a simple circuit model showing feature-rich Bogdanov-Takens bifurcation

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***Abstract*—A circuit model is proposed for studying the global behavior of the normal form describing the Bogdanov-Takens bifurcation, which is encountered in the study of autonomous dynamical systems arising in different branches of science and engineering. The circuit is easy-to-implement and one can experimentally study the rich dynamics and bifurcations simply by altering the values of some linear circuit elements (R, L, C) and the e.m.f. of a d.c. voltage source. It is shown that the system exhibits three local (saddle-node, Andronov-Hopf, spiral-to-node) bifurcations and one global (Homoclinic) bifurcation. The phase portraits associated with each of these bifurcations are presented, which serve to illustrate the qualitative changes in the system's dynamics across a bifurcation curve. The implications of these changes on the system's stability are discussed.**

***Index Terms*—Circuit modeling; bifurcation; Bogdanov-Takens; nonlinear differential equations; dynamical system**

I. Introduction

The theory of dynamical systems and bifurcations finds wide usage in diverse fields of engineering and the natural sciences. From Chua's circuit to Lotka-Volterra models of population biology, whenever a system is modelled by a set of differential equations, the phenomena of bifurcation is regarded as a powerful tool to investigate its dynamics [1]–[12]. Knowing the location of a bifurcation point is of great importance, because it marks the transition from one regime of dynamics to another. Bifurcation can be classified as either local or global. Bifurcations such as the saddle-node and Hopf bifurcations are local because they can be detected by a local analysis about an equilibrium point. On the other hand, a bifurcation involving the collision between a steady-state solution and a periodic solution (e.g., homoclinic bifurcation) is global as it can cannot be identified by a local analysis alone.

The Bogdanov-Takens (BT) bifurcation [13]–[15], which involves the simultaneous ocurrence of both these classes, has attracted increasing attention of the scientific community in the past few decades [1], [4]–[9], [16]. It has been argued that the dynamical behaviors of power-system models attempting to explain voltage collapse can be best understood in terms of Bogdanov-Takens bifurcation points and the associated Šil'nikov homoclinicity [1]. The BT bifurcation plays a major role in the excitability of neurons in a 2-compartment neuronal model [4]. It also arises in other diverse subjects such as the indirect field oriented control of induction motor drives [5], multimolecular biochemical reactions [6], predator-prey systems in population biology [7], [8] and the study of weakly coupled nonlinear oscillators [9].

In all of the above systems, the governing equations are topologically equivalent to the normal form for the BT bifurcation in the neighbourhood of a fixed point. The importance of studying the global characteristics of the BT bifurcation curve has also been stressed [16]. Despite this interest in the BT bifurcation, a circuit model for studying the overall dynamics of the BT normal form is lacking in the literature. Circuit models are important mainly because they serve to illustrate simply the salient features of a complex system and are easier for mathematical and experimental study [10], [11].

Thus, keeping in mind the wide applicability and the intrinsic mathematical interest, I propose in this article a simple electrical circuit model, which is governed by the normal form of the BT bifurcation. Apart from illustration purposes, this model should also provide quantitative details from experimental measurements about the Homoclinic bifurcation curve, which cannot be derived by direct analytical methods. The overall bifurcation diagram and the associated phase portraits are also presented for illustrating the qualitatively different dynamics the system exhibits in different zones of the parameter space.

II. The Circuit model

The Bogdanov-Takens bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ordinary differential equations at which the critical equilibrium has a zero eigenvalue of (algebraic) mulitplicity two [17], [18]. The normal form for the BT bifurcation is given by,

$$\dot{y}_1 = y_2 \tag{1}$$

$$\dot{y}_2 = \beta_1 + \beta_2 y_1 + y_1^2 \pm y_1 y_2 \tag{2}$$

where β_1 and β_2 are two independent parameters which can take both positive and negative values. Here we choose the '$-$' sign in the 2nd equation and propose the following circuit realization (Fig. 1).

The boxed element in the circuit can be viewed as a nonlinear resistor governed by $v = ki^2$. This block can be readily implemented using the squaring circuits available in the literature [19], [20]. We have used two voltage-controlled

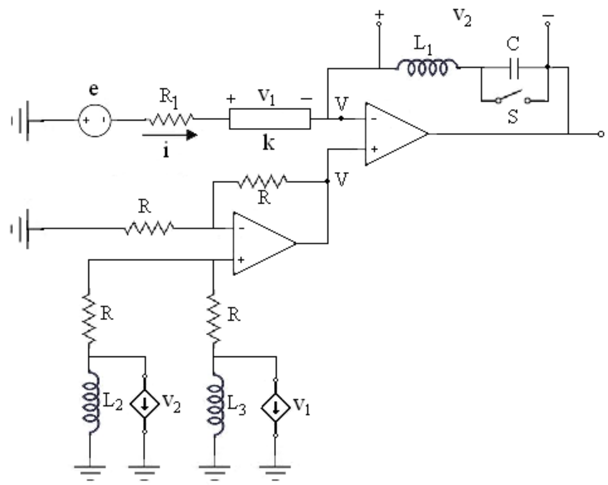


Fig. 1. Circuit model for the Bogdanov-Takens Bifurcation

current sources v_1 and v_2 whose magnitudes are given by,

$$v_1 = ki^2 \tag{3}$$

$$v_2 = L_1 \frac{di}{dt} + S \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau \tag{4}$$

where S is the switch variable which takes values 0 and 1 for closed and open positions respectively. The lower op-amp acts as a non-inverting summer producing the output,

$$\begin{aligned} V &= -2\Big(L_2 \frac{dv_2}{dt} + L_3 \frac{dv_1}{dt}\Big) \\ &= -2\Big(L_1 L_2 \frac{d^2 i}{dt^2} + S \frac{L_2}{C} i + 2L_3 k i \frac{di}{dt}\Big) \text{ [using (3) and (4)]} \end{aligned}$$

Again, application of Kirchoff's voltage law to the branch carrying the current i yields,

$$V = -(e + R_1 i + ki^2)$$

Comparing these two equations, we get,

$$2\Big(L_1 L_2 \frac{d^2 i}{dt^2} + S \frac{L_2}{C} i + 2L_3 k i \frac{di}{dt}\Big) = (e + R_1 i + ki^2) \tag{5}$$

Dividing (5) by k and defining the 'normalized' time as, $\tau = t/4L_3$, the equation can be simplified as,

$$\Big(\frac{L_1 L_2}{8kL_3^2}\Big) \frac{d^2 i}{d\tau^2} = \frac{e}{k} + \frac{R_1 - S(2L_2/C)}{k} i + i^2 - i \frac{di}{d\tau} \tag{6}$$

Let us agree to choose the parameter values in such a way that $L_1 L_2 = 8kL_3^2$. Then, we can define two phase variables, $y_1 \equiv i$ and $y_2 \equiv di/d\tau$, in terms of which, the above equation can be written as,

$$\dot{y}_1 = y_2 \tag{7}$$

$$\dot{y}_2 = \frac{e}{k} + \frac{R_1 - S(2L_2/C)}{k} y_1 + y_1^2 - y_1 y_2 \tag{8}$$

which is exactly identical to (1)-(2) with the minus sign in the 2nd equation. The parameters β_1 and β_2 are given by,

$$\beta_1 = \frac{e}{k}\,, \qquad \beta_2 = \frac{R_1 - S(2L_2/C)}{k} \tag{9}$$

By changing the polarity and varying the e.m.f. of the voltage source, the parameter β_1 can be varied over both positive and negative values. For positive values of β_2, we may simply close the switch S, which makes $S = 0$ and alter the resistance value R_1. However, if we want β_2 to be negative, then we must keep the switch open and change the ratio of L_2 and C instead.

III. Phase portraits and Bifurcation

We proceed by discussing the nature of equilibrium points of the system. At the equilibrium points (y_1^*, y_2^*), both the time derivatives in (1)-(2) vanish. Hence,

$$\beta_1 + \beta_2 y_1^* + y_1^{*2} = 0 \tag{10}$$

$$y_2^* = 0 \tag{11}$$

Equation (10) does not admit any real solution if $\beta_1 > \beta_2^2/4$. Hence, no fixed points exist in this region of the parameter space. On the other hand, if $\beta_1 < \beta_2^2/4$, there are two solutions given by $(y_1^*, y_2^*) = (-\beta_2/2 \pm \sqrt{\beta_2^2/4 - \beta_1}, 0)$. If we linearize the governing equations about these fixed points, we obtain the Jacobian matrix as,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ \beta_2 + 2y_1^* & -y_1^* \end{pmatrix} \tag{12}$$

For the fixed point at $(-\beta_2/2 + \sqrt{\beta_2^2/4 - \beta_1}, 0)$, $Det(\mathbf{J}) = -\sqrt{\beta_2^2 - 4\beta_1} < 0$. Consequently, this point is a saddle [12]. On the other hand, for the fixed points $(-\beta_2/2 - \sqrt{\beta_2^2/4 - \beta_1}, 0)$, $Det(\mathbf{J}) = \sqrt{\beta_2^2 - 4\beta_1} > 0$ and $Tr(\mathbf{J}) = \beta_2/2 + \sqrt{\beta_2^2/4 - \beta_1}$. We see that, $Tr(\mathbf{J})$ is negative only if $\beta_2 < 0$ and $\beta_1 > 0$ and is positive otherwise. We know that the trace of $\mathbf{J}$ is related to the stability of the fixed point. Positive trace implies that the fixed point is unstable and vice-versa. In addition, to determine whether the fixed point is a spiral or a node, we need to evaluate the sign of the quantity $D = (Tr(\mathbf{J}))^2 - 4Det(\mathbf{J})$. In our case, D equals zero along the curve denoted by L in Fig. 2.

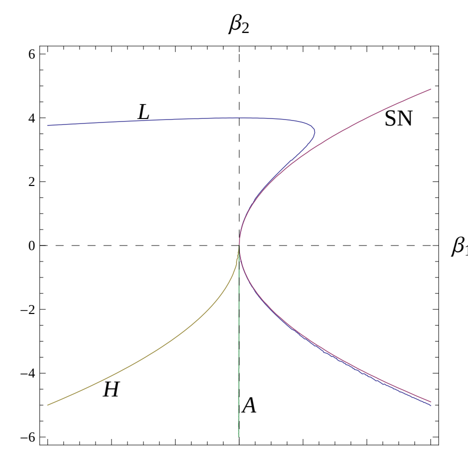
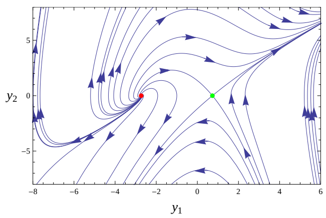
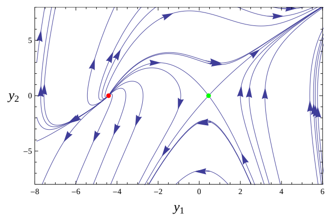


Fig. 2. Division of parameter space by the bifurcation curves

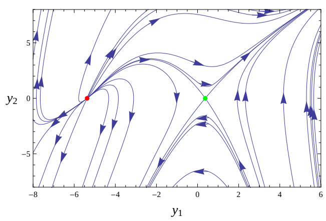
For points of the parameter space above this curve, $D > 0$ and the fixed point is a node, whereas below it, the point is a spiral. Accordingly a spiral-to-node local bifurcation takes place as one crosses this curve (Fig. 3).



(a) $\beta_2 = 2$: unstable spiral



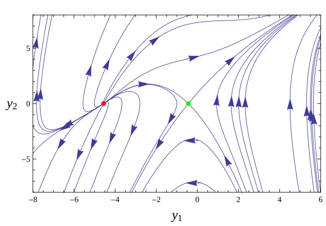
(b) $\beta_2 \approx 3.957$: unstable degenerate node



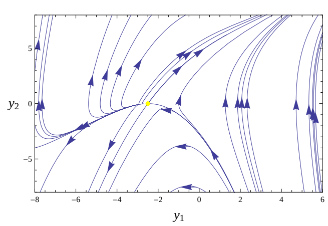
(c) $\beta_2 = 5$: unstable node

Fig. 3. Spiral-to-node bifurcation for $\beta_1 = -2$

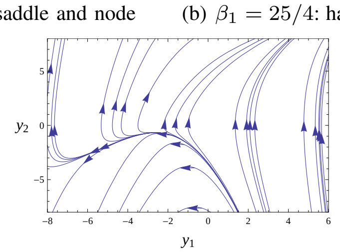
Two other types of local bifurcations occur as one crosses the curves SN and A. SN is a parabola given by $\beta_2^2 = 4\beta_1$. As we traverse SN from left, the saddle and the node approach each other, coalesce and then disappear, exhibiting the so-called saddle-node bifurcation (Fig. 4).



(a) $\beta_1 = 2$: saddle and node



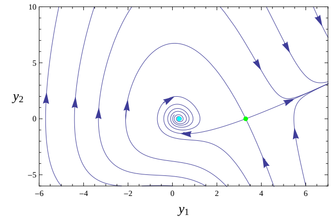
(b) $\beta_1 = 25/4$: half-saddle half-node



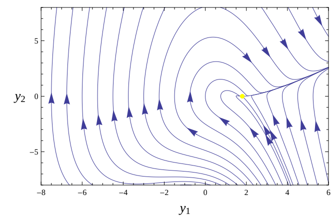
(c) $\beta_1 = 8$: no fixed points

Fig. 4. Saddle-node bifurcation for $\beta_2 = 5$

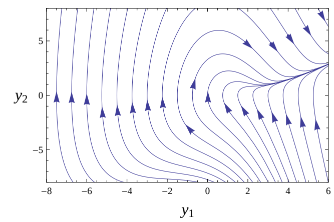
Fig. 5 shows the changes of dynamics one observes if one crosses SN keeping $\beta_2 < 0$. The key thing to notice is that to the left of SN, there exists a stable fixed operating point (stable spiral) whereas no such operating points exist on the other side. This has alarming consequences in power-system models, where voltage collapse can occur either when there is no attracting bounded solution or the initial state lies outside the basin of attraction of such a bounded state. Thus the computation of saddle-node bifurcation curve is important, as it defines the maximum possible region of safe operation [1].



(a) $\beta_1 = 1$: stable spiral



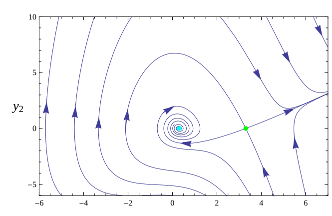
(b) $\beta_1 = 3.24$: half-stable point



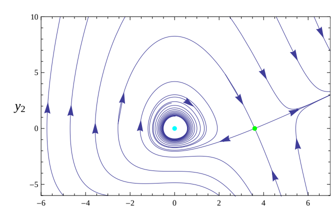
(c) $\beta_1 = 5$: no fixed points

Fig. 5. Loss of stable operating point as one crosses SN with $\beta_2 = -3.6$

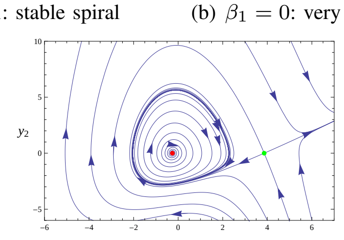
Another local bifurcation involving changes of system stability takes place if one crosses the β_2 axis, keeping $\beta_2 < 0$. We had already noted that for $\beta_1 > 0$, there exists a stable spiral. If β_1 is made negative, this spiral becomes unstable and a stable limit cycle appears around it. Thus the system undergoes a supercritical Andronov-Hopf bifurcation (Fig. 6). Consequently, all the initial states belonging to the basin of attraction of the stable spiral now gives rise to sustained oscillations of the state variables around the limit cycle. In many situations, oscillatory response is considered undesirable, and therefore, Hopf bifurcation may be looked upon as having a destabilising effect on the system's dynamics [1].



(a) $\beta_1 = 1$: stable spiral



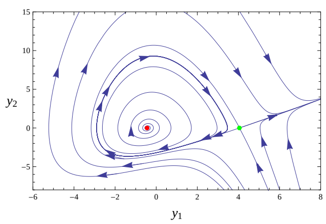
(b) $\beta_1 = 0$: very weak stable spiral



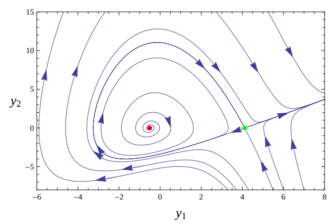
(c) $\beta_1 = -1$: unstable spiral surrounded by a limit cycle

Fig. 6. Andronov-Hopf bifurcation for $\beta_2 = -3.6$

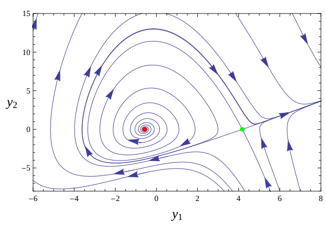
All the bifurcations discussed so far are local bifurcations which could be forecasted by linear stability analysis of the equilibrium points. Apart from these, the system also exhibits an important type of global bifurcation - the saddle homoclinic bifurcation. As β_1 is made more negative keeping β_2 fixed at some negative value, the stable limit cycle grows and the period of oscillation increases. It eventually collides with the saddle, forming a homoclinic orbit of infinite period, on the curve H in Fig. 2. If β_1 is made more negative, the limit cycle disappears, leaving two unstable fixed points - an unstable spiral and a saddle (Fig. 7).



(a) $\beta_1 = -1.8$: stable limit cycle and saddle



(b) $\beta_1 \approx -2.107$: homoclinic orbit of infinite period



(c) $\beta_1 = -2.4$: unstable spiral and saddle

Fig. 7. Saddle Homoclinic bifurcation for $\beta_2 = -3.6$

These Homoclinic orbits often have profound effect on the overall system stability [1] and a lot of work has been devoted to studying the behaviour near these orbits [21]. The physical variables such as currents and voltages executing sustained oscillation along the limit cycle, change exceedingly slowly as it approaches the homoclinic orbit. Then suddenly, as the homoclinic orbit disappears, all these state variables increase in an unbounded manner. Thus, the system suddenly changes its nature from marginally stable to potentially unstable.

The bifurcation curves corresponding to this global bifurcation and all three types of local bifurcations discussed previously, meet at the Bogdanov-Takens bifurcation point $(\beta_1, \beta_2) = (0, 0)$ (see Fig. 2). Accordingly, slight alterations of parameters, such as the resistance R_1 and voltage source e in our circuit model, in the neighbourhood of this critical point, lead to drastic qualitative changes in the system's dynamics. Fig. 2 shows the global bifurcation diagram for the system studied.

IV. Concluding Remarks

In this paper, I have proposed a simple circuit model, which is governed globally by the normal form of the Bogdanov-Takens bifurcation. It is quite surprising that such a seemingly simple circuit can exhibit a feature-rich dynamics with changes of system parameters. Three types of local bifurcation and one global bifurcation are observed, which can be summarized by the overall bifurcation diagram in Fig. 2. The circuit should be both useful for illustration purposes and amenable to experimental study of the various aspects of BT bifurcation arising in diverse contexts.

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