A Simpler Alternative to Wave Tracing in Solving Transmission Lines

Clayton R. Paul,
Mercer University, Macon, GA (USA), paul_cr@Mercer.edu

Abstract—A simple method for rapidly sketching the terminal voltage and current waveforms of a lossless, two-conductor transmission line is given. The resulting simple equations for the terminal voltages and currents of the line are given in terms of symbols and do not have to be rederived for every problem. The method easily accommodates source waveforms having arbitrary wave shapes.

Index Terms—transmission lines, time-domain solution, wave tracing

I. The Basic Transmisssion-Line Problem

The basic transmission line considered is shown in Fig. 1 and consists of two parallel conductors having uniform cross sections along their total length $\mathscr{L}$ that are directed along and parallel to the z axis. A source represented with a Thevenin equivalent representation consists of an open-circuit voltage waveform, $V_S(t)$, and a source resistance R_S. The line is terminated with a load consisting of a resistance R_L. The line parameters such as length, spacing of the two conductors, and all

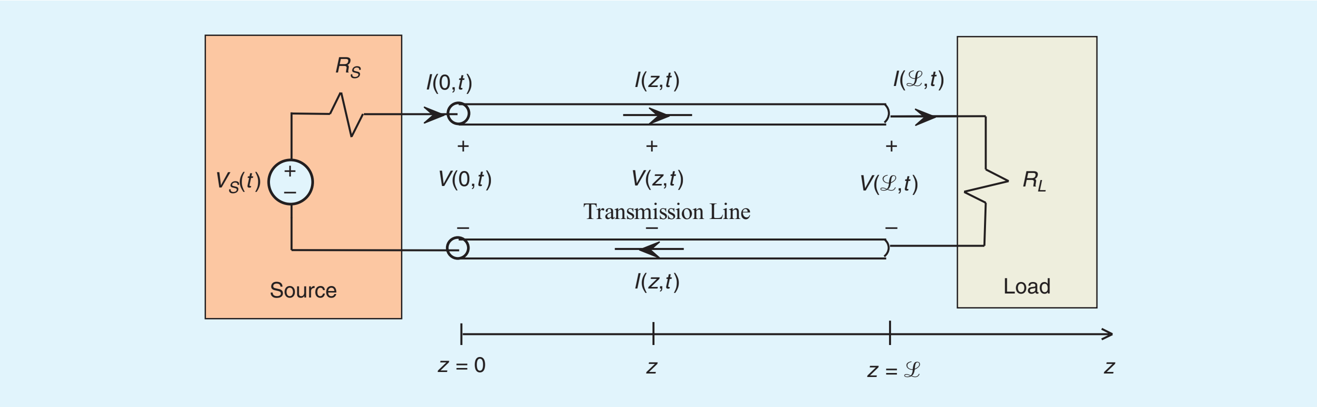

Fig. 1. The general configuration of the two-conductor, transmission lines to be studied.

conductor dimensions are assumed known. The terminal impedances R_S and R_L as well as the waveform of the open-circuit voltage source, $V_S(t)$, are also assumed known. The line voltage and current are functions of position along the line, z, and time, t, as $V(z,t)$ and $I(z,t)$. The task here is to produce the solution waveforms of the terminal voltages and currents: $V(0,t)$, $V(\mathscr{L},t)$, $I(0,t)$, and $I(\mathscr{L},t)$. This is the primary task in determining the *signal integrity* of a digital system.

Waves of voltage and current propagate along the conductors where they are reflected at the terminals. In order to sketch the terminal solution waveforms, we can trace the individual waves as they are reflected at the terminals and then sum in time all the waves present as illustrated in reference [1]. In this paper we will derive general equations for the terminal voltage and current waveforms in terms of symbols rather than numerical values for a specific problem so that the resulting equations apply for all problems and do not have to be rederived for a specific problem. The results are predetermined for any source waveform, $V_S(t)$ and don't require any further wave tracing or other derivations for a new problem. Replacing $V_S(t)$ with a different waveform but retaining the other problem parameters doesn't change the process.

II. The Transmisssion-Line Equations

The line voltages and currents, $V(z,t)$ and $I(z,t)$, are governed by the *transmission-line* equations [1]:

$$\frac{\partial V(z,t)}{\partial z} = -l\frac{\partial I(z,t)}{\partial t} \tag{1a}$$

$$\frac{\partial I(z,t)}{\partial z} = -c\frac{\partial V(z,t)}{\partial t} \tag{1b}$$

which are a set of *coupled*, partial differential equations. All cross-sectional dimensions of the line are contained in its per-unit-length inductance, l, and its per-unit-length capacitance, c. The electric and magnetic fields along the line lie in the x-y plane transverse to the axis of propagation, the z axis. Hence the field structure is referred to as the Transverse Electro Magnetic (TEM) mode of propagation. The velocity of propagation of the TEM waves along the line is

$$v = \frac{1}{\sqrt{lc}} \quad \frac{\text{m}}{\text{s}} \tag{2}$$

So we can obtain one parameter from the other:

$$l = \frac{1}{cv^2} \quad \frac{\text{H}}{\text{m}} \tag{3a}$$

$$c = \frac{1}{lv^2} \quad \frac{\text{F}}{\text{m}} \tag{3b}$$

The general solution of the transmission line equations is

$$V(z,t) = V^+\left(t - \frac{z}{v}\right) + V^-\left(t + \frac{z}{v}\right) \tag{4a}$$

$$I(z,t) = \frac{1}{Z_C}V^+\left(t - \frac{z}{v}\right) - \frac{1}{Z_C}V^-\left(t + \frac{z}{v}\right) \tag{4b}$$

where Z_C is the *characteristic impedance of the line*:

$$Z_C = \sqrt{\frac{l}{c}} \;\; \Omega = vl = \frac{1}{vc} \tag{5}$$

The V^+ and V^- are, as yet, undetermined *functions* but depend on z, t, and v only as $t + (z/v)$ and $t - (z/v)$. These functions are determined by the source and load: $V_S(t)$, R_S, and R_L. Also note that there is an important negative sign in the solution for the current. The V^+ represent forward-traveling waves traveling in the $+z$ direction, whereas the V^- represent backward-traveling waves traveling in the $-z$ direction. So in general we have waves of voltage and current (or equivalently waves of electric and magnetic fields) traveling back and forth down the line. We see that the voltage and current waves are in general being reflected at the source and at the load, and the combination of these waves determine the total voltage and current solution waveforms at the source and the load ends of the line.

III. Reflections At the Line Terminations

The time required to transit the line from one end to the other is the *one-way time delay* on the line:

$$T_D = \frac{\mathscr{L}}{v} \tag{6}$$

At the load end of the line, $z = \mathscr{L}$, the voltages and currents are denoted as

$$V(\mathscr{L},t) = V^+(t - T_D) + V^-(t + T_D) \tag{7a}$$

$$I(\mathscr{L},t) = \frac{1}{Z_C}V^+(t - T_D) - \frac{1}{Z_C}V^-(t + T_D) \tag{7b}$$

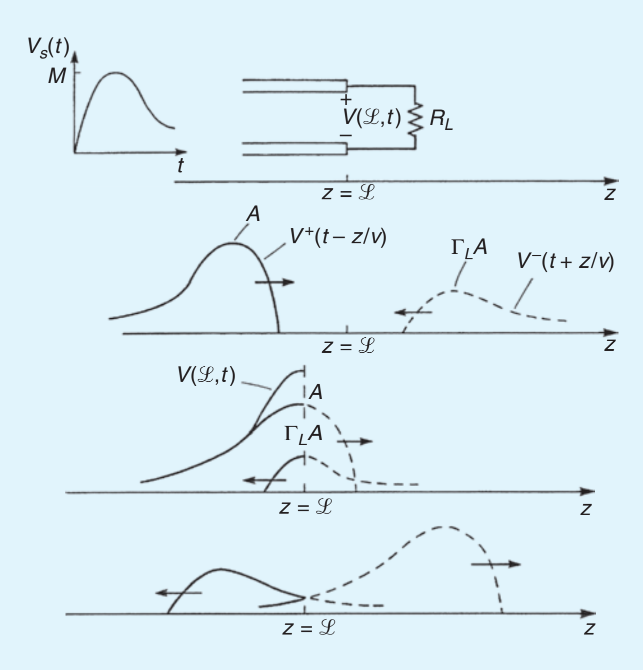

Fig. 2. Illustration of reflection at a mismatched load.

If the load is *matched*, i.e., $R_L = Z_C$, then we will only have a forward-traveling (incoming) wave at the load and there will be no reflected wave at the load. But for some general load that is not matched, $R_L \neq Z_C$, we must have an incident (forward-traveling) wave and a reflected (backward-traveling) wave at the load in order to satisfy Ohm's law. Define the *voltage reflection coefficient at the load* as the ratio of the reflected and incident voltage waves:

$$\Gamma_L = \frac{V^-(t + T_D)}{V^+(t - T_D)} \tag{8}$$

If we know the load reflection coefficient, Γ_L, we can determine the reflected voltage wave knowing the incident voltage wave. The *total* voltage and current at the load can then be written in terms of the load reflection coefficient as

$$\begin{aligned} V(\mathcal{L}, t) &= V^+(t - T_D)\,[1 + \Gamma_L] \\ I(\mathcal{L}, t) &= \frac{1}{Z_C} V^+(t - T_D)\,[1 - \Gamma_L] \end{aligned} \tag{9}$$

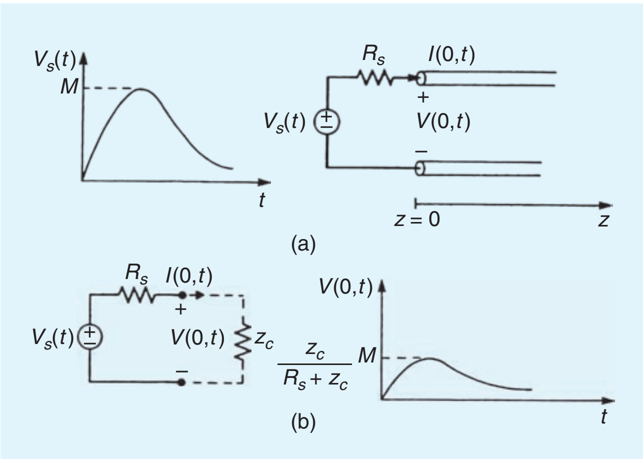

Fig. 3. Illustration of the input impedance to the line for $0 \le t < 2T_D$.

Taking the ratio of these two relations gives

$$\frac{V(\mathcal{L}, t)}{I(\mathcal{L}, t)} = R_L = Z_C \left[\frac{1 + \Gamma_L}{1 - \Gamma_L}\right] \tag{10}$$

Solving this gives the *voltage reflection coefficient at the load* as

$$\Gamma_L = \frac{R_L - Z_C}{R_L + Z_C} \tag{11}$$

Observe that since there is a minus sign in the current relation, the current reflection coefficient is the negative of the voltage reflection coefficient:

$$\Gamma_L\big|_{\text{current}} = -\Gamma_L\big|_{\text{voltage}} \tag{12}$$

The process of reflection at the load is like a mirror: the reflected wave is coming out of the mirror and the incident wave is going in as illustrated in Fig. 2. *The total voltage is the sum of the incident and reflected waves.*

The voltage or current wave that was reflected at the load travels back to the source in another time delay of T_D where it is reflected with a voltage reflection coefficient at the source of

$$\Gamma_S = \frac{R_S - Z_C}{R_S + Z_C} \tag{13}$$

and sent back to the load. The current reflection coefficient at the source is, again, the negative of the voltage reflection coefficient at the source:

$$\Gamma_S\big|_{\text{current}} = -\Gamma_S\big|_{\text{voltage}} \tag{14}$$

Finally we obtain the initially sent out wave. We reason that when the source voltage is initially applied, an initial forward-traveling wave is sent out towards the load. This initial wave will take a time delay of T_D to get to the load. Any reflections of this initial wave at a mismatched load will require another one-way time delay of T_D to get back to the source. Hence no reflected wave will have arrived at the source over the time interval $0 < t < 2T_D$. So the total voltage at the source is just the initially sent out forward traveling wave and hence the ratio of the total voltage to total current at the source end of the line, $z = 0$, will just be

$$\frac{V(0, t)}{I(0, t)} = \frac{V^+(t - 0)}{\frac{1}{Z_C} V^+(t - 0)} = Z_C \qquad 0 < t < 2T_D \tag{15}$$

So the input impedance to the line initially appears to be Z_C. Hence we can calculate the initially sent out voltage and current waves from

$$V_{\text{init}} = \frac{Z_C}{R_S + Z_C} V_S(t) \tag{16}$$

$$I_{\text{init}} = \frac{V_S(t)}{R_S + Z_C} \tag{17}$$

as illustrated in Fig. 3.

IV. Closed-Form General Solutions of the Terminal Voltages and Currents

We can trace the incident and reflected waves giving closed-form solutions for the terminal waveforms in terms of symbols

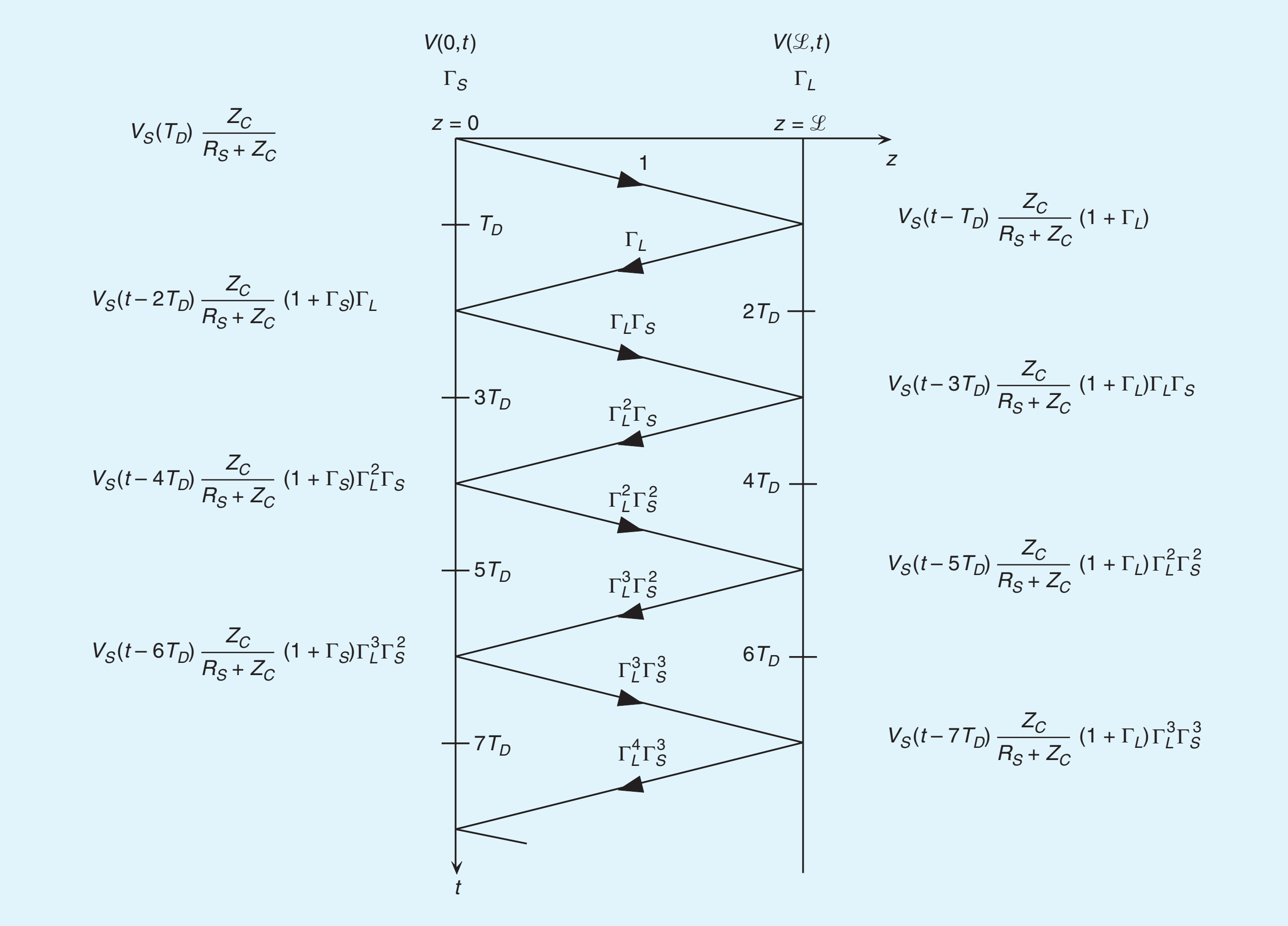

Fig. 4. The lattice diagram.

which do NOT have to be repeated for every different problem. To obtain these solutions we will use a form of a "lattice diagram" shown below in Fig. 4 that is *normalized* for a unity wave that is launched initially. Time is recorded on the vertical axis in increments of the one-way time delay T_D, and positions along the line are recorded on the horizontal axis. At a time point on the vertical axis where an incident and a reflected wave are present, the incident wave is multiplied by the reflection coefficient and the two waves are added.

The following TOTAL solutions have identical FORMS for ALL problems. For the terminal voltages these are

$$V(0,t) = \frac{Z_C}{R_S+Z_C}V_S(t) + \frac{Z_C}{R_S+Z_C}(1+\Gamma_S)\,\Gamma_L[V_S(t-2T_D) + (\Gamma_S\Gamma_L)V_S(t-4T_D) + (\Gamma_S\Gamma_L)^2V_S(t-6T_D) + \cdots] \tag{18a}$$

and

$$V(\mathscr{L},t) = \frac{Z_C}{R_S+Z_C}(1+\Gamma_L)\,[V_S(t-T_D) + (\Gamma_S\Gamma_L)V_S(t-3T_D) + (\Gamma_S\Gamma_L)^2V_S(t-5T_D) + (\Gamma_S\Gamma_L)^3V_S(t-7T_D) + \cdots] \tag{18b}$$

The terminal current solutions are similarly obtained from the voltage solutions but with the reflection coefficients for currents being the negative for those for the voltages as shown in (12) and (14) and the initially sent out wave is given by (17). Hence the symbolic solutions for the total terminal currents are

$$I(0,t) = \frac{1}{R_S+Z_C}V_S(t) + \frac{1}{R_S+Z_C}(1-\Gamma_S)(-\Gamma_L)[V_S(t-2T_D) + (\Gamma_S\Gamma_L)V_S(t-4T_D) + (\Gamma_S\Gamma_L)^2V_S(t-6T_D) + \cdots] \tag{19a}$$

and

$$I(\mathscr{L},t) = \frac{1}{R_S+Z_C}(1-\Gamma_L)\,[V_S(t-T_D) + (\Gamma_S\Gamma_L)V_S(t-3T_D) + (\Gamma_S\Gamma_L)^2V_S(t-5T_D) + (\Gamma_S\Gamma_L)^3V_S(t-7T_D) + \cdots] \tag{19b}$$

where Γ_S and Γ_L in the current expressions are the voltage reflection coefficients but their signs reversed.

Observe in these expressions that the total voltages and currents at the input and the output to the transmission line are combinations of the source waveform, $V_S(t)$, that are delayed by two time delays. Also note that the magnitudes of the source and load reflection coefficients are less than or equal to unity:

$$|\Gamma_S| \le 1 \tag{20a}$$

$$|\Gamma_L| \le 1 \tag{20b}$$

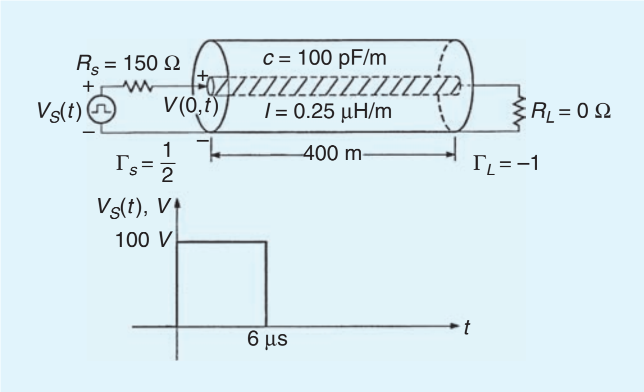

Fig. 5. An example.

Observe that the total terminal voltage waveforms, $V(0, t)$ and $V(\mathcal{L}, t)$, in (18) and total terminal current waveforms, $I(0, t)$ and $I(\mathcal{L}, t)$, in (19) are sums of delayed replicas of $V_S(t)$ multiplied by products of the source and load reflection coefficients, $(\Gamma_S \Gamma_L)^n$, which are also progressively less than unity. Hence if the source resistor is less than the characteristic impedance, $R_S < Z_C$, and the load resistor is greater than the load resistor, $R_L > Z_C$, or vice-versa, the source and load reflection coefficients are of *opposite sign*. Hence a resulting terminal voltage will have a portion added to it and subtracted from it resulting in *oscillations*. On the other hand if the source resistor and the load resistor are both less than the characteristic impedance, $R_S < Z_C$, $R_L < Z_C$, or are both greater than the characteristic impedance, $R_S > Z_C$, $R_L > Z_C$, the source and load reflection coefficients are of the *same sign* and the terminal voltage will steadily build up to its steady-state value. These observations apply also to the terminal currents.

The voltage results in (18) are multiplied by a factor representing the voltage division at the source that was used to determine the initially sent out voltage: $Z_C/(R_S + Z_C)$. Similarly the current results in (19) are multiplied by $1/(R_S + Z_C)$. Finally, each result is multiplied by a constant: $(1 + \Gamma_S)\,\Gamma_L$ for $V(0, t)$ and $(1 + \Gamma_L)$ for $V(\mathcal{L}, t)$ and $(1 - \Gamma_S)\,(-\Gamma_L)$ for $I(0, t)$ and $(1 - \Gamma_L)$ for $I(\mathcal{L}, t)$ where these current coefficients result from negating the voltage reflection coefficients to give the corresponding current reflection coefficients. Once

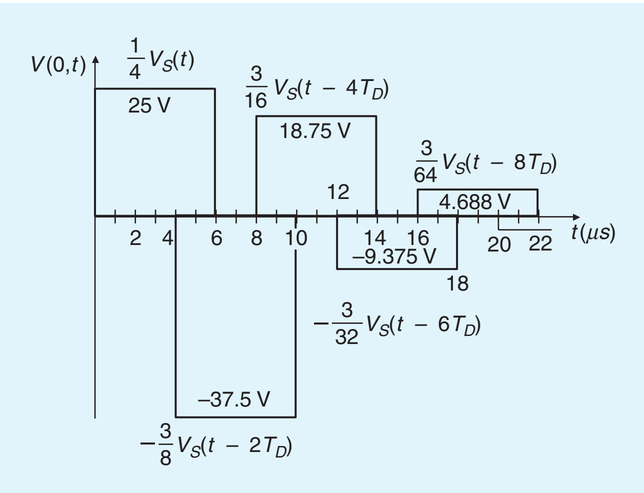

Fig. 6.

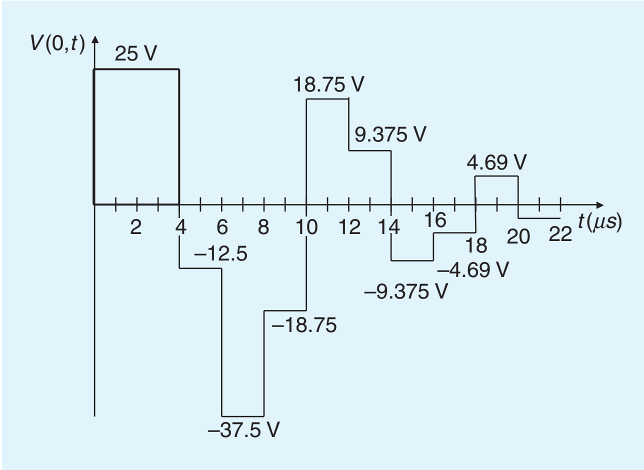

Fig. 7.

these coefficients are determined, the component waveforms are plotted in terms of the delayed source waveform $V_S(t)$. Once this is completed, these waveforms are summed to give the total waveforms $V(0, t)$, $V(\mathcal{L}, t)$, $I(0, t)$, and $I(\mathcal{L}, t)$ and the analysis is complete.

V. Examples

Sketch the voltage at the input, $V(0, t)$, and the current at the output, $I(\mathcal{L}, t)$ of the line versus time for the problem in Fig. 5.

This problem illustrates the case where the source voltage waveform, $V_S(t)$, is a pulse of amplitude of 100 V and 6 μs duration that is several one-way time delays of the line, $T_D = \mathcal{L}/v = 2\ \mu$s, in duration. Hence the incident and reflected pulses from opposite terminations overlap in time and combine to give very complicated total wave shapes at those terminations. First perform the initial computations:

$$Z_C = \sqrt{\frac{l}{c}} = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = 50 \quad \Omega$$

$$v = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{(0.25 \times 10^{-6})(100 \times 10^{-12})}} = 200 \quad \frac{\text{m}}{\mu\,\text{s}}$$

$$T_D = \frac{\mathcal{L}}{v} = 2\ \mu\,\text{s}$$

Perform the initial computations for the voltage:

$$V_{\text{init}} = \frac{Z_C}{R_S + Z_C} V_S(t) = \frac{50}{150 + 50} 100 = 25\ \text{V}$$

$$\Gamma_S = \frac{R_S - Z_C}{R_S + Z_C} = \frac{150 - 50}{150 + 50} = \frac{1}{2}$$

$$\Gamma_L = \frac{R_L - Z_C}{R_L + Z_C} = \frac{0 - 50}{0 + 50} = -1$$

For this example we compute the factors $Z_C/(R_S + Z_C) = 1/4$, $(1 + \Gamma_S)\,\Gamma_L = -3/2$ for $V(0, t)$ and $(1 + \Gamma_L) = 0$ for $V(\mathcal{L}, t)$. For the plot of $V(0, t)$ the series expression in (18a) becomes

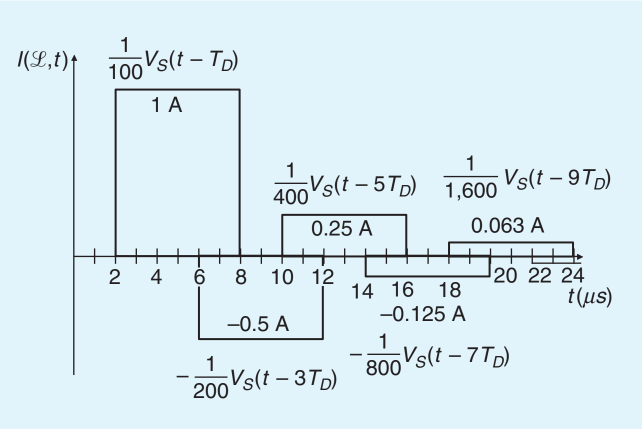

Fig. 8.

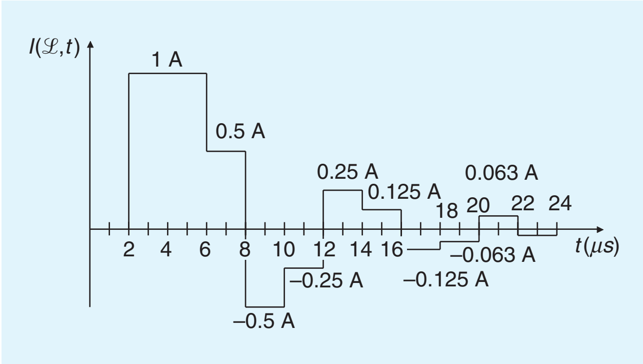

Fig. 9

$$V(0,t) = \frac{1}{4}V_S(t) - \frac{3}{8}\Big[V_S(t-2T_D) - \frac{1}{2}V_S(t-4T_D) + \frac{1}{4}V_S(t-6T_D) - \frac{1}{8}V_S(t-8T_D) + \cdots\Big]$$

$$= \frac{1}{4}V_S(t) - \frac{3}{8}V_S(t-2T_D) + \frac{3}{16}V_S(t-4T_D) - \frac{3}{32}V_S(t-6T_D) + \frac{3}{64}V_S(t-8T_D) + \cdots$$

which is shown in Fig. 6:

Adding the pulses gives the solution for $V(0,t)$ shown in Fig. 7.

The series solution for voltages in (18) can be easily modified for currents as in (19) by (1) negating the voltage reflection coefficients to give the current reflection coefficients AND (2) using

$$I_{\text{init}} = \frac{1}{R_S + Z_C}V_S(t) \tag{21}$$

as shown in (19). For the plot of current the coefficients are $1/(R_S + Z_C) = 1/200$, $(1 + -\Gamma_S)(-\Gamma_L) = 1/2$, for $I(0,t)$ and $(1 + -\Gamma_L) = 2$ for $I(\mathcal{L},t)$. The series expression for $I(\mathcal{L},t)$ in (19b) becomes

$$I(\mathcal{L},t) = \frac{1}{200}2\Big[V_S(t-T_D) - \frac{1}{2}V_S(t-3T_D) + \frac{1}{4}V_S(t-5T_D) - \frac{1}{8}V_S(t-7T_D) + \cdots\Big]$$

$$= \frac{1}{100}V_S(t-T_D) - \frac{1}{200}V_S(t-3T_D) + \frac{1}{400}V_S(t-5T_D) - \frac{1}{800}V_S(t-7T_D) + \cdots$$

Figure 8 shows this summation in terms of the source pulse, $V_S(t)$.

Adding the pulses gives the solution for $I(\mathcal{L},t)$ shown in Fig. 9.

As another example and one in which $V_S(t)$ is complicated, suppose $V_S(t)$ is again a pulse of 100 V and duration of 6 μs but steadily ramps from 0 V at $t = 0$ s to 100 V at $t = 6$ μs at which time it goes to zero as shown in Fig. 10.

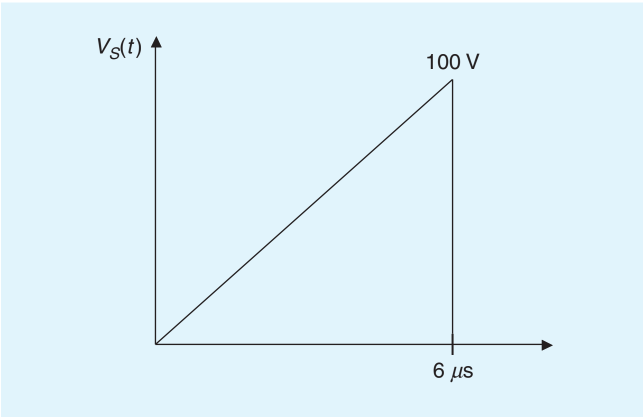

Fig. 10.

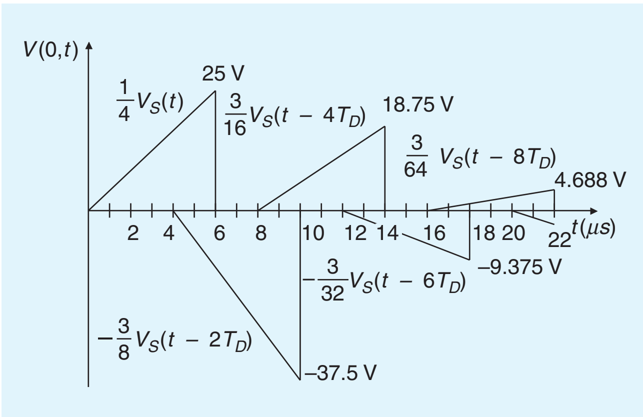

Fig. 11.

Since none of the parameters except the waveform for $V_S(t)$ has been changed, the equation for $V(0,t)$ is *unchanged*:

$$V(0,t) = \frac{1}{4}V_S(t) - \frac{3}{8}\Big[V_S(t-2T_D) - \frac{1}{2}V_S(t-4T_D) + \frac{1}{4}V_S(t-6T_D) - \frac{1}{8}V_S(t-8T_D) + \cdots\Big]$$

$$= \frac{1}{4}V_S(t) - \frac{3}{8}V_S(t-2T_D) + \frac{3}{16}V_S(t-4T_D) - \frac{3}{32}V_S(t-6T_D) + \frac{3}{64}V_S(t-8T_D) + \cdots$$

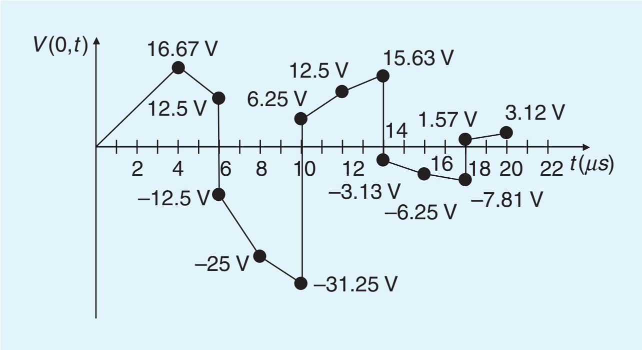

Fig. 12.

Plotting the individual components is shown in Fig. 11.

Adding the pulses gives the solution for $V(0, t)$ shown in Fig. 12.

All of the results of the above examples were validated using the PSPICE circuit analysis computer program. Use of the *exact lossless transmission line* model in PSPICE for more complicated *lossless* transmission-line problems is highly recommended [1].

VI. References

[1] C.R. Paul, "Transmission Lines in Digital and Analog Electronic Systems: Signal Integrity and Crosstalk," John Wiley, Hoboken, NJ, 2010.

Biography

Clayton R. Paul *received the B.S. degree, from The Citadel, Charleston, SC, in 1963, the M.S. degree, from Georgia Institute of Technology, Atlanta, GA, in 1964, and the Ph.D. degree, from Purdue University, Lafayette, IN, in 1970, all in Electrical Engineering. He is an Emeritus Professor of Electrical Engineering at the University of Kentucky where he was a member of the faculty in the Department of Electrical Engineering for 27 years retiring in 1998. Since 1998 he has been the Sam Nunn Eminent Professor of Aerospace Systems Engineering and a Professor of Electrical and Computer Engineering in the Department of Electrical and Computer Engineering at Mercer University in Macon, GA. He has published numerous papers on the results of his research in the Electromagnetic Compatibility (EMC) of electronic systems and given numerous invited presentations. He has also published 18 textbooks and Chapters in 4 handbooks. Dr. Paul is a Life Fellow of the Institute of Electrical and Electronics Engineers (IEEE) and is an Honorary Life Member of the IEEE EMC Society. He was awarded the IEEE Electromagnetics Award in 2005 and the IEEE Undergraduate Teaching Award in 2007.*