1 Linear Systems, Superposition, and Convolution

In this section we provide a brief introduction to linear systems theory. It is based on the following definition:

Let $L$ denote a \textit{time-invariant, linear system} with input $x$ and output $y$:

$$x \rightarrow [L] \rightarrow y$$

Such systems are \textit{additive} (the response to the sum of two inputs $x_1$ and $x_2$ equals the sum of the responses to each input taken individually):

$$L(x_1 + x_2) = L(x_1) + L(x_2)$$

and \textit{homogeneous} (the system can be scaled by the magnitude of the input $\alpha$, where $\alpha$ is a scalar):

$$L(\alpha x) = \alpha L(x)$$

Taken together, they enjoy the \textit{Principle of Superposition}:

$$L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2)$$

Several idealized input functions are of special importance in analyzing systems, the \textit{Dirac delta function} and the \textit{unit step function}. The unit step is defined as:

$$u(t) = \begin{cases} 
0, & \text{if } t < 0; \\
1, & \text{if } t > 0; \\
\text{undefined}, & \text{if } t = 0 
\end{cases} \quad (1)$$

and the Dirac delta function is its derivative:

$$\delta(t) \triangleq \frac{du(t)}{dt}.$$

Of course, since the step function is not continuous (notice the value at 0), one has to be careful in defining exactly what is meant by the above equation. Formally, the $\delta(t)$ is a \textit{distribution} defined by the integral equation:

$$\int_{-\infty}^{\infty} v(t)\delta(t)dt = v(0).$$

It can be represented as the limit of a sequence of functions such as:

$$\delta_\epsilon(t) = \frac{1}{\pi t} \sin \frac{\pi t}{\epsilon}$$

with $\epsilon \to 0$, see Fig. 1; During this limiting process we have something that resembles a physical approximation to the unit impulse, the first test pattern used to evaluate the lateral inhibitory network. The result is called the \textit{impulse response}

$$h(t) = L[\delta(t)]$$
Similarly, since the step function is the integral of the impulse function, the step response, or the response to a unit step function

\[ S(t) = \mathcal{L}(u(t)) \]

is the integral of the impulse response function:

\[ S(t) = \int_{-\infty}^{t} h(\lambda) d\lambda. \]

In general, of course, we are interested in the response of a system not to these special functions, but to an arbitrary input. The trick is to approximate this arbitrary input as a sequence of step functions (Fig. 2.12 in Oppenheim and Willsky, p91), and then to use superposition to obtain the overall response. It is necessary to assume that the input function \( x(t) \) is continuous, so that the approximation is meaningful. Now, suppose the input starts at \( t = 0 \), and time is discretized into bins \( \Delta \) seconds apart. The first step in the approximation has height \( x(0) \), a constant, and the additional step at \( t = k\Delta \) has height \( x(k\Delta) - x((k - 1)\Delta) \). Remembering that \( u(0) = 1 \), the signal \( x(t) \) can thus be approximated by

\[
\dot{x}(t) = x(0)u(t) + [x(\Delta) - x(0)]u(t - \Delta) + [x(2\Delta) - x(\Delta)]u(t - 2\Delta) + \cdots \tag{2}
\]

\[
= x(0)u(t) + \sum_{k=1}^{\infty} \{x(k\Delta) - x((k - 1)\Delta)\} u(t - k\Delta) \tag{3}
\]

The approximate output is then given by

Figure 1:
\[ \hat{y}(t) = x(0)S(t) + \sum_{k=1}^{\infty}\{x(k\Delta) - x((k-1)\Delta)\}S(t-k\Delta) \]  
\[ = x(0)S(t) + \sum_{k=1}^{\infty}\left\{ \frac{x(k\Delta) - x((k-1)\Delta)}{\Delta} \right\}S(t-k\Delta) \Delta \]  

Now, introduce a limiting process in which the time steps become vanishingly close, so that \( \Delta \to d\tau, \ k\Delta \to \tau \), the sum becomes an integral and we have:

\[ \lim_{\Delta \to d\tau} \hat{y}(t) \to y(t) \]  
\[ = x(0)S(t) + \int_{0^+}^{\infty} \frac{dx(\tau)}{d\tau} S(t-\tau)d\tau \]  

Repeating this approximation procedure for a continuous input \( x(t) \) starting at \( t' > -\infty \) and such that \( x(t')S(t-t') \to 0 \) as \( t' \to -\infty \) yields:

\[ y(t) = \int_{-\infty}^{\infty} \frac{dx(\tau)}{d\tau} S(t-\tau)d\tau \]  
\[ = \int_{-\infty}^{\infty} x(\tau) \frac{dS(\tau)}{d\tau} d\tau \]  
\[ = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)d\tau \]  

Thus the output can be computed as an integral function of the impulse response. Such an integral is called a convolution integral, and the above expression is often abbreviated:

\[ y(t) = x(t) \ast h(t) \]

Reference