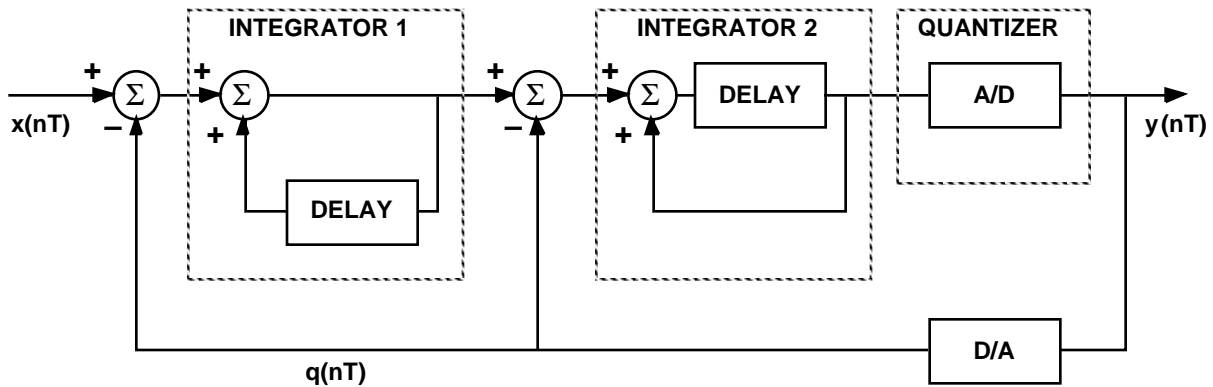
which can be rearranged as



The above topology reduces to

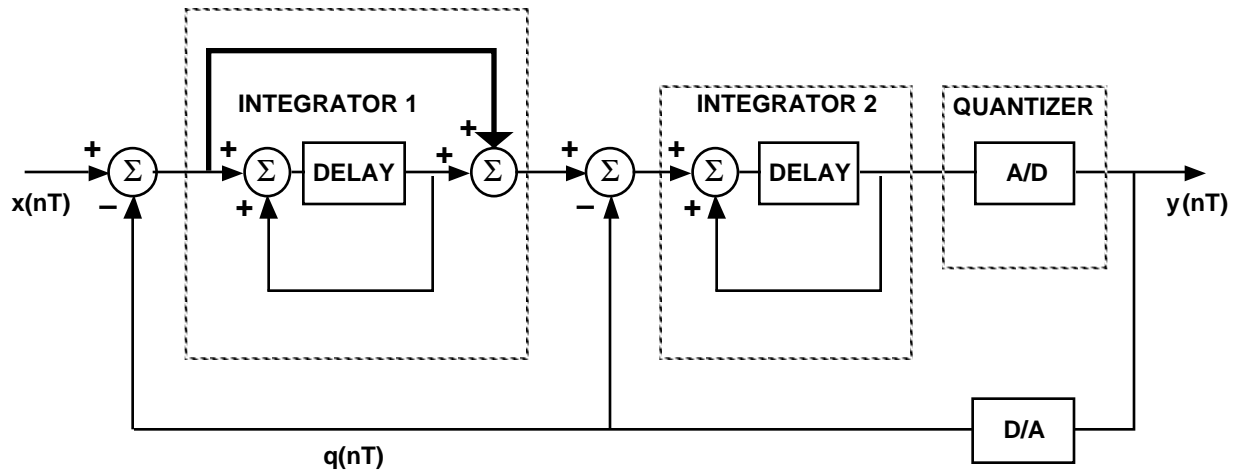


Thus, the classical topology for a 2nd-order $\Sigma\Delta$ modulator is

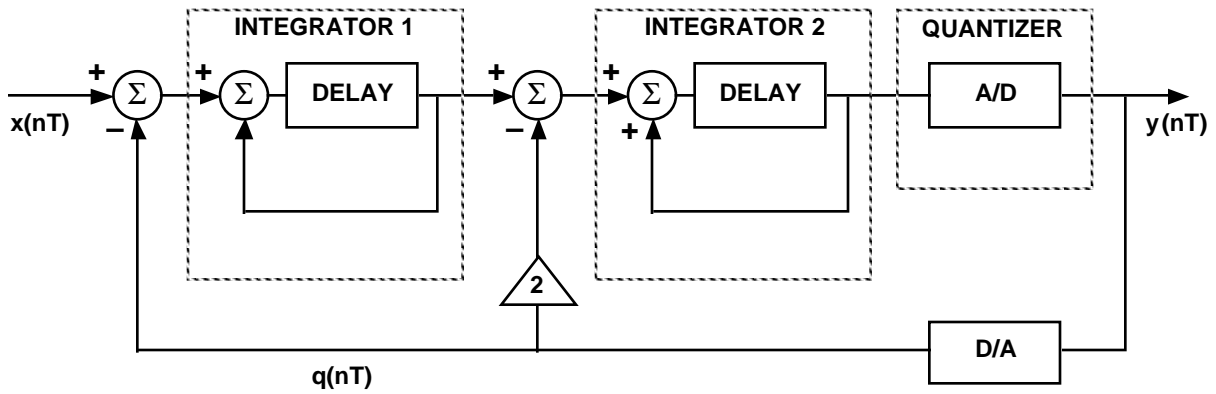


$$Y(z) = z^{-1} X(z) + (1 - z^{-1})^2 E(z)$$

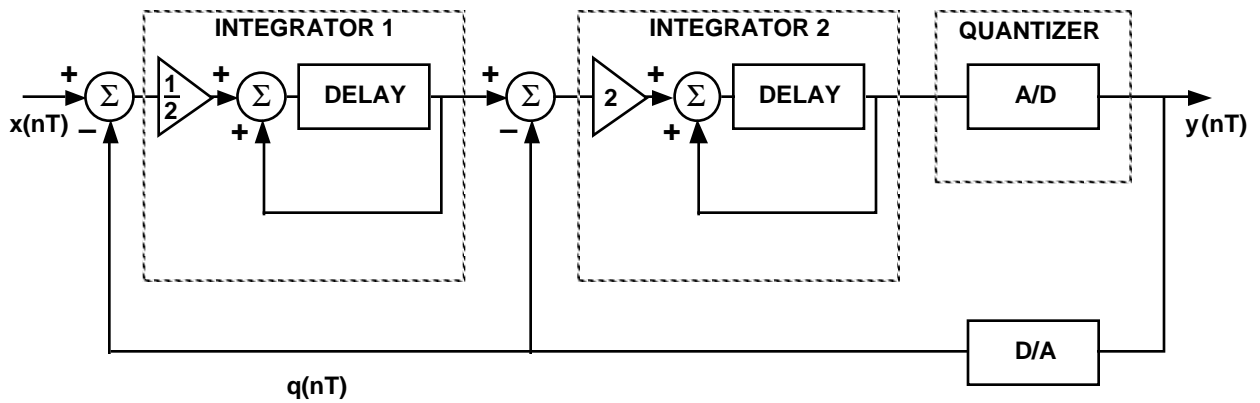
The first integrator in this configuration is nondelaying. Such an integrator can be realized using a delaying integrator identical to that used in the second stage with the following configuration.



which can be rearranged as:



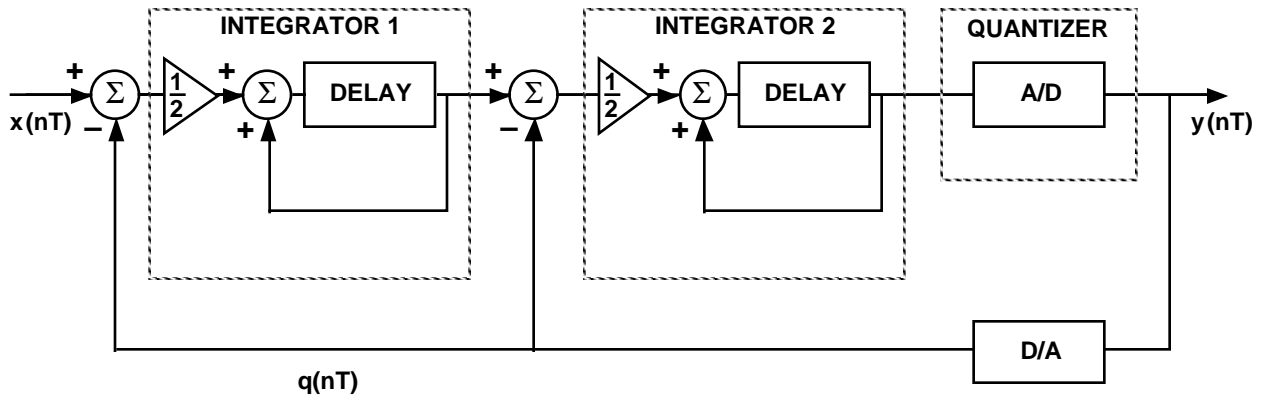
To eliminate the multiply by 2:



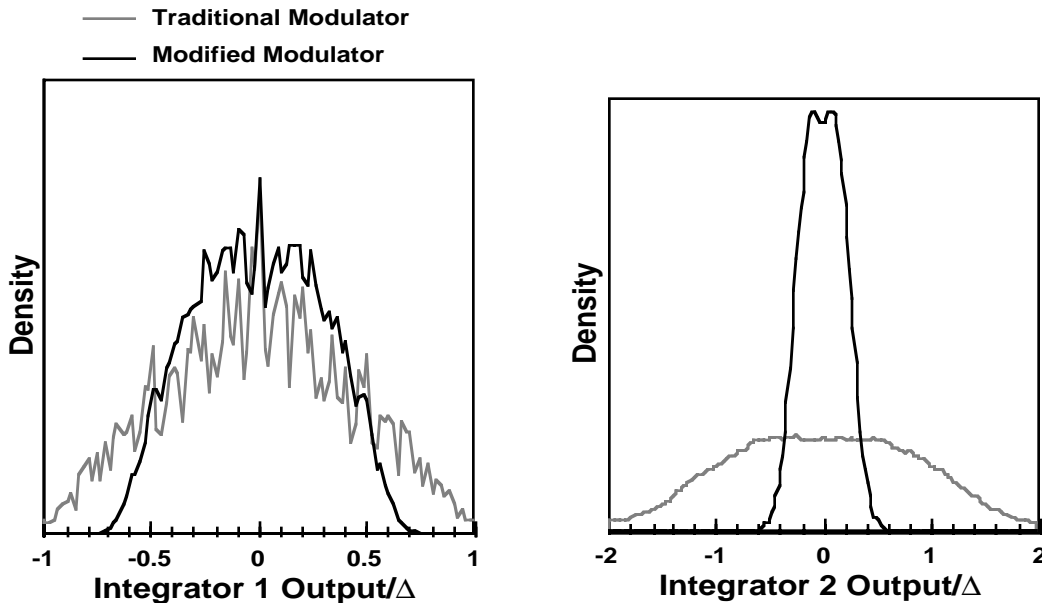
The gain of 2 preceding the second integrator stage results in the need for a large dynamic range at the output of this stage, in turn requiring a signal swing at the input that is well below the supply voltage.

However, if a 2-level quantizer is used, then the “gain” preceding the second stage can be adjusted arbitrarily. In that case, the second stage can be implemented using the same topology as the first stage.

Second-Order $\Sigma\Delta$ Modulator (w/ 2-level quantizer)

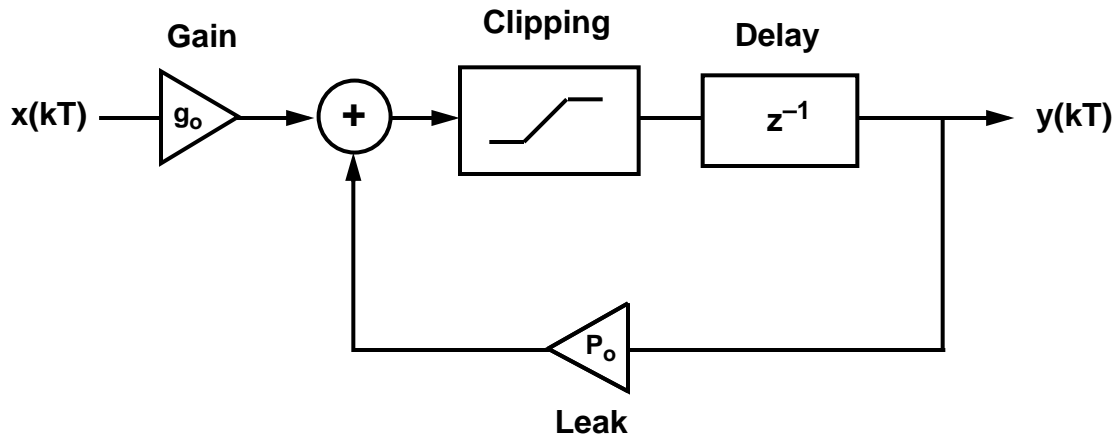


Integrator Dynamic Range

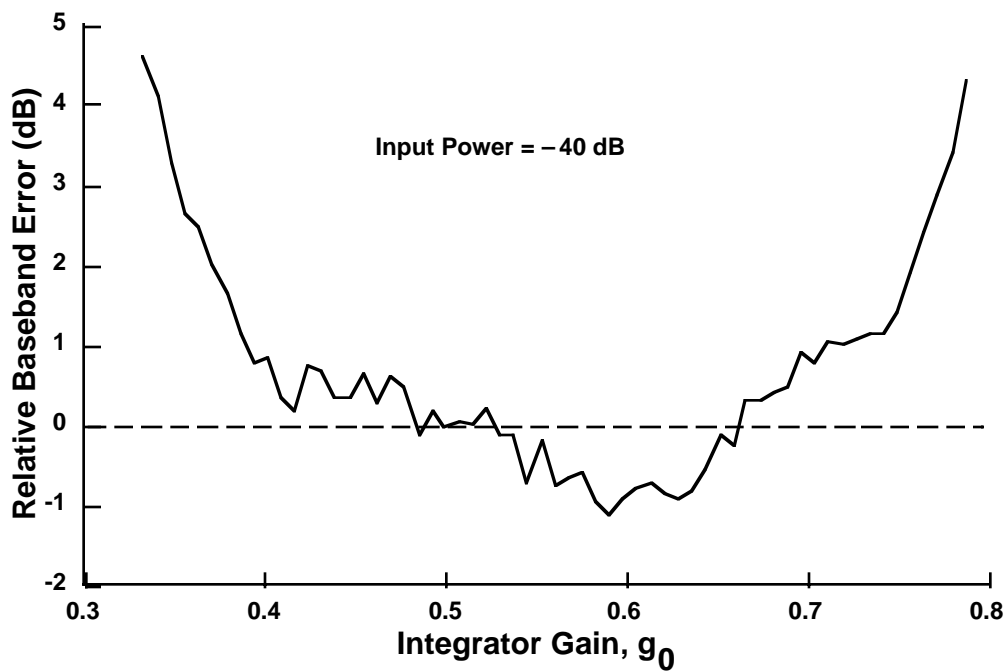


Integrator Gain

Integrator model:



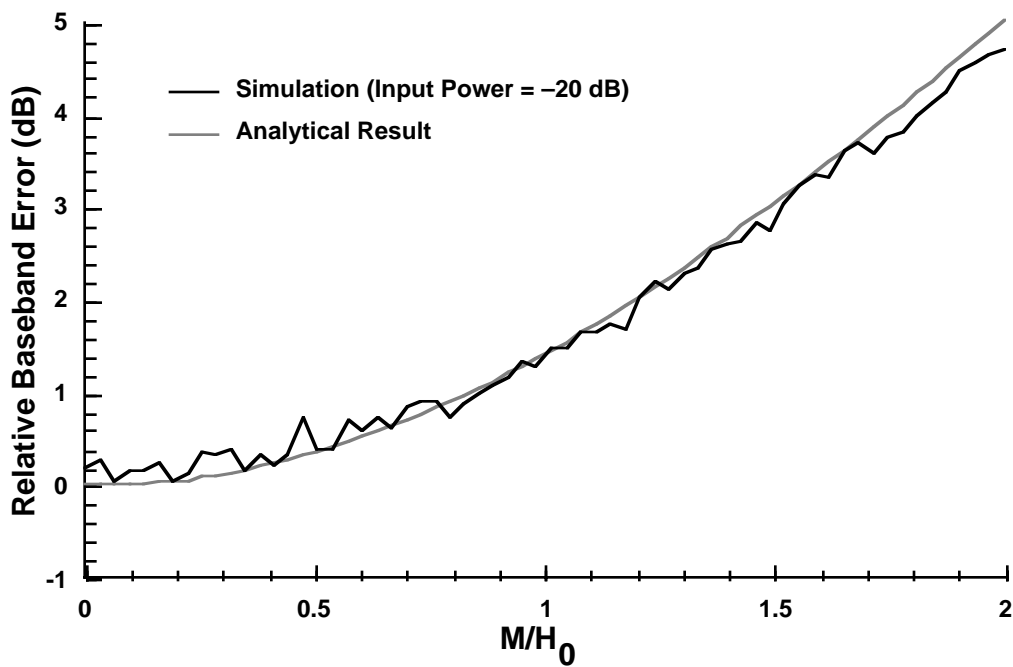
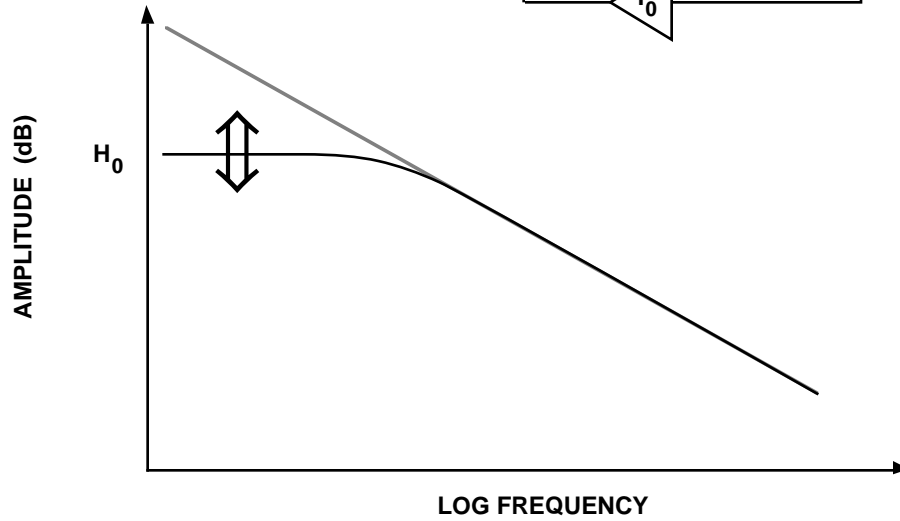
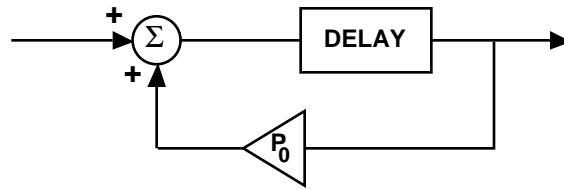
Sensitivity of baseband noise to integrator gain:



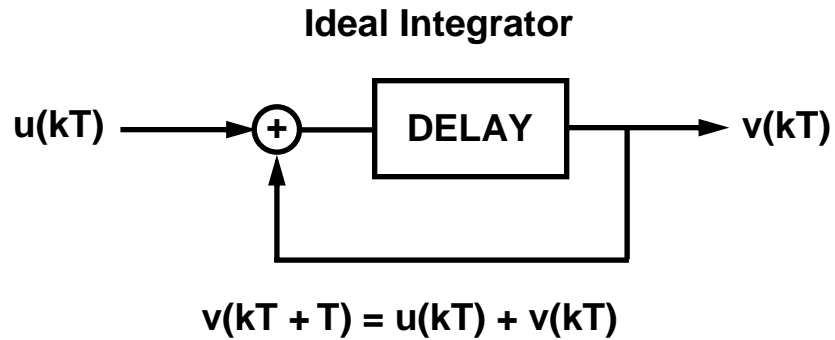
Integrator leak

Integrator “leak” refers to the finite dc gain of a practical integrator

$$H(z) = \frac{z^{-1}}{1 - P_0 z^{-1}}$$



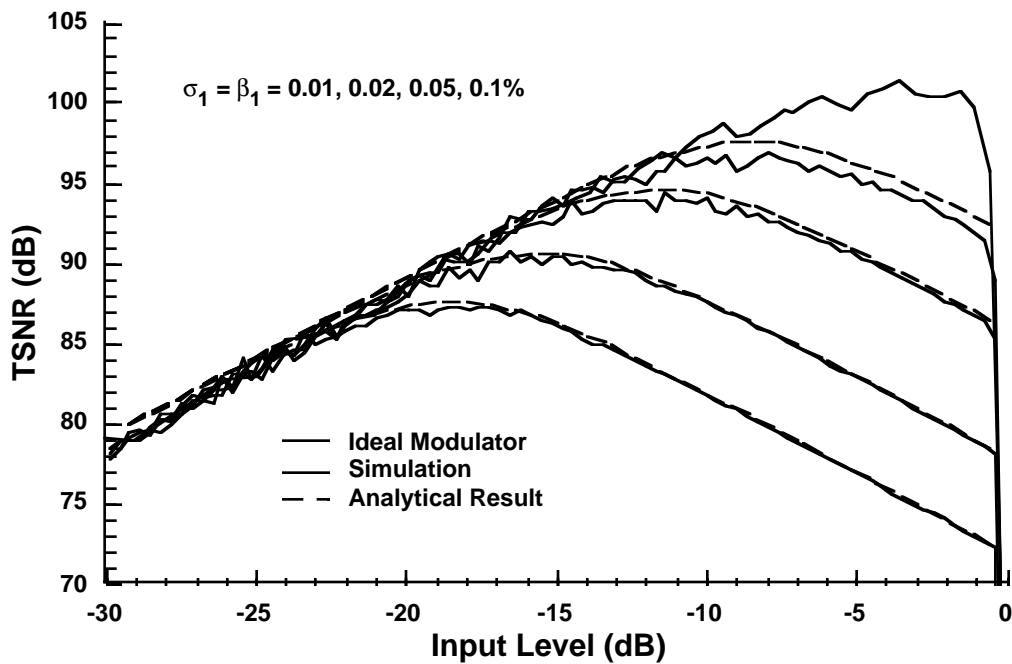
Integrator Linearity

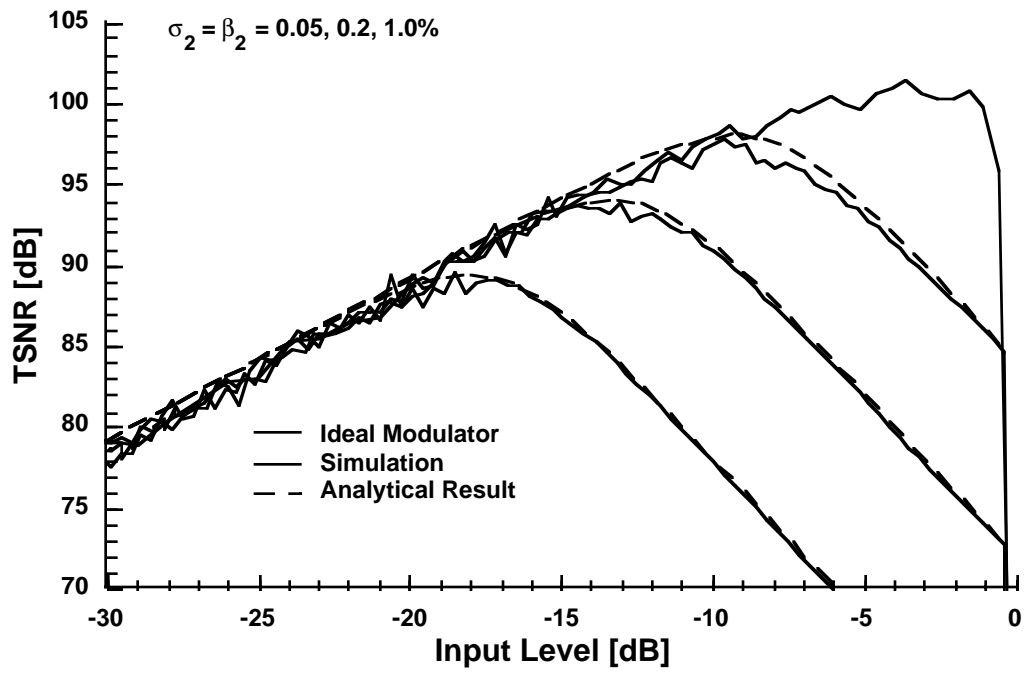


with nonlinearity

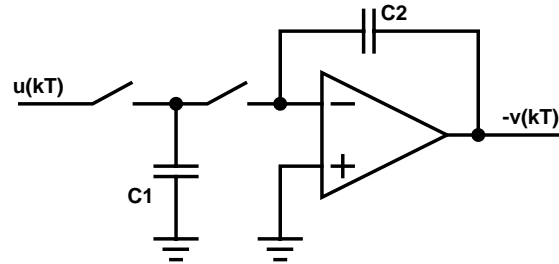
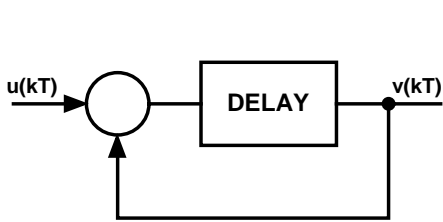
$$v(kT + T) = u(kT) + \alpha_1[u(kT)]^2 + \alpha_2[u(kT)]^3 + \dots$$

$$+ v(kT) + \beta_1[v(kT)]^2 + \beta_2[v(kT)]^3 + \dots$$

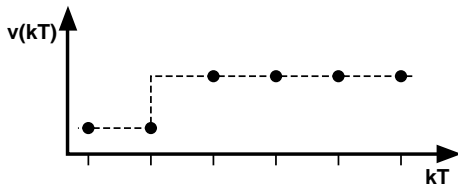




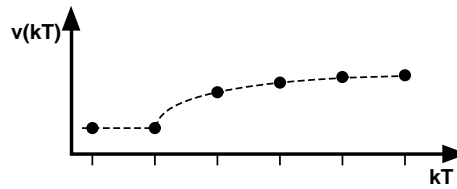
Integrator Slewing



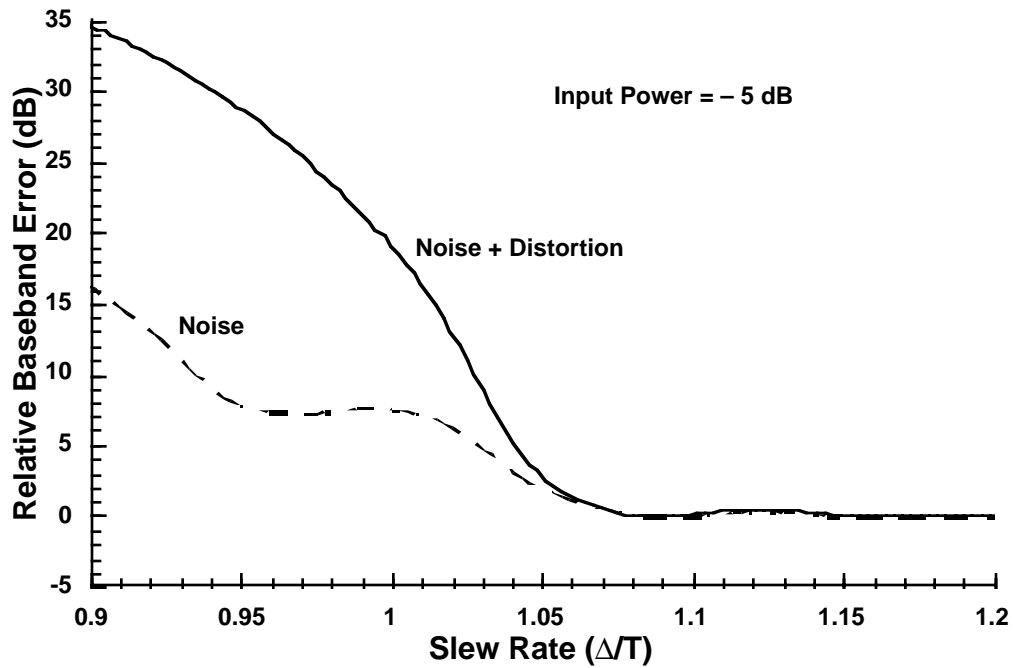
STEP RESPONSE



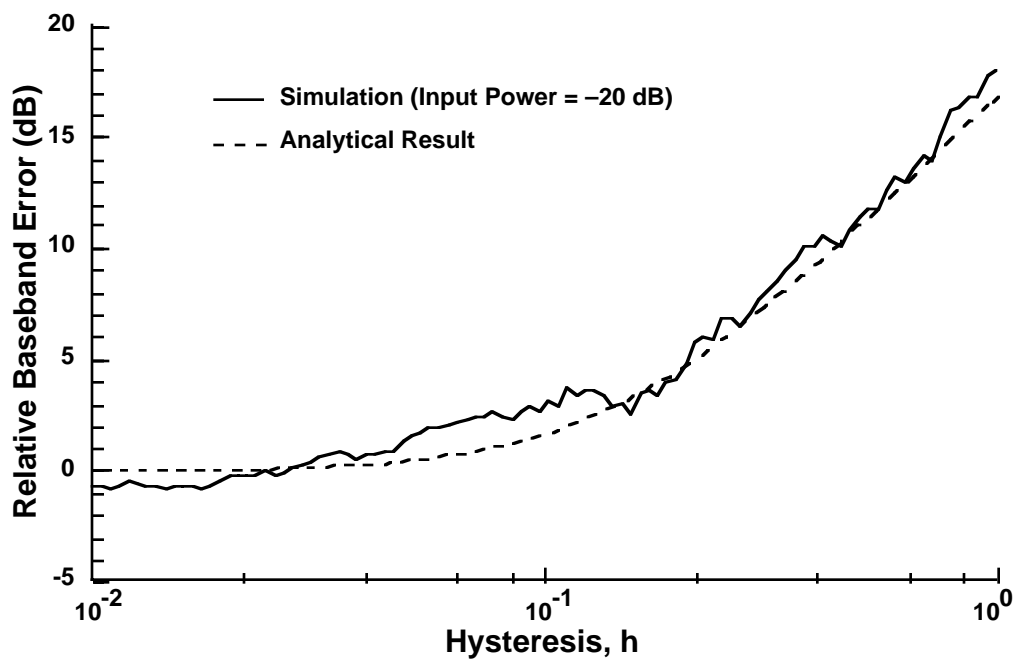
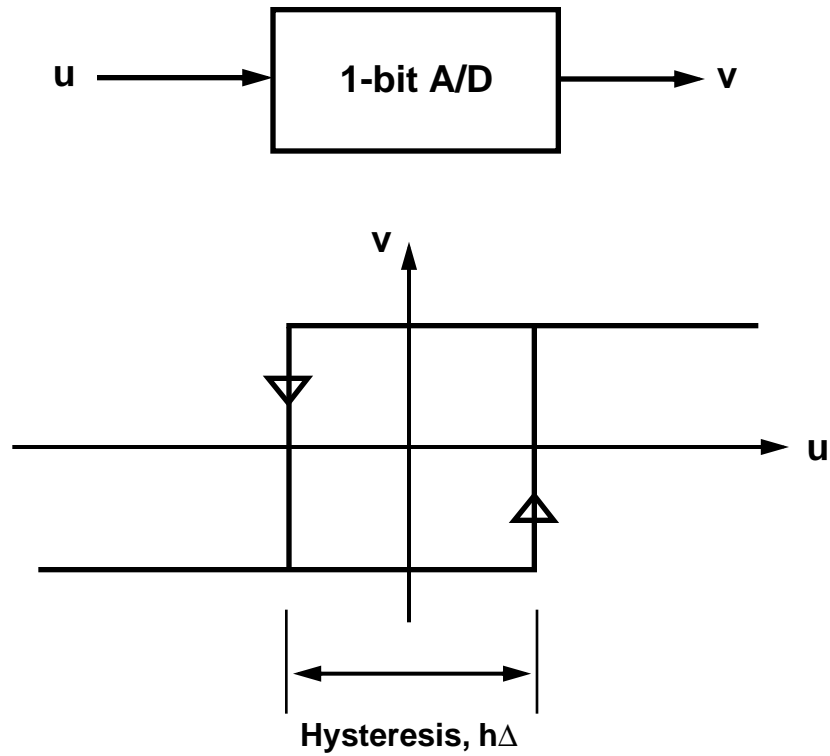
STEP RESPONSE



Integrator Slewing



Comparator Hysteresis



Analog Integration

Two basic circuit approaches to realizing analog integrators in a CMOS technology are **CONTINUOUS TIME** and **SAMPLED-DATA**

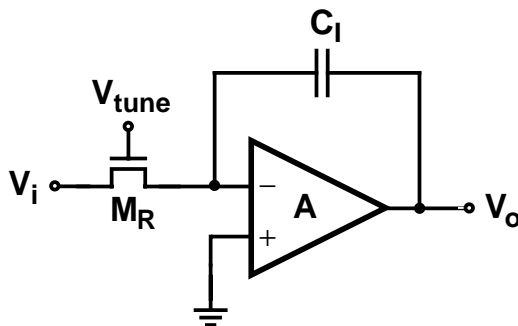
Continuous Time Integration

- g_m -C
- MOSFET-C

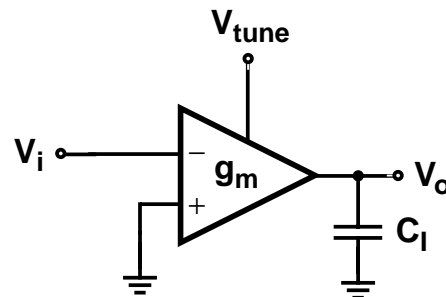
Sampled-Data Integration

- Switched-Current
- Switched-Capacitor

Continuous-Time Integrators



MOSFET - C

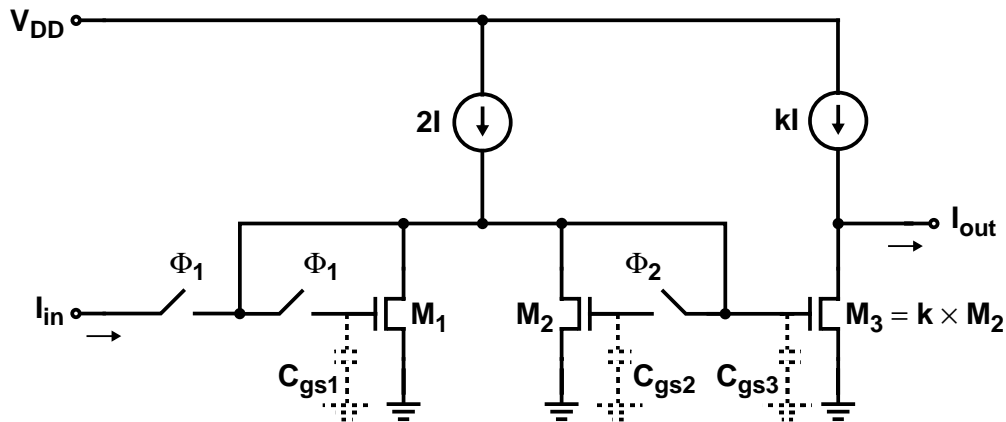


g_m - C

Limitations

- Integrator output is sensitive to timing jitter
- Sensitive to waveform asymmetry
 - e.g. response to ...011000... differs from response to ...010100...
- Waveform asymmetry can enhance spurious noise tones
- Frequency response of loop governed by capacitors and MOS transconductance or resistance
- Poor linearity

Switched-Current Integrator

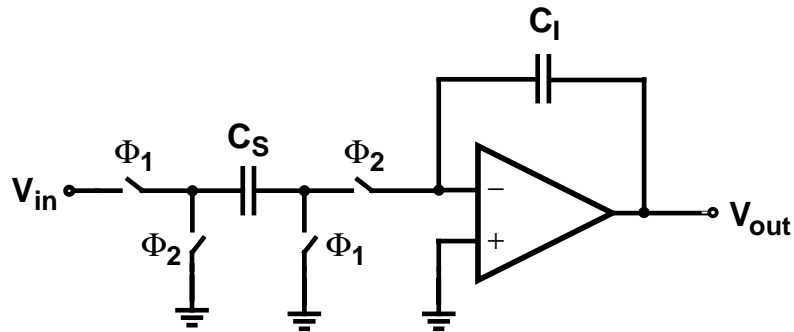


$$\frac{I_{out}}{I_{in}} = \frac{kz^{-1}}{1 - z^{-1}}$$

Limitations

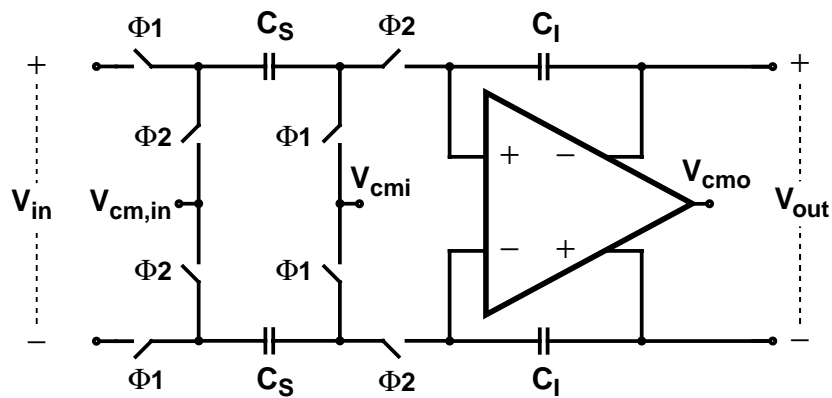
- Current sources must be cascode to reduce output conductance ==> high supply voltage
- Large $V_{gs} - V_T$ needed to reduce sensitivity to V_T mismatch ==> high power dissipation
- Sensitive to switch parasitics and charge injection

Switched-Capacitor Integrator



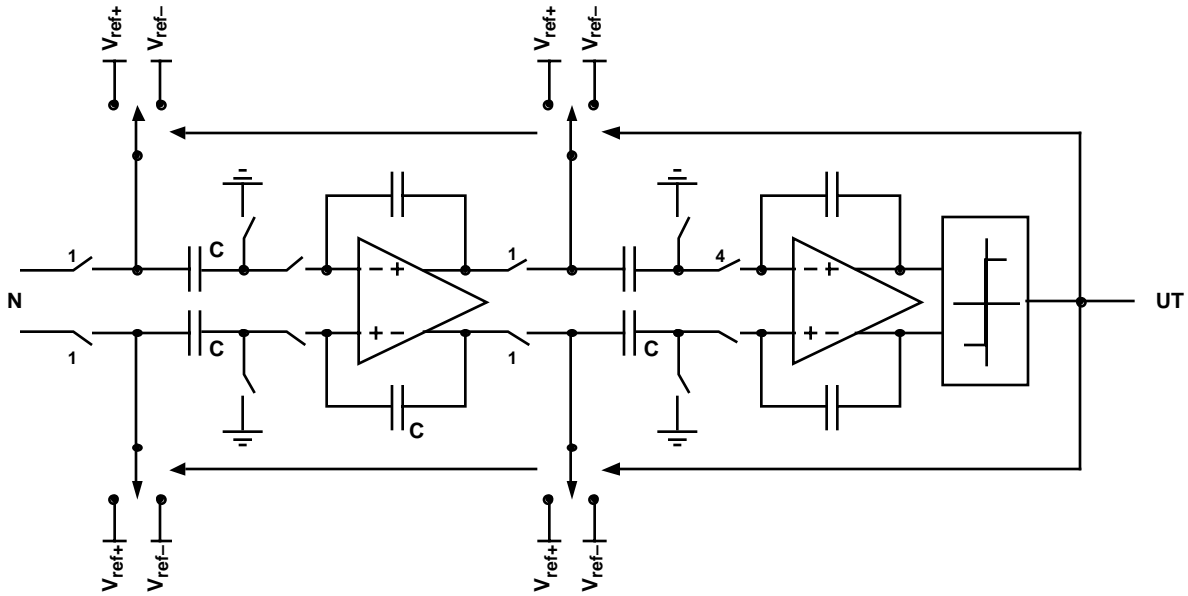
$$\frac{V_{out}}{V_{in}} = \frac{C_S}{C_I} \left(\frac{z^{-1}}{1 - z^{-1}} \right)$$

Fully-differential switched-C integrator

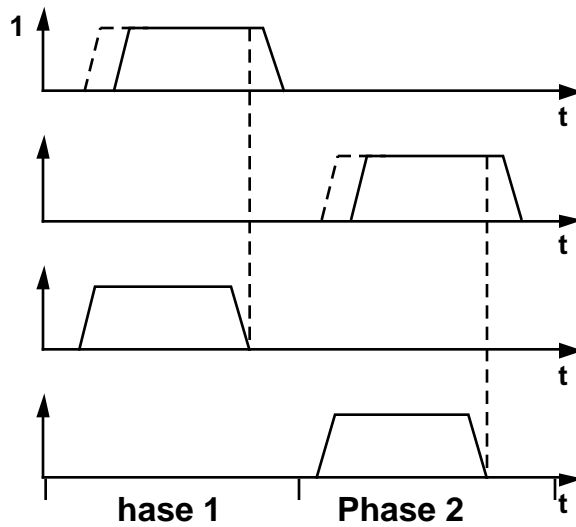


- Common-mode input and output levels can be set independently

CMOS Implementation of a 2nd-Order $\Sigma\Delta$ Modulator

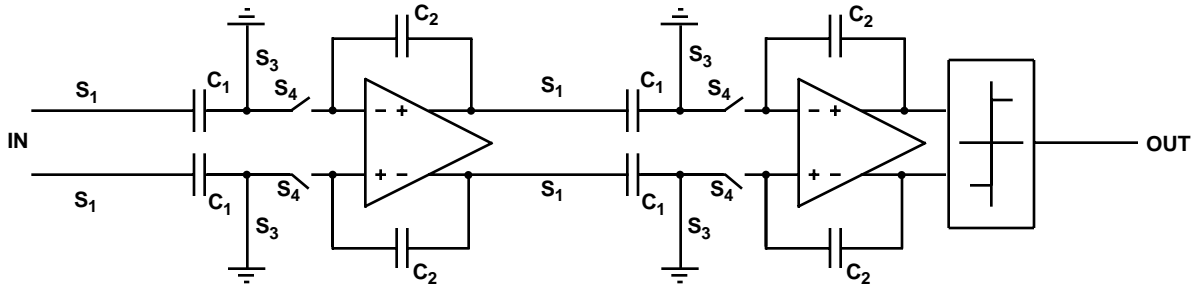


Timing Diagram

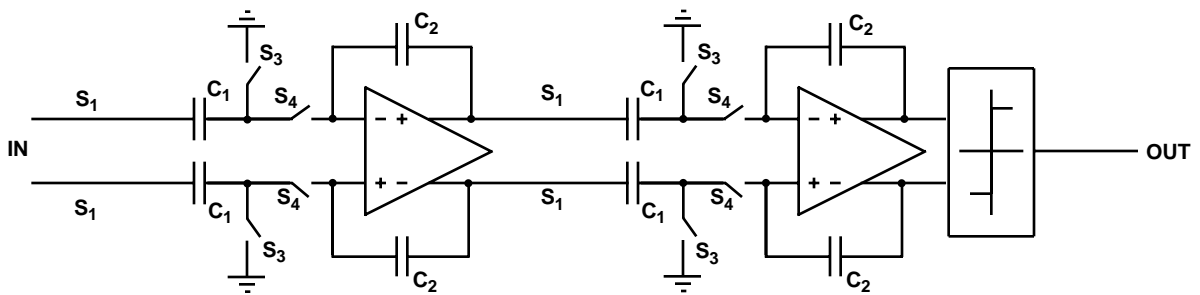


Phase 1

- Sample inputs
- Compare outputs

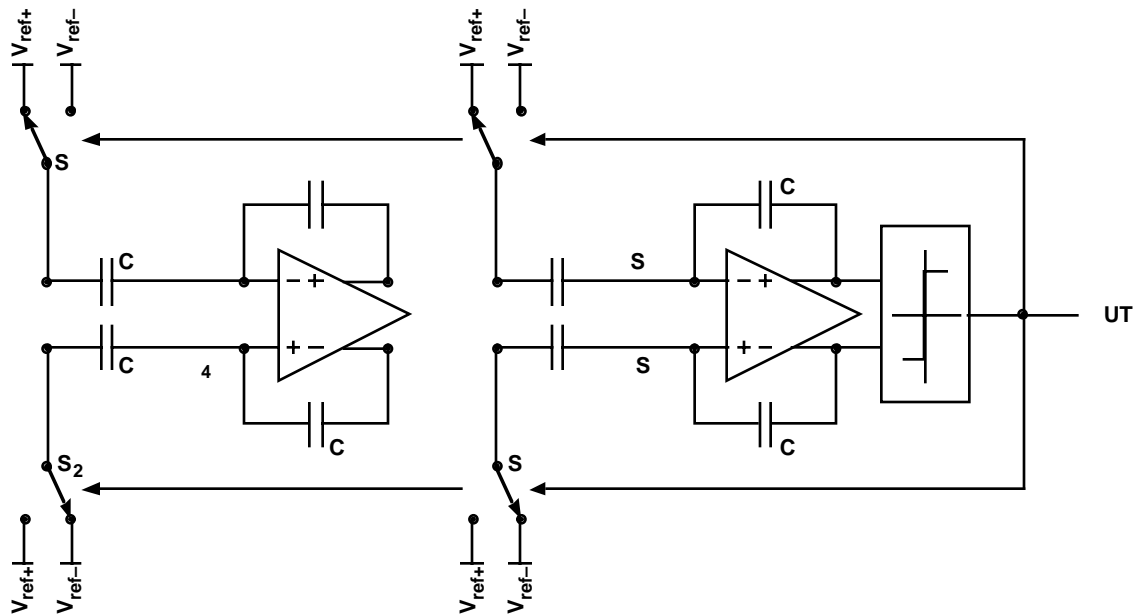


- S3 opens before S1

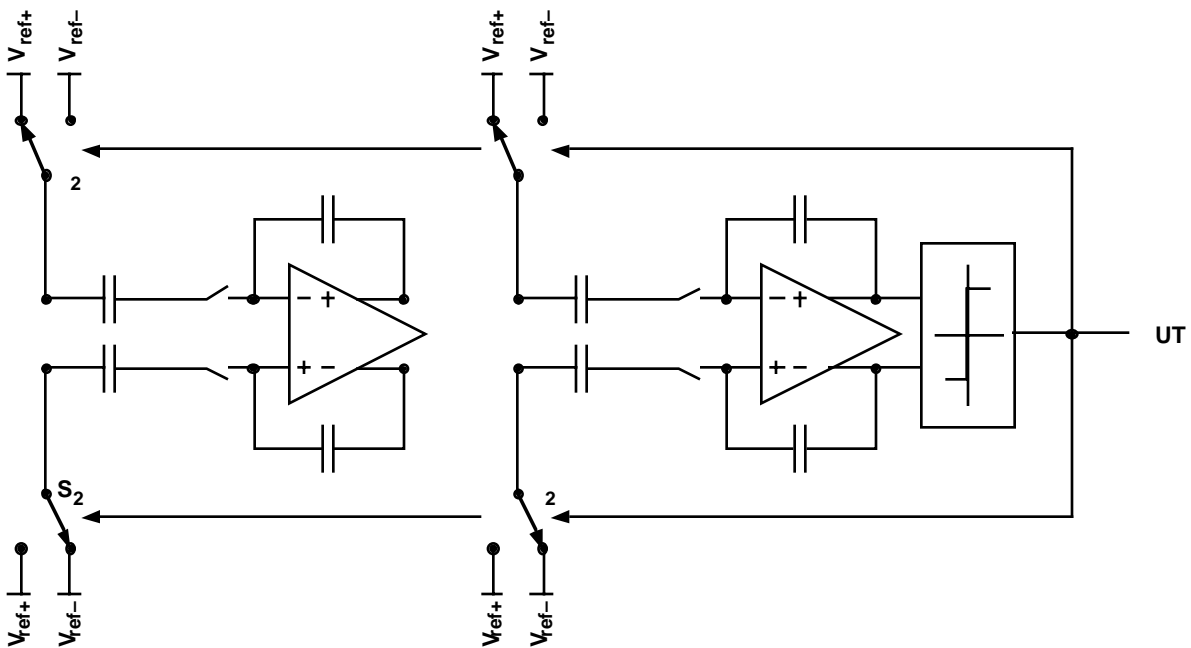


Phase 2

- Enable feedback
- Integrate
- Reset comparator



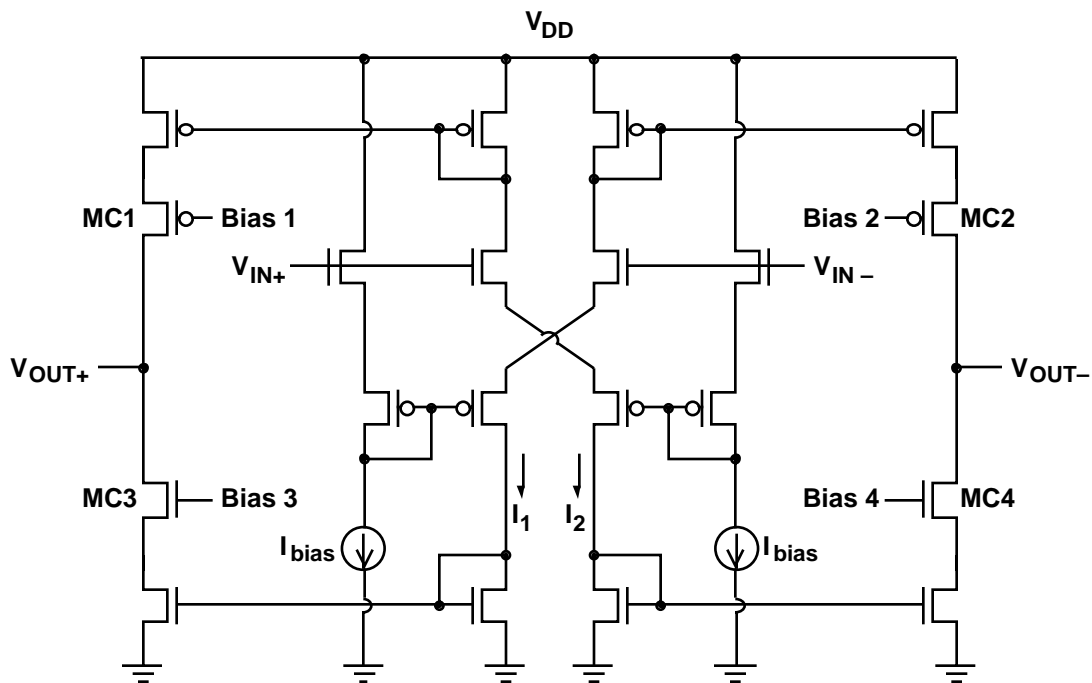
S4 opens before S2



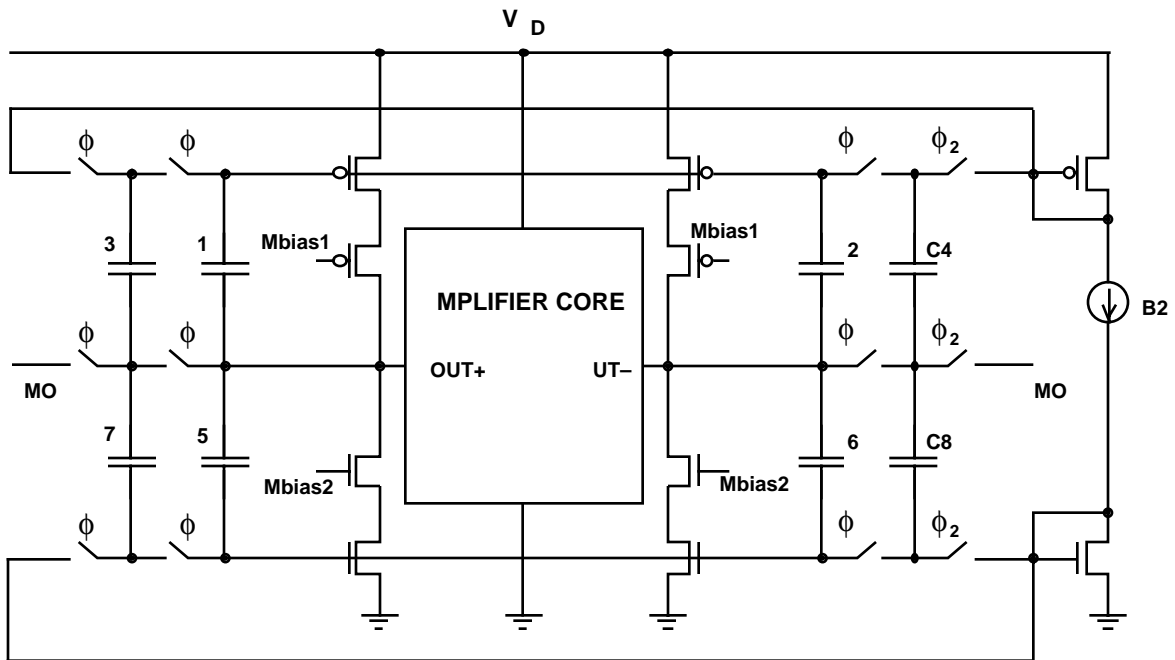
Operational Amplifier Requirements

- Linear settling \Rightarrow high slew rate
- High speed \Rightarrow single-stage amplifier
- Low distortion \Rightarrow gain $> 60\text{dB}$
- Wide dynamic range \Rightarrow low noise and large output swing
- Differential architecture

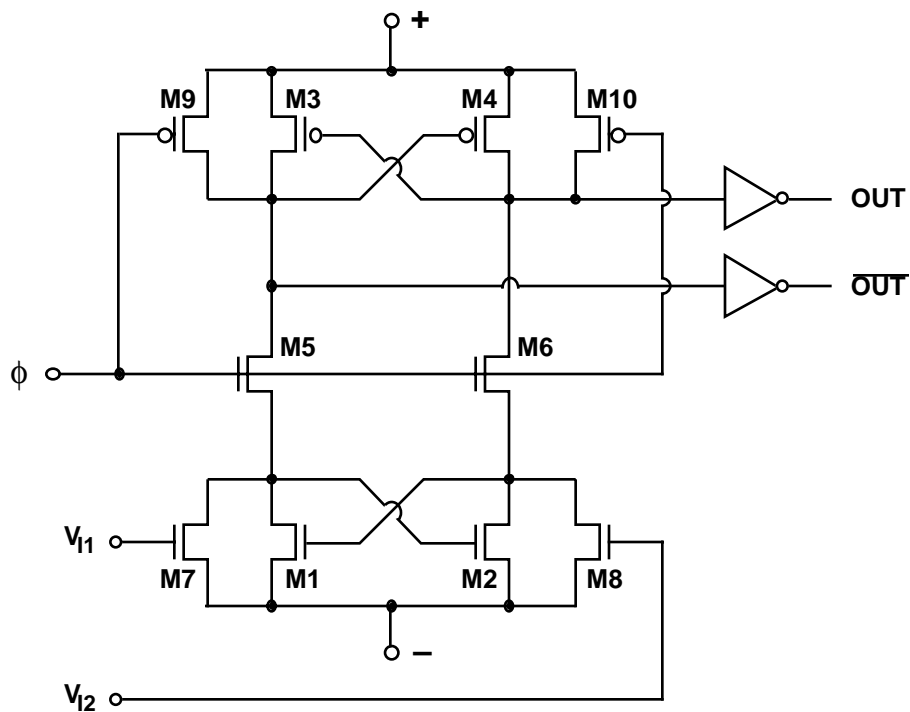
Class AB Op Amp



Common-mode biasing:



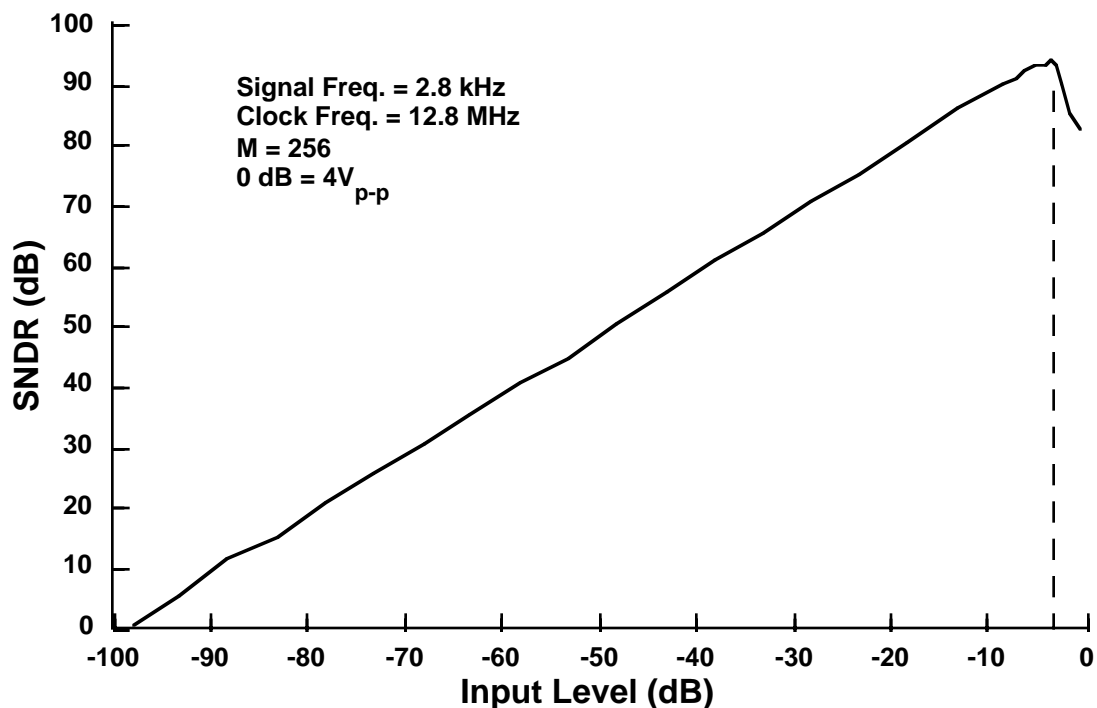
Comparator



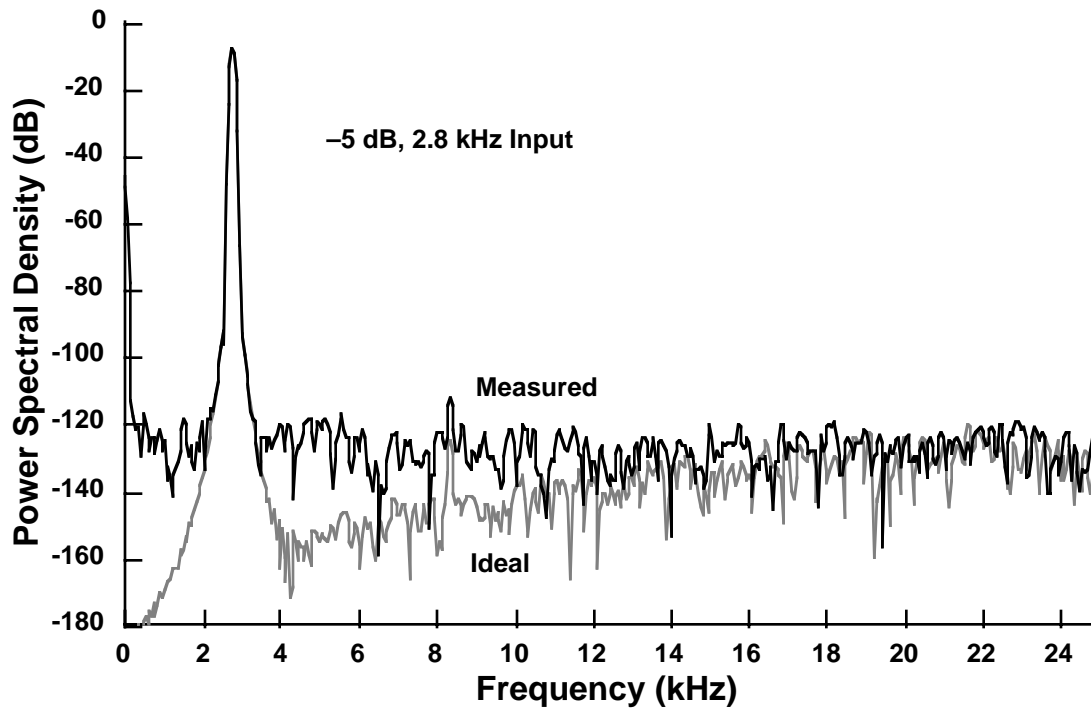
Summary of Measured Performance

Dynamic Range	98 dB (16 bits)
Peak SNDR	94 dB (0.002% THD)
Nyquist rate	50 kHz
Sampling rate	12.8 MHz
Oversampling Ratio	256
Differential input range	4 V
Supply voltage	5 V
Supply rejection	60 dB
Power dissipation	13.8 mW
Active Area	0.39 mm²
Technology	1-μm CMOS

Measured SNDR



Baseband Spectrum



Subcircuit Performance

Operational Amplifier

DC gain	67 dB
Unity-gain frequency	50 MHz
Slew rate	350 V/ μ sec
Linear output range	6 V
Sampling rate	12.8 MHz

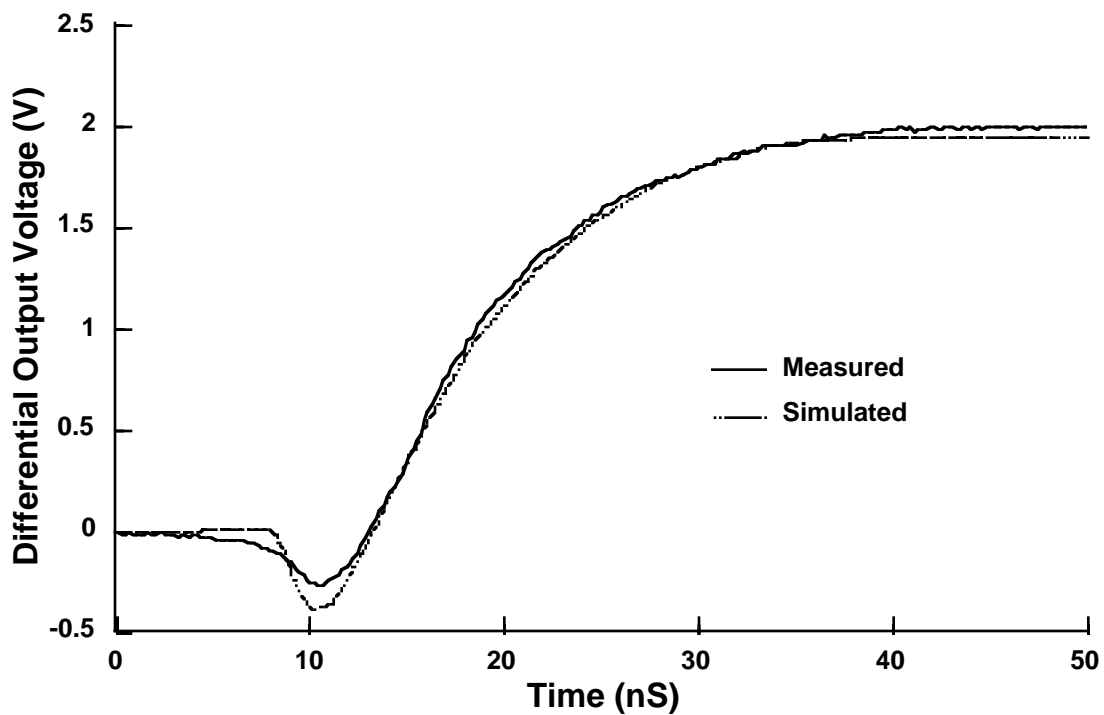
Integrator

Settling time constant	7.25 nsec
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Comparator

Offset	13 mV
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Integrator Step Response



Attributes of 2nd-order $\Sigma\Delta$ modulator:

- Simple architecture
- Very tolerant of component mismatch
- Stable operation
- High linearity
- Small area, low power

Quantization Noise Spectra of $\Sigma\Delta$ Modulators

Owing to the correlation of the quantization error with the input in oversampling modulator employing a single quantizer, discrete noise peaks (tones) appear in the output spectrum for certain inputs.

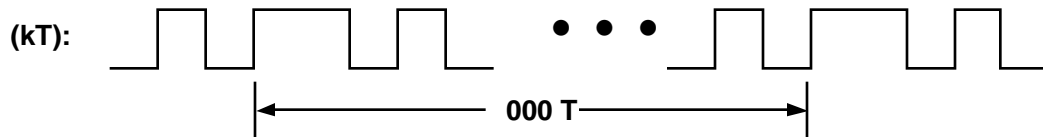
The following simple example illustrates the origin of the noise tones:

- $x(kT) = 0$ (Midrange Input)



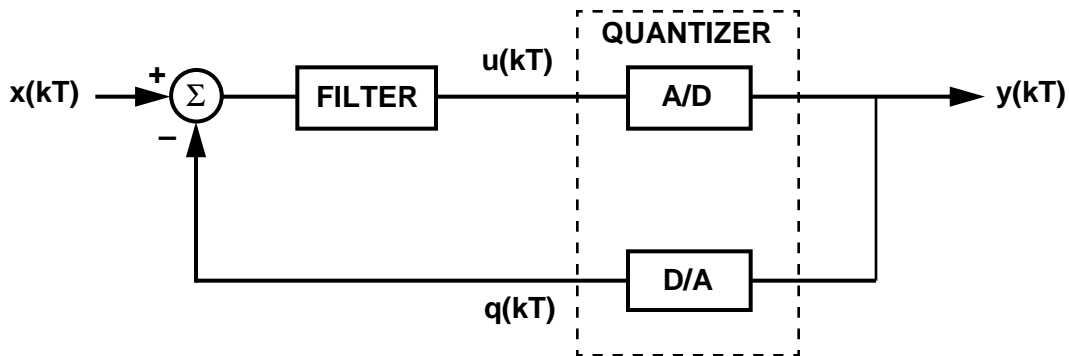
⇒ No Low Frequency Component

- $x(kT) = (0.001) \Delta/2$



⇒ Frequency Component in Baseband

For a single-quantizer modulator, the quantizer error is defined as the difference between the analog input and output of the quantizer:



$$e(kT) = q(kT) - u(kT)$$

System analysis (R. Gray, "Spectral Analysis of Sigma-Delta Quantization Noise) of an ideal first-order $\Sigma\Delta$ modulator indicates that the spectrum of the quantizer noise is not white. For a dc input, x_{dc} , the spectrum consists solely of impulses with power

$$S_e(k) = \frac{\Delta^2}{(2\pi k)^2}$$

at frequencies

$$f_k = \left\langle k \left(\frac{x_{dc}}{\Delta} + \frac{1}{2} \right) \right\rangle f_s$$

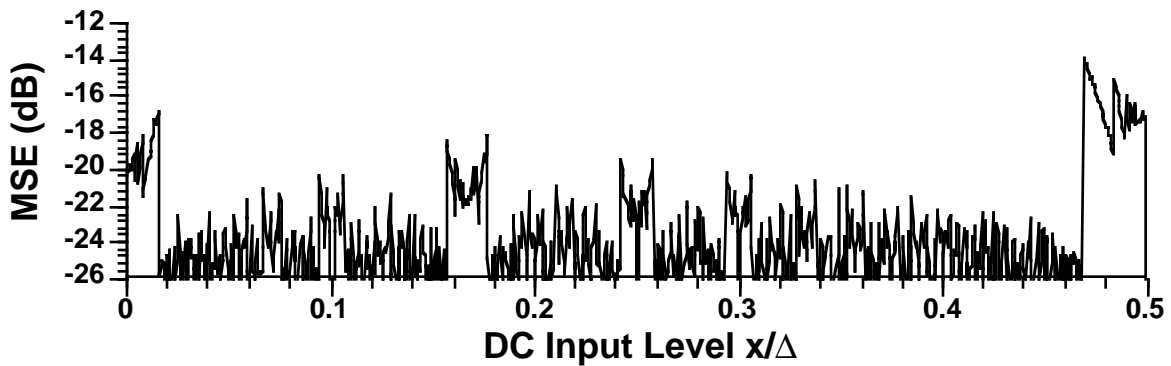
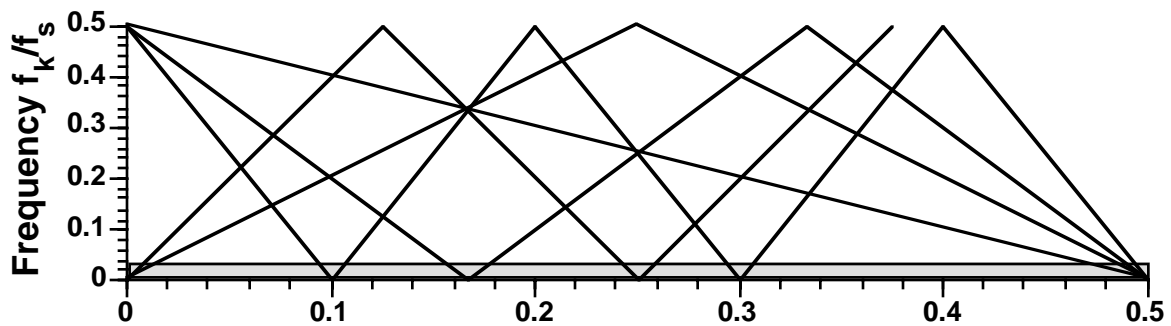
where k is a nonzero integer and $\langle w \rangle$ represents the fractional part of w .

The strongest tones occur for small values of k because of the reciprocal dependence in the expression for $S_e(k)$.

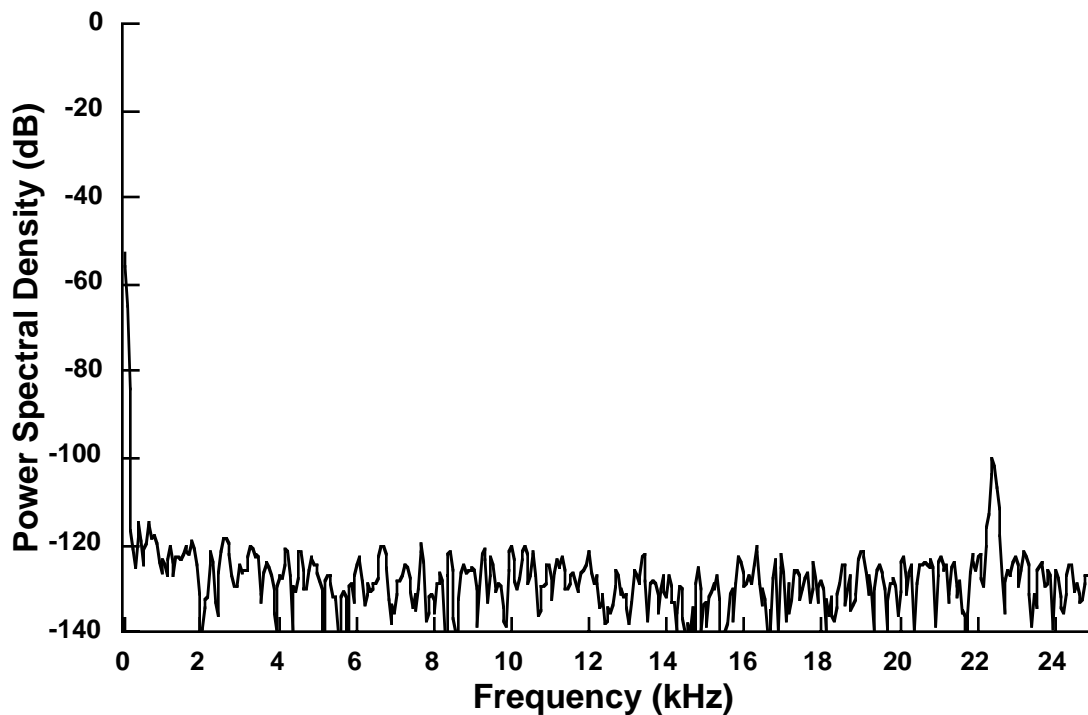
The power in the quantization noise tone in the output of the modulator that corresponds to $S_e(k)$ is

$$S_y(k) = \frac{\Delta^2 [\sin(\pi f_k T)]^2}{(\pi k)^2}$$

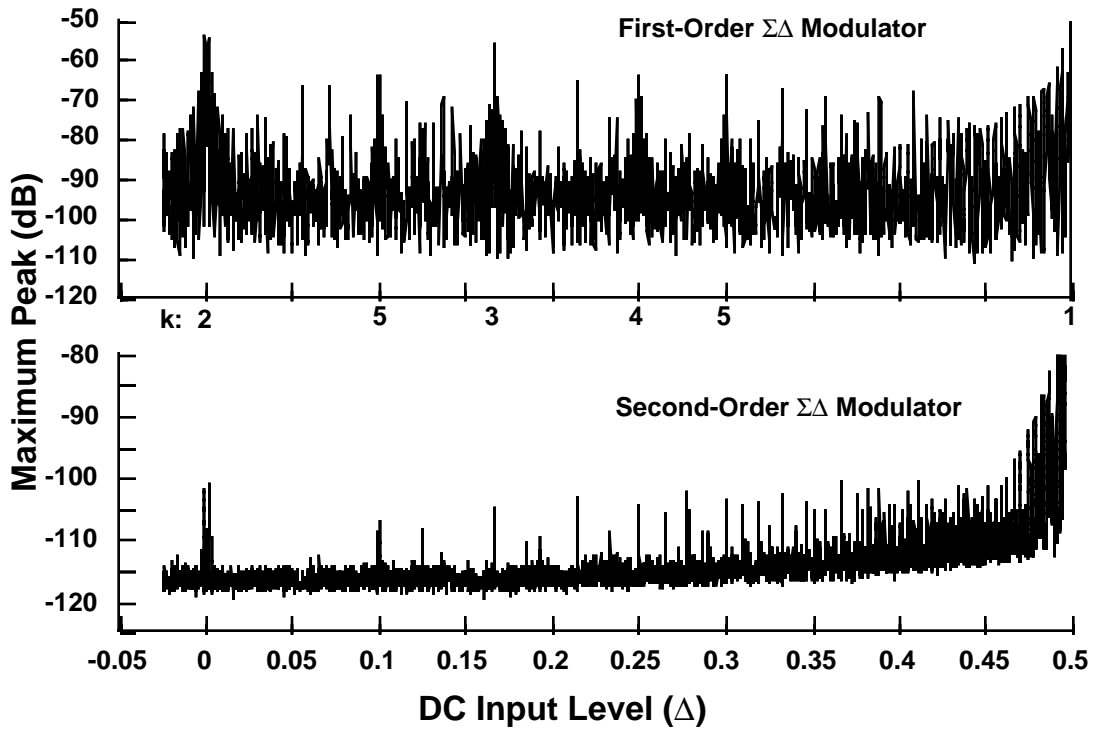
Quantization noise spectrum for a first-order modulator:



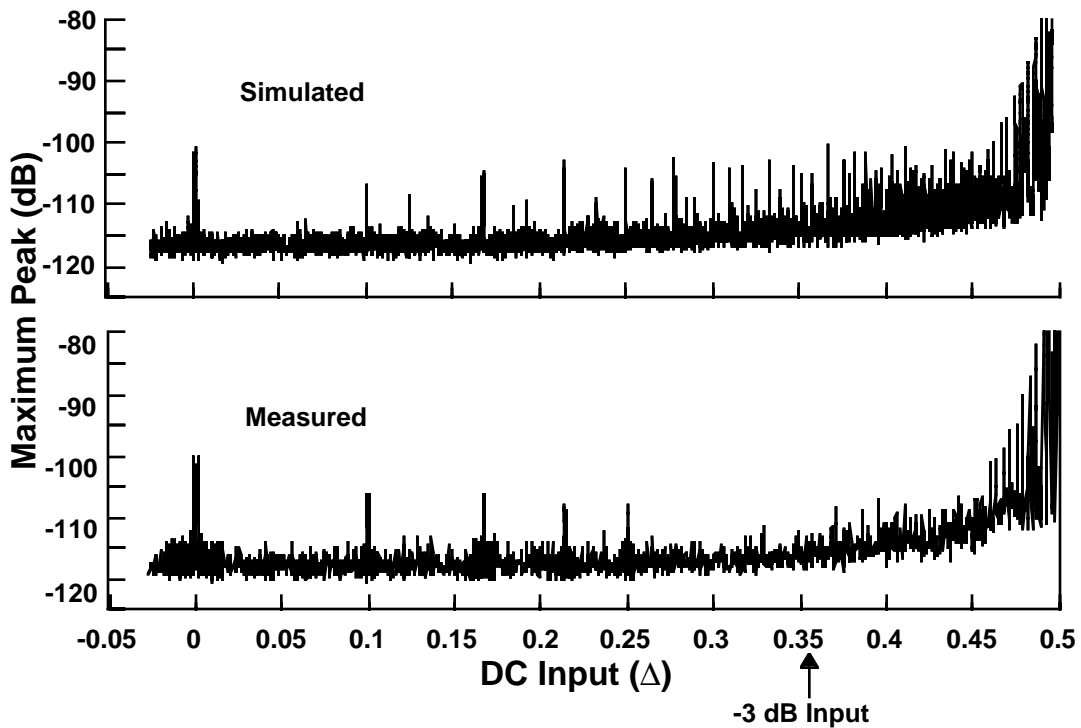
Discrete tone in a 2nd-order modulator spectrum:



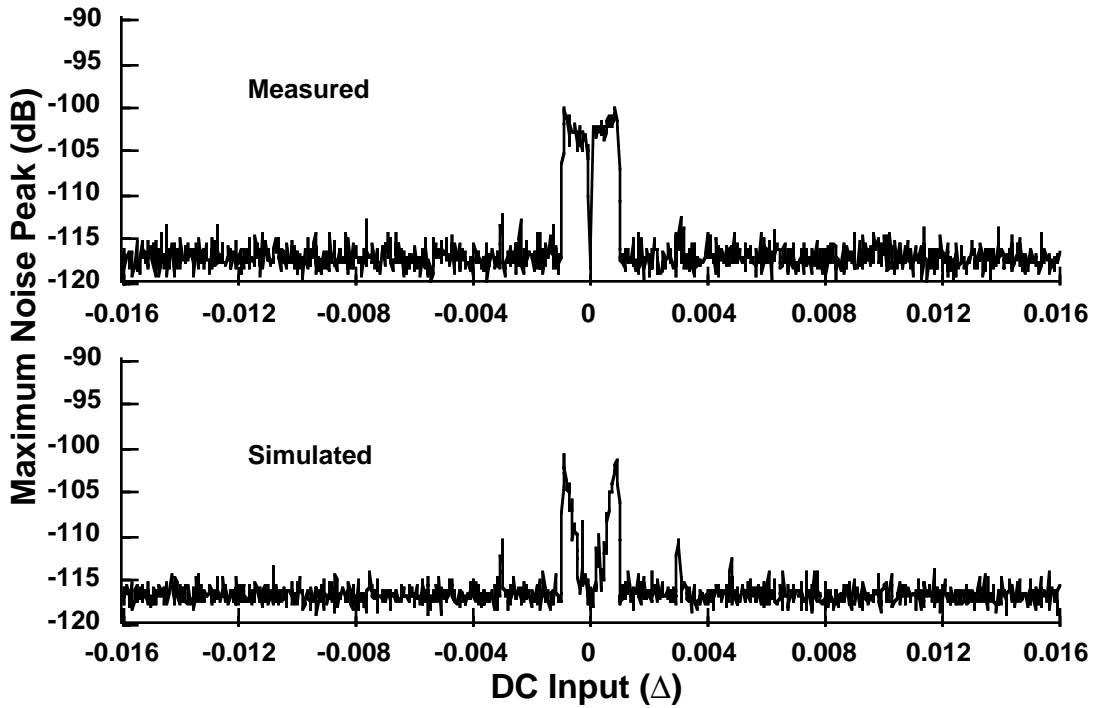
Quantization noise spectra for $M = 256$:



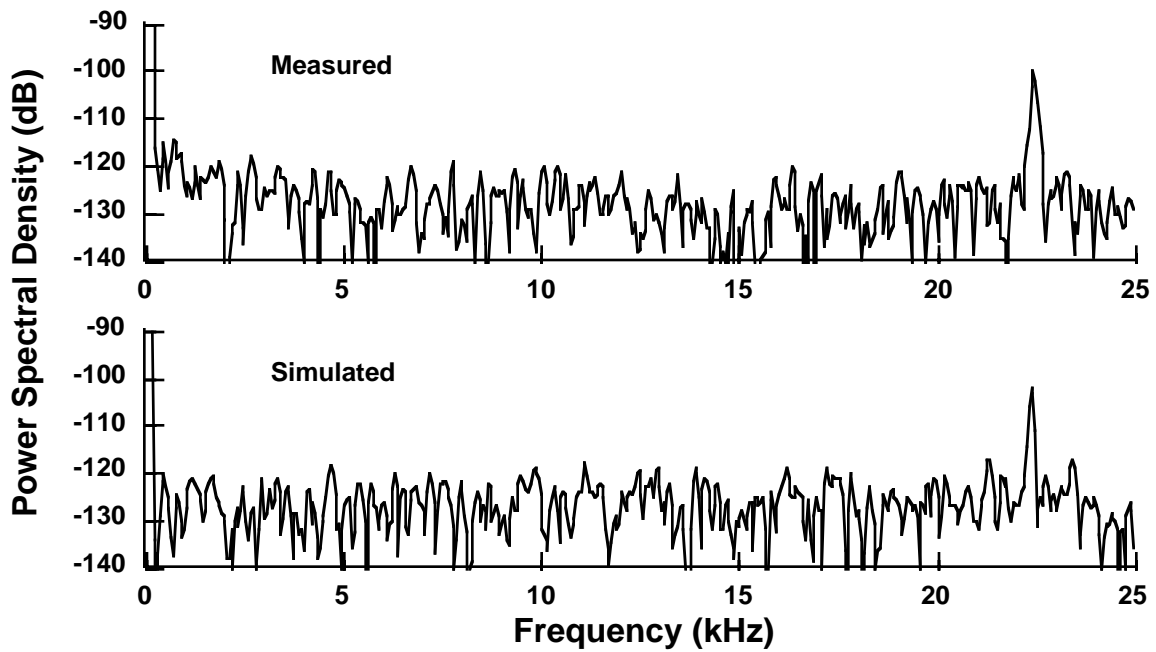
Noise spectrum of 2nd-order modulator:



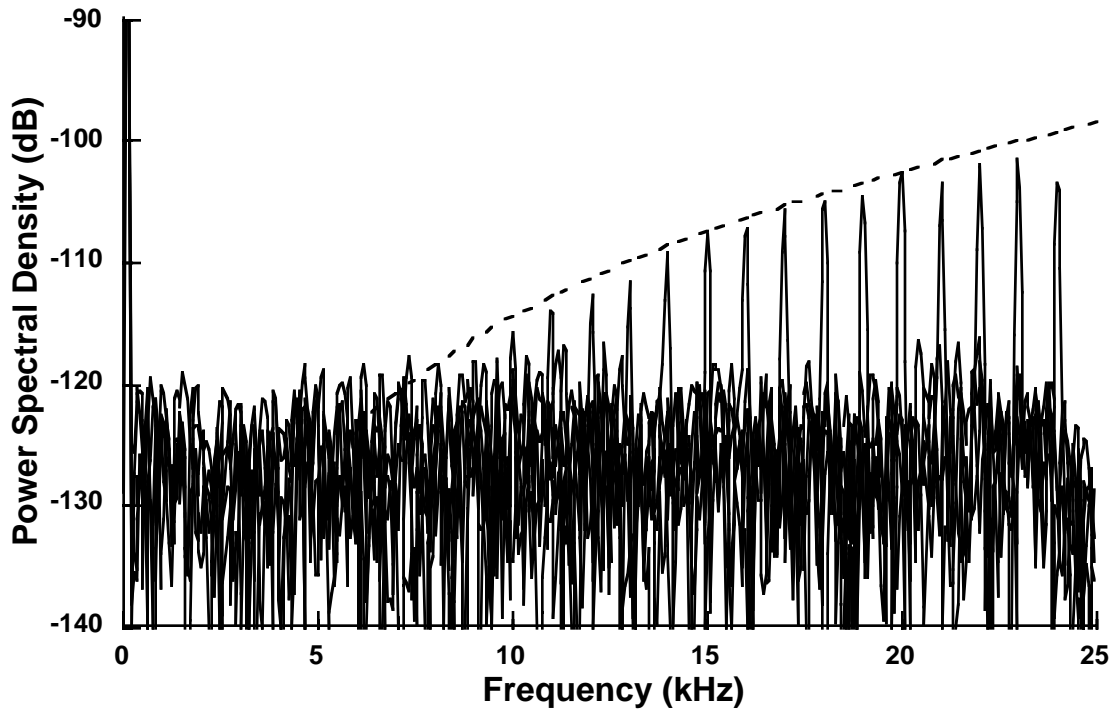
Tones near midrange in 2nd-order modulator:



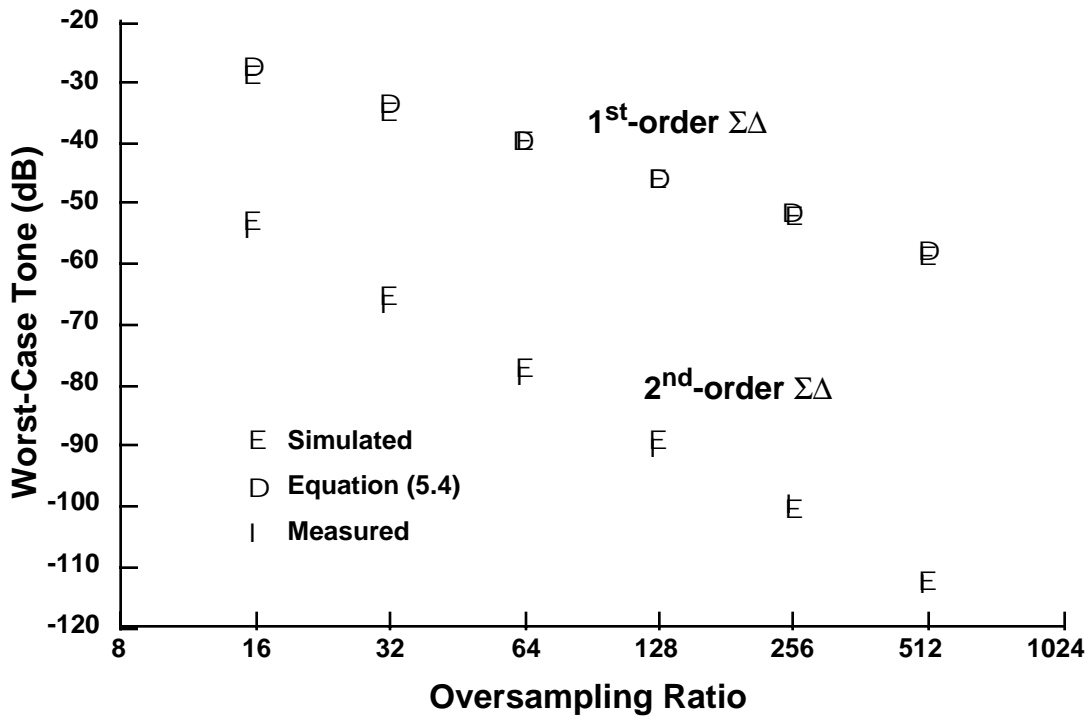
Worst tone a 2nd-order modulator:



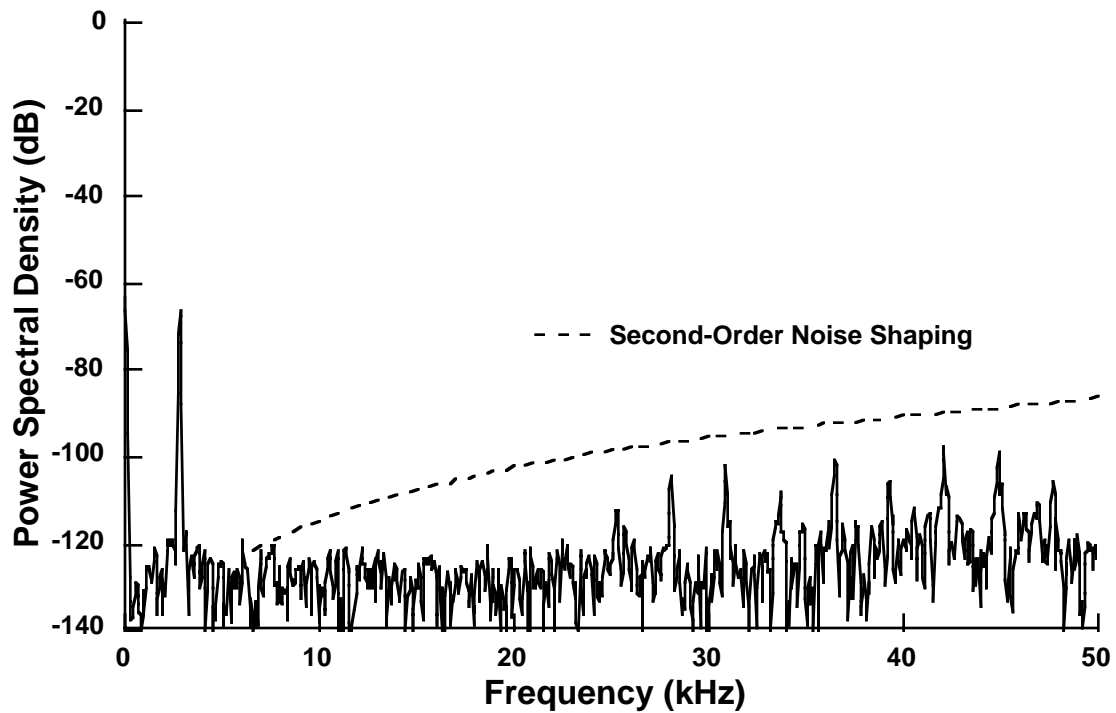
Composite spectra for several DC inputs:



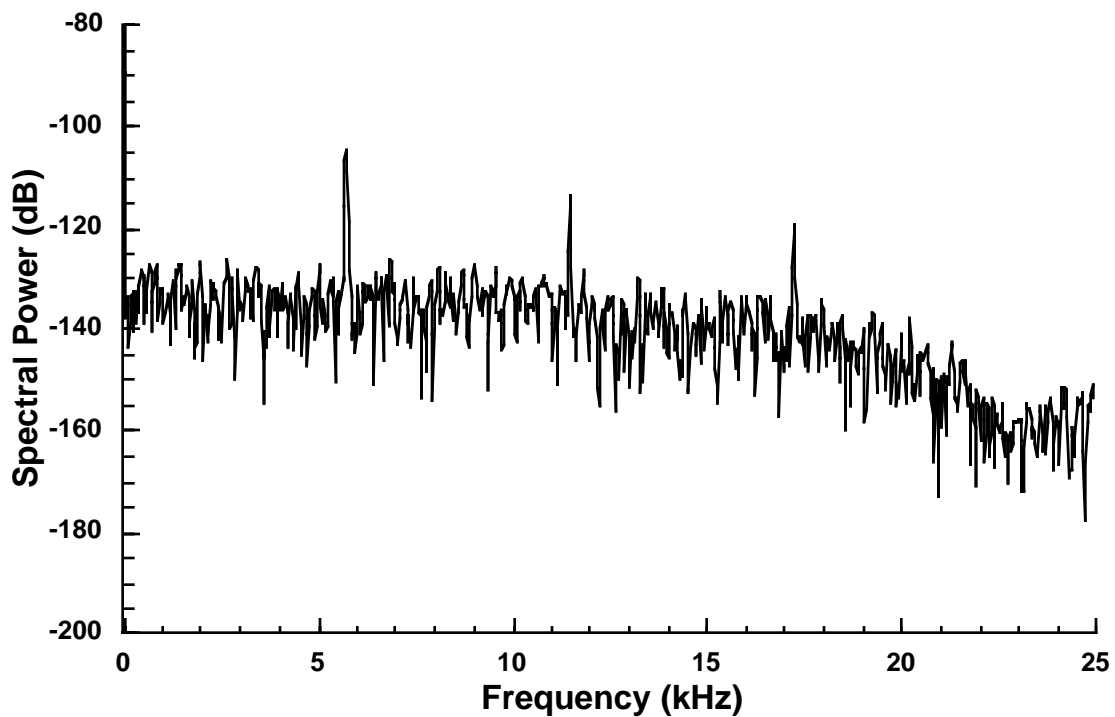
Worst-tones as a function of oversampling ratio:



Spectra for a sinusoidal input:



Spectral tones in a single-quantizer, 4th-order $\Sigma\Delta$ modulator:



Approaches to “whitening” the error spectrum

- **Cascaded sigma-delta modulation**
- **Multibit quantizers**
- **Dither**

Simulation Requirements

The design of oversampling data converters requires a behavioral simulation capability because of the need to simulate long data traces.

The results presented here were obtained using the program MIDAS, a functional simulator for mixed digital and analog sampled data systems.

MIDAS

- **System described by netlist**
- **Simulate in discrete time**
- **Includes tools for performance analyses such as spectral analysis, distortion and statistical functions**
- **Flexible I/O, including interface to test environment**
- **Written in C++**
- **User extensible with C-like code**
- **Not an “expert” tool for novice designers; intended for use by designers who understand what models should be used or constructed**