Seminar

Modeling and Simulation of Dynamical Systems

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Session 1

Mathematical Models of Dynamical Systems For Control System Design

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Outline

- Mathematical Models:
  - Intuitive Definition, Static and Dynamic Models
- Static Models
  - Crisp/Fuzzy Logic Models, Algebraic Models
- Dynamic Physical Models
  - Mechanical, Electrical, Fluid Flow Systems
  - Discrete Time Systems
- Transformations and Solutions
  - Transfer Functions
  - Numerical Solution
  - Discrete Time, Discrete State Systems
- Summary

Model: Intuitive Definition

Real Boeing 787

Flight Simulator Model of Boeing 787

- Captures certain desirable characteristics of the physical system (Geometric proportions, in the case of Desktop model)
- Provides a more easily manipulatable form for conceptualization, experimentation and design
- Desktop model Scaling Law: $\mathbf{d_m} = \frac{1}{200}\mathbf{d}$

Desktop Model of Boeing 787

Mathematical Models…

Symbolic Expressions, Data Tables and Computer Programs that describe certain features of a physical system can be considered as Mathematical Models

Word Problems- The basis for building all Mathematical Models

1. Joe wants to build a rectangular deck which is 6 feet longer than it is wide, covering an area of 280 feet. What are the dimensions of his deck?

Define: Width W, Length L, From Geometry: Area = Length×Width

Model: $(w+6)w = 280$

$$\text{width} = 14', \quad \text{length} = 20'$$

2. Newton's Second Law: The product of the acceleration *a* experienced by the body with its mass *m,* is equal to the applied force *F*.

Model: $F = ma$

Since acceleration a is the time rate of change of velocity v, and v is the rate of change of displacement x, this model can also be written as:

$$F = m\frac{dv}{dt} = m\dot{v} \qquad F = m\frac{d^2x}{dt^2} = m\ddot{x}$$

Mathematical Models

Control Systems Engineering is based on "Input-Output" Models, describing the relationship between *Causes* we can "choose" and the *Effects* we are interested in.

Volume Control (Input)

Sound(Output)

Carrier Frequency, Temperature,
Fan speed, Current in the devices…
(Internal "States" of the Radio)

Mathematical model relates the inputs to the system states, and the system states to the outputs.

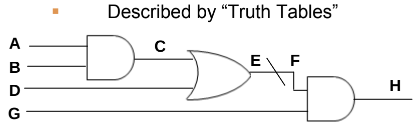
- Static Model: The system output response to inputs and starting values of the states do not depend on time
- Dynamic Model: The system response to inputs and starting values of the states depend on time

Steps in Mathematical Modeling

- Identify Inputs, Outputs and States
- Identify the Physical Laws/ Operating Principles/ Heuristics(such as analogies) that relate the Inputs, States and the Outputs
- Translate these into Logical / Algebraic / Differential/Difference expressions
- Convert to Manipulatable Form
- Analyze the system

Static Models: Used in "Slow" Control Problems

- Logic Models
  - Assembled using "And, Or, Not, Exclusive Or, Nand, Nor" Logic
  - Described by "Truth Tables"



A	B	C	D	E	F	G	H
0	0	0	0	0	1	1	1
1	0	0	1	1	0	0	0
0	1	0	1	1	0	1	0
1	1	1	1	1	0	1	0
0	0	0	0	0	1	0	0

- *"Fuzzy"* Logic Models
  - Assembled from "Membership" functions
    - Inputs are classified as belonging to different classes to certain degrees: Input is 80% Cold, 15% Warm, 5% Hot...(Fuzzification)
    - If-then-else conditions are used to select from a set of fuzzy logic rules: If input is 80% Cold, then heating coil must be at 90%, or if input is 15% Warm, heating coil must be at 50%...
    - A rule combining algorithm is used to arrive at a final decision on the control action
- Algebraic Models (Generally derived from experimental data)
  - Inputs and outputs are related through algebraic functions

$$\mathbf{y} = \mathbf{tanh}(\mathbf{ax})$$

For large a, x≠0, Approximates a switch:
if x>0, y=1else if x<0, y=-1

Dynamic Models: Spring-Mass-Damper

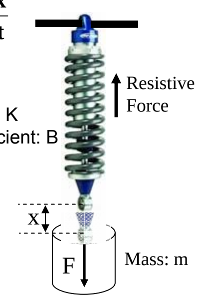
- Input: Force, Output: Displacement, States: Displacement, Velocity
- Physical Laws: The resistance to a longitudinal force applied at the tip of a spring-mass-damper assembly is the sum of the product of spring stiffness and spring deflection, the product of viscous damping coefficient and the rate of spring deflection, and the inertia.

Displacement: x, rate: dx/dt, acceleration: d^2x/dt^2, force: F

Inertia: $m\frac{d^2x}{dt^2}$ Resistance from Spring: kx Resistance from Damper: $b\frac{dx}{dt}$

Putting it all together: $F = kx + b\frac{dx}{dt} + m\frac{d^2x}{dt^2}$

Spring stiffness: K
Damping Coefficient: B

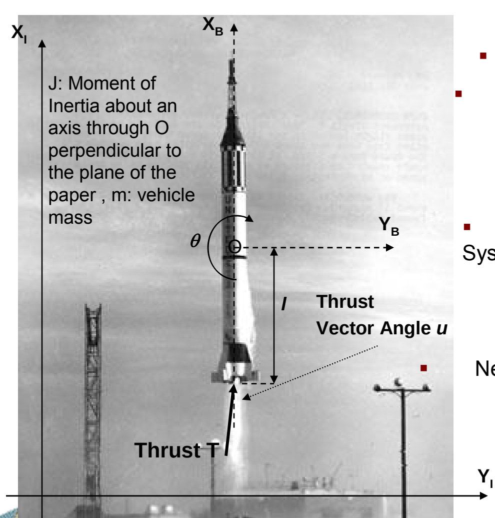


More Concisely: $m\ddot{x} + kx + b\dot{x} = F$

“Linear” differential equation, with forcing function (input) F.

Dynamic Models: An Aerospace Example

- Mercury Launch Vehicle in ascent



- Input: u, Outputs: x, y
  States: $x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}$
- Physical Laws: Newton’s (Euler’s) Law
- Resolve the thrust in the X_B, Y_B coordinate system

$$F_{XB} = T\cos u \cong T \quad F_{YB} = T\sin u \cong Tu$$
$$M = -lT\sin u \cong -lTu$$

- Resolve the forces in the X_B, Y_B Coordinate System to X_I, Y_I Coordinate system

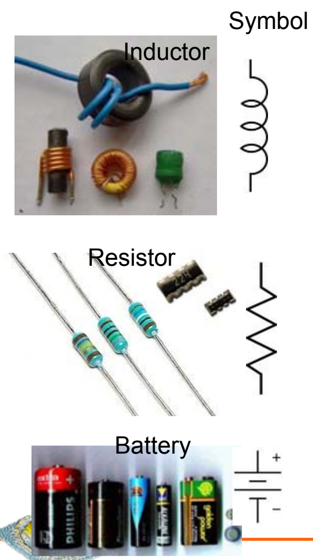
$$F_{XI} = T\cos\theta - Tu\sin\theta - g \cong T - g$$
$$F_{YI} = T\sin\theta + Tu\cos\theta \cong Tu$$

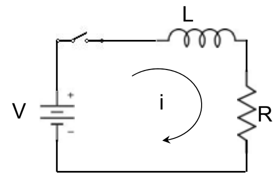
- Newton’s (Euler’s) Law:

$$\ddot{x} = \frac{T-g}{m} \quad \ddot{y} = \frac{Tu}{m} \quad \ddot{\theta} = -\frac{Tlu}{J}$$

“Linear” differential equations, with forcing function (input) u.

Dynamic Models: A Passive Electric Circuit





- Input: Voltage V, Output: Current, State: Current,
- Physical Laws:
  - The product of the rate of change of current through an inductor and its inductance is equal to the potential difference.
  - The potential difference across resistor is the product of the current through the resistor and its resistance.
  - Kirchoff's voltage law

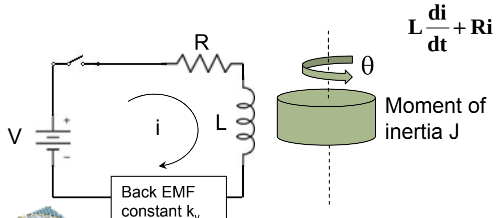
$$\mathbf{L\frac{di}{dt} + Ri = V}$$

Or $$\mathbf{\frac{di}{dt} = \frac{1}{L}(V - Ri)}$$

Dynamic Models: DC (PM) Motor

- Input: Voltage V, Output: Motor shaft position, States: Current, Motor speed, Motor shaft position (we will assume perfect commutation)
- Physical Laws:
  - The product of the rate of change of current through an inductor and its inductance is equal to the potential difference; The potential difference across resistor is the product of the current through the resistor and its resistance; Kirchoff's voltage law
  - Back EMF is proportional to the motor speed (Lenz's law/Faraday's law of induction)
  - Torque acting on the armature (rotor) is proportional to the field current
  - Euler's law: Angular acceleration is equal to the ratio of Torque and the moment of inertia
  - Viscous friction in the bearings is proportional to the shaft speed

$$\mathbf{L\frac{di}{dt} + Ri + k_v\frac{d\theta}{dt} = V} \qquad \mathbf{\frac{d^2\theta}{dt^2} = \frac{T}{J} = \frac{1}{J}\left(k_T i - B\frac{d\theta}{dt}\right)}$$



$$\mathbf{\dot{i} = \frac{1}{L}\left(V - Ri - k_v\dot{\theta}\right)}$$

$$\mathbf{\ddot{\theta} = \frac{1}{J}\left(k_T i - B\dot{\theta}\right)}$$

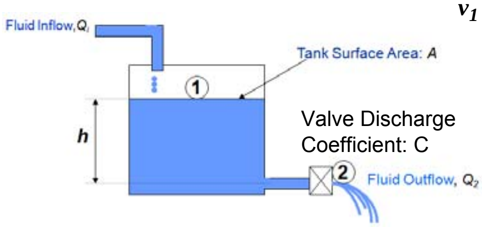
Dynamic Models: Liquid Level in a Tank

- Input: Fluid inflow Q_i, Output: Liquid level in the tank, State: Liquid level in the tank (Fluid out flow from the tank Q_2 through the fixed orifice is assumed to be given, non-viscous liquid) is assumed to)
- Physical Laws:
  - Law of conservation of mass
  - Bernoulli's equation (conservation of energy)

Conservation of Mass: $\dot{h} = \frac{1}{A}(Q_i - Q_2)$

Bernoulli's Equation: $\frac{v_1^2}{2g} + \frac{p_1}{\gamma} + h = \frac{v_2^2}{2g} + \frac{p_2}{\gamma}$ $\Rightarrow v_2 = \sqrt{2gh}$

$$v_1 = 0, p_1 = p_2$$



Outflow: $Q_2 = C A_o v_2$

System Model (Nonlinear)

$$\dot{h} = \frac{1}{A}\left(Q_i - C A_o \sqrt{2gh}\right)$$

Dynamic Models: State Variable Description

- So far, the dynamic models were given in the form of differential equations.
- Any n[th] Order Differential Equation can be expressed by n-first order differential equations.

Second-Order Ordinary Differential Equation describing the spring-mass-damper $\ddot{x} = \frac{F}{m} - \frac{k}{m}x - \frac{b}{m}\dot{x}$

Let: $x_1 \equiv x, \quad x_2 \equiv \dot{x} \quad \Rightarrow \dot{x}_1 \equiv x_2, \quad \dot{x}_2 \equiv \ddot{x}$

The System can be expressed by: $\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{F}{m} - \frac{k}{m}x_1 - \frac{b}{m}x_2$

- If the system is linear (Products and Powers of the Dependent Variables {outputs or states} do not appear in the model), the set of first-order differential equations can be represented in Vector-Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- In addition to being compact, the system model in this form can be readily manipulated using linear algebra. Software packages such a Scilab©, MATLAB® provide numerical methods to carryout these operations

Dynamic Models: Transfer Functions

- Transfer Function is another representation of the dynamic model, useful for analysis and design
- Given a differential equation with all initial conditions set to zero:

$$\ddot{x} + 3\dot{x} + 2x = 5u(t)$$

- Use the Laplace (or Heaviside-Laplace) Transform to change the independent variable from time t to the complex variable **s=σ+jω**

$$F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Unit Step Function: $u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ $\quad L(u(t)) = \frac{1}{s}$

Exponential Function: e^{-at} $\quad L(e^{-at}) = \frac{1}{s+a}$

Monomial in Time: $t^n, \quad n > 0$ $\quad L(t^n) = \frac{n!}{s^{n+1}}$

Sine Function: $\sin \omega t$ $\quad L(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$

Dynamic Models: Transfer Functions

Differentiation: $L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots f^{(n-1)}(0)$

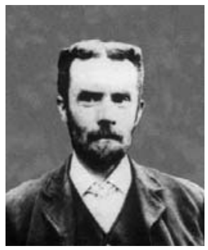
Integration: $L\left[\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(\tau)\,d\tau\,dt_1\ldots dt_{n-1}\right] = \frac{F(s)}{s^n}$

- Transform the differential equation : $\ddot{x} + 3\dot{x} + 2x = 5u(t)$

To the form: $s^2X(s) + 3sX(s) + 2X(s) = 5U(s)$

- After algebraic manipulations: $\frac{X(s)}{U(s)} = \frac{5}{s^2 + 3s + 2}$ Transfer Function
- The Transfer Function form of the dynamic model can be readily converted to the State Variable form and vice versa.

Heaviside-Laplace Transforms



Oliver Heaviside (1850-1925)

$$F(s) = L\big(f(t)\big) = \int_0^{\infty} f(t) e^{-st} dt$$

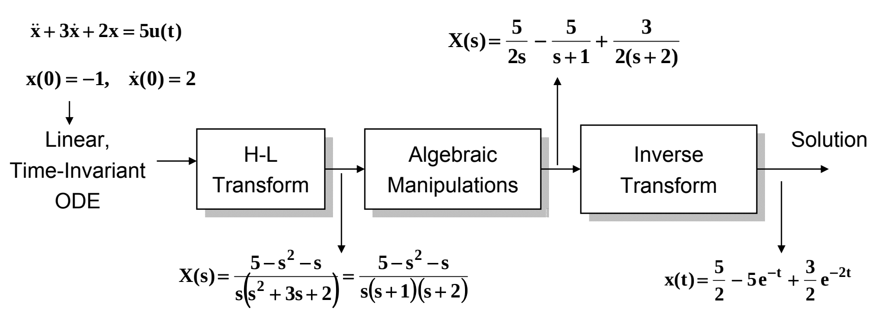
Piere-Simon Laplace (1749-1827)

$$F(s) = L\big(f(t)\big) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Heaviside.html

Dynamic Models: Solutions

- The dynamic models are given in the form of differential equations.
  - Linear ODE Can be “Solved” for any given input and any set of initial conditions using Laplace (Heaviside) Transforms



- This process is difficult if the inputs have complicated forms, and impossible if the dynamic models are nonlinear (as in the case of the liquid level example)
  - Numerical solutions can be obtained using computers

Dynamic Models: Numerical Solution (Simulation)

- Numerical integration techniques are used to set up computer Simulations.
- Some Popular Numerical Integration Techniques:
  - Euler's Method
  - Runge-Kutta Methods
  - Adams-Bashforth Integration Formulas
- Euler Integration Example

$$\dot{x} = a x^3 + b u \qquad \text{x(0), u(t) Given}$$

- Use Finite Difference approximation to the derivative:

$$\dot{x} \cong \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad \text{For "small" } \Delta t$$

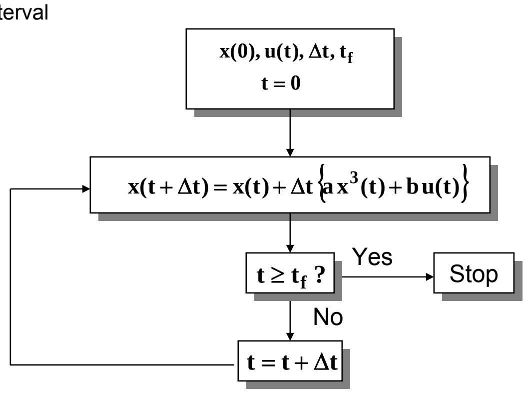
- Rewrite the differential equation as a "Difference Equation":

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = a x^3(t) + b u(t) \qquad \text{Or} \qquad x(t+\Delta t) = x(t) + \Delta t \left\{ a x^3(t) + b u(t) \right\}$$

Recursive "Marching" formula

Dynamic Models: Numerical Solution (Simulation)

- A "For – Next" loop can be written to evaluate the recursive formula for a desired time interval



- If due care is exercised in the selection Δt, this technique can be used to simulate most dynamic systems
- Runge-Kutta and Adams-Bashforth techniques can deliver higher accuracy for the same Δt. However, the computational effort will be much higher.

Dynamic Models: Discrete Time

- We "Converted" a differential equation to a "Difference Equation" by assuming that the time can be "Sliced" into "Discrete Chunks".
  - Effectively, we converted "Continuous Time" into "Discrete Time".
  - Since Digital Computers operate based on a "Clock", dynamic models based on discrete time are more natural representations.
- If the difference equations describing the dynamic model is linear, we can use "z" transforms to obtain solutions (similar to the H-L transform in continuous time)
  - Loosely speaking, the idea is to use the symbol z to denote a "Unit Time Advance", and z^{-1} to denote a "Unit Time Delay". Consider an example, with initial condition x(0) = 0:

$$x(t+\Delta t) = x(t) + \Delta t\left\{a\,x(t) + b\,u(t)\right\} = (1 + a\Delta t)\,x(t) + b\Delta t\;u(t)$$

Let $\alpha \equiv (1 + a\Delta t), \quad \beta \equiv b\Delta t \qquad\qquad x(t+\Delta t) = \alpha\; x(t) + \beta\; u(t)$

z Transform: $z\;X(z) = \alpha\;X(z) + \beta\;U(z) \quad \Rightarrow X(z) = \alpha\;z^{-1}X(z) + \beta\;z^{-1}U(z)$

Discrete-time Transfer Function: $\dfrac{X(z)}{U(z)} = \dfrac{\beta\,z^{-1}}{1 - \alpha\,z^{-1}}$

Note: Difference equations can also be represented in State Variable form.

Dynamic Models: Discrete-Time Systems

- Just as in the case of Continuous Time linear differential equations,
  - Linear Difference Equations can be "Solved" for any given input and any set of initial conditions using the z-Transforms

$x(t+2\Delta t) + 3x(t+\Delta t) + 2x(t) = 0$

$x(0) = 0, \quad x(1) = 1$

Linear, Time-Invariant Difference Equation → z Transform → Algebraic Manipulations → Inverse Transform → Solution

$$X(z) = \frac{z}{z^2 + 3z + 2}$$

$$X(z) = \frac{1}{z^{-1}+1} - \frac{1}{1+2z^{-1}}$$

$$x(k) = (-1)^k - (-2)^k, \quad k = 0, 1, 2, \ldots$$

- This process is difficult if the inputs have complicated forms, or impossible if the dynamic models are nonlinear
  - Numerical solutions can always be obtained using computers

Discrete Time Model: Dynamic Model of a Portfolio

- Input: Buy/Sell securities, State: Cash position, securities position, Output: Portfolio value
  - Facts (Laws): Cash in the portfolio earns a daily interest rate of r, the securities provide a daily dividend rate of d. Every security buy/sell decision involves a commission rate of m

A cash/security model is given by:

$$\begin{bmatrix} C_{i+1} \\ S_{i+1} \end{bmatrix} = \begin{bmatrix} (1+r) & 0 \\ 0 & (1+d) \end{bmatrix} \begin{bmatrix} C_i \\ S_i \end{bmatrix} + \begin{bmatrix} -1 & (1-m)\ P_i \\ \dfrac{(1-m)}{P_i} & -1 \end{bmatrix} \begin{bmatrix} \beta_i \\ \sigma_i \end{bmatrix}$$

C_i	Cash available on day i
S_i	Number of security shares held on day i
P_i	Security price on day i
β_i	Buy command on day i
σ_i	Sell command on day i
r	Interest rate on cash holdings
d	Dividend rate on security holdings
m	Transaction commission

Discrete-Time Dynamic Systems: Finite Impulse Response Representation

- Transfer functions such as:

$$\frac{X(z)}{U(z)} = \frac{2-0.6z^{-1}}{1+0.5z^{-1}}$$

are called Infinite Impulse Response (IIR) systems because the effect of an input impulse will persist in the system for infinite time (theoretically)

- Engineers in the Signal Processing area are fond of using "Finite Impulse Response" (FIR) Transfer Functions due to their interesting properties. An IIR Transfer Function can be converted to an FIR Transfer Function using long division (Numerical methods are also available)

$$\begin{array}{r|l} & 2-1.6z^{-1}+0.8z^{-2}-0.4z^{-3}+0.2z^{-4}-0.1z^{-5}+0.05z^{-6}\dots \\ \hline 1+0.5z^{-1} & 2-0.6z^{-1} \\ & \underline{2+z^{-1}} \\ & \quad -1.6z^{-1} \\ & \quad \underline{-1.6z^{-1}-0.8z^{-2}} \\ & \qquad\qquad 0.8z^{-2} \\ & \qquad\qquad \underline{0.8z^{-2}+0.4z^{-3}\dots} \\ & \qquad\qquad \vdots \end{array}$$

$$\frac{X(z)}{U(z)} = 2-1.6z^{-1}+0.8z^{-2}-0.4z^{-3}+0.2z^{-4}-0.1z^{-5}+0.05z^{-6}\dots$$

Truncate to retain any desired number of terms to obtain the FIR Transfer Function of the system

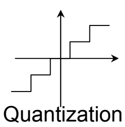
Discrete-Time, Discrete-State Dynamic Models

- We "Converted" a differential equation to a "Difference Equation" by assuming that the time can be "Sliced" into "Discrete Chunks".

$$\dot{x} = ax + bu \quad \underset{t \cong 0, \Delta t, 2\Delta t \ldots}{\Rightarrow} \quad x(t + \Delta t) = \alpha \, x(t) + \beta \, u(t)$$

- In many applications, the model will have to be evaluated on computers with limited precision. In those cases, the dependent variable x as well as the input u will be "Quantized" into few, finite number of levels

$$x_n, \; n = 1, 2, 3, 4, 5, \ldots m \qquad u_i, \; i = 1, 2, 3, 4, 5, \ldots j$$



Quantization Function

- The dynamic model can only transition between these "Finite" states. A discrete time equation can still be written, but the variables will be integers, and the process of addition and multiplication will be governed by the rules of integer arithmetic.

- Such dynamic models are called "Finite State Machines" and their behavior is generally characterized by "State (transition) Diagrams", and by Truth Tables in the case of systems governed by Binary (two state) arithmetic.

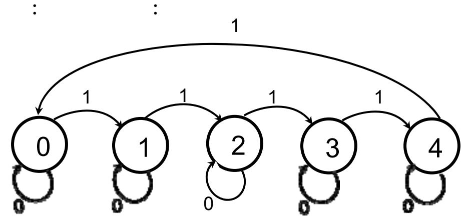
Discrete-Time, Discrete-State Models

- An example of a finite state machine

$$x(k+1) = x(k) + u(k), \quad x \equiv 0, 1, 2, 3, 4 \quad u \equiv 0, 1$$

$$\begin{array}{ll} x(0) = 3 & u(0) = 1 \\ x(1) = 4 & u(1) = 1 \\ x(2) = 0 & u(2) = 1 \\ \vdots & \vdots \end{array}$$

- State Diagram



- If the machine is linear, *D*-Transform methods can be used to investigate its characteristics without exhaustive enumeration

Summary

Presented a birds-eye-view of mathematical modeling process. Discussed:

- Static models: Logic/Fuzzy Logic, Algebraic
- Dynamic models: Spring-mass-damper, rocket, passive electric circuit, DC motor, liquid level in a tank
- State variable and transfer function representations, solution using H-L transform, numerical solution methodology
- Discrete-time model, transfer functions based on z-transform, Finite Impulse Response model
- Finite state machine

We did not discuss: Models described by partial differential/difference equations, stochastic models.

Recommended Reading

1. K. Ogata, System Dynamics, Prentice-Hall, Englewood Cliffs, NJ, 1978.
2. R. H. Canon, Dynamics of Physical Systems, McGraw-Hill, New York, NY, 1967.
3. D. Rowell and D. N. Wormley, System Dynamics-An Introduction, Prentice-Hall, Upper Saddle River, NJ, 1998.
4. F. Morrison, The Art of Modeling Dynamic Systems, John Wiley, New York, NY, 1991.
5. C-M Ong, Dynamic Simulation of Electrical Machinery, Prentice-Hall, Upper Saddle River, NJ, 1998.
6. F. D. Norvelle, Electrohydraulic Control Systems, Prentice-Hall, Upper Saddle River, NJ, 2000.
7. Z. Kohavi, Switching and Finite Automata Theory, McGraw-Hill, New York, NY, 1978.

Session 1: Mathematical Models

Q & A

Program

Welcome	08:45 – 09:10am	Coffee and bagels, Seminar kickoff at 9:00am
Session 1	09:10 – 10:00am	Mathematical models of dynamical systems Dr. P.K. Menon, Optimal Synthesis
Session 2	10:10 – 11:00am	System Identification - Theory and Practice Dr. Mark B. Tischler, Ames Research Center
Session 3	11:10 – 12:00am	Visualization and Virtual Environments Dr. Hadi Aggoune, Cogswell Polytech. College
Lunch	12:00 – 12:40pm	Sandwiches, sodas, discussions and product demos
Session 4	12:40 – 01:30pm	Applications of Hardware-in-the-Loop Simulators Christoph Wimmer, National Instruments
Session 5	01:40 – 2:30pm	Simulation with Software Tools Elliot English, Dr. Martin Aalund, Dr. Karl Mathia

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