Mathematical Programming Techniques in Multiobjective Optimization

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1. Introduction
   - Problem Formulation and Definitions of Optimality
2. Finding Efficient Solutions – Scalarization
   - The Idea of Scalarization
   - Scalarization Techniques and Their Properties
3. Multiobjective Linear Programming
   - Formulation and the Fundamental Theorem
   - Solving MOLPs in Decision and Objective Space
4. Multiobjective Combinatorial Optimization
   - Definitions Revisited and Characteristics
   - Solution Methods
5. Applications
6. Commercials

Overview

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Mathematical Formulation

$$\begin{aligned} \min f(x) & \\ \text{subject to } g(x) &\leqq 0 \\ x &\in \mathbb{R}^n \end{aligned}$$

$x \in \mathbb{R}^n \longrightarrow n$ variables, $i = 1, \ldots, n$

$g : \mathbb{R}^n \to \mathbb{R}^m \longrightarrow m$ constraints, $j = 1, \ldots, m$

$f : \mathbb{R}^n \to \mathbb{R}^p \longrightarrow p$ objective functions, $k = 1, \ldots, p$

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Feasible Sets

- $X = \{x \in \mathbb{R}^n : g(x) \leqq 0\}$
  feasible set in decision space
- $Y = f(X) = \{f(x) : x \in X\}$
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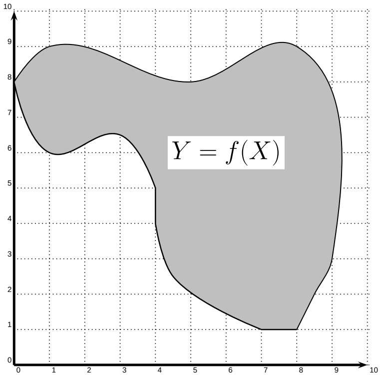

Notation

- $y^1 \leqq y^2 \Leftrightarrow y^1_k \leqq y^2_k$ for $k = 1, \ldots, p$
- $y^1 < y^2 \Leftrightarrow y^1_k < y^2_k$ for $k = 1, \ldots, p$
- $y^1 \leq y^2 \Leftrightarrow y^1 \leqq y^2$ and $y^1 \neq y^2$
- $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$
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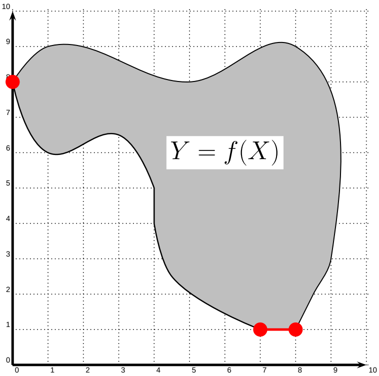
Lexicographic Optimality

- Individual minima
  $f_k(\hat{x}) \leqq f_k(x)$ for all $x \in X$
- Lexicographic optimality (1)
  $f(\hat{x}) \leq_{lex} f(x)$ for all $x \in X$
- Lexicographic optimality (2)
  $f^\pi(\hat{x}) \leq_{lex} f^\pi(x)$ for all $x \in X$ and some permutation f^π of $(f_1, \ldots, f_p)$



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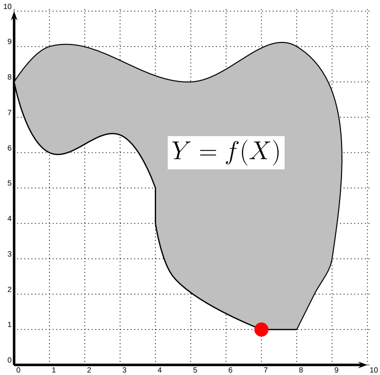

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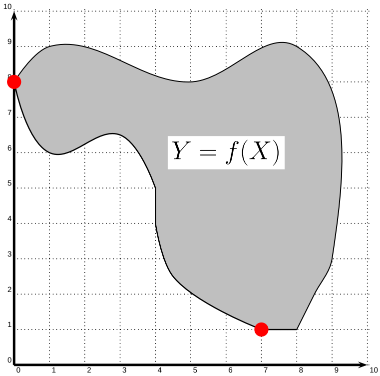

(Weakly) Efficient Solutions

- Weakly efficient solutions X_{wE}
  There is no x with $f(x) < f(\hat{x})$
  $f(\hat{x})$ is weakly nondominated
  $Y_{wN} := f(X_{wN})$
- Efficient solutions X_E
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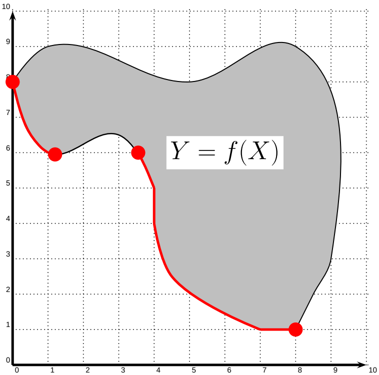

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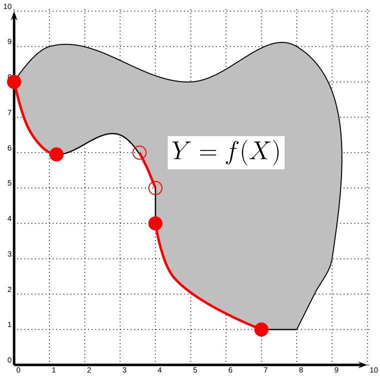

Properly Efficient Solutions

- Properly efficient solutions X_{pE}
  - $\hat{x}$ is efficient
  - There is $M > 0$ such that for each k and x with $f_k(x) < f_k(\hat{x})$ there is l with $f_l(\hat{x}) < f_l(x)$ and

$$\frac{f_k(\hat{x}) - f_k(x)}{f_l(x) - f_l(\hat{x})} \leq M$$

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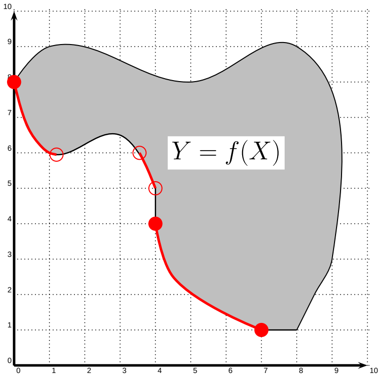

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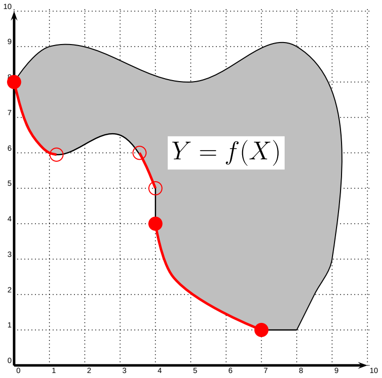
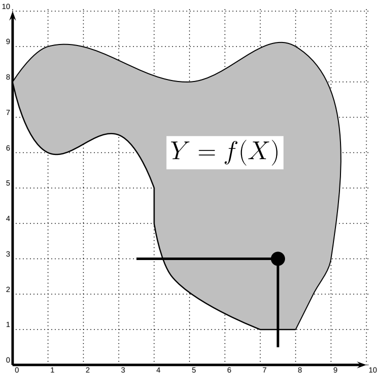


Existence

- $Y_N \neq \emptyset$ if for some $y^0 \in Y$ the section $\left(y^0 - \mathbb{R}_{\geqq}\right) \cap Y \neq \emptyset$ is compact
- $X_E \neq \emptyset$ if X is compact and f is ($\mathbb{R}_{\geqq}$-semi-)continuous



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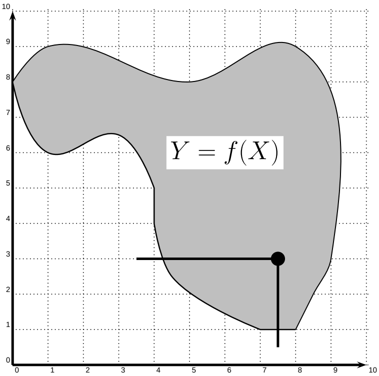
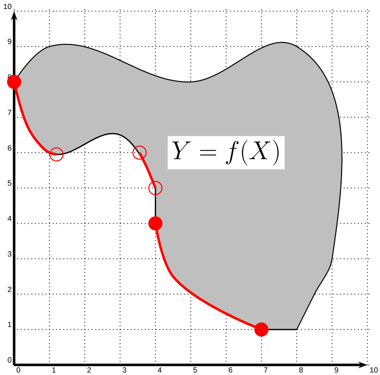


Relationships of Solution Sets

$X_{pE} \subseteq X_E \subseteq X_{wE}$
$Y_{pN} \subseteq Y_N \subseteq Y_{wN}$

It is possible that
$Y_N = Y$ but $Y_{pN} = \emptyset$
$Y = \left\{(y_1, y_2) : y_2 = \frac{1}{y_1}, y_1 < 0\right\}$



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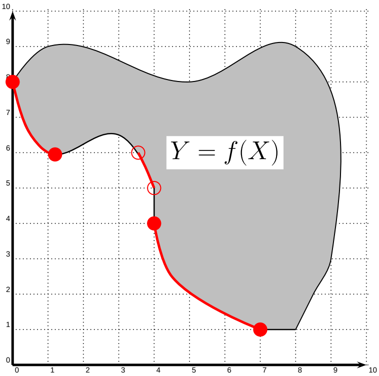

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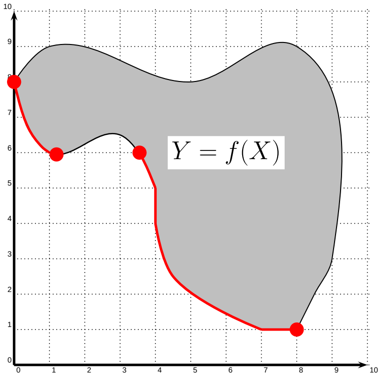

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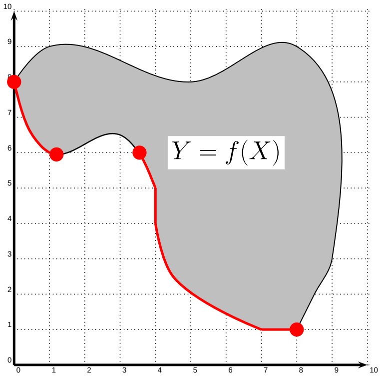

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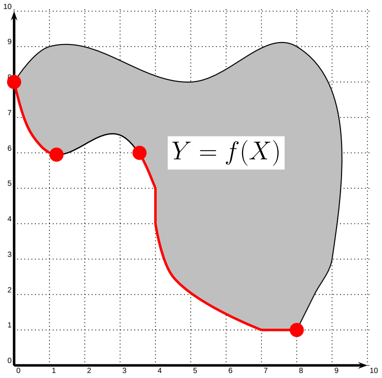

Ideal and Nadir Points

Ideal point y^I

- $y^I_k = \min\{y_k : y \in Y\}$

Nadir point y^N

- $y^N_k = \min\{y_k : y \in Y_N\}$

Anti-ideal point y^{AI}

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Utopia point y^U

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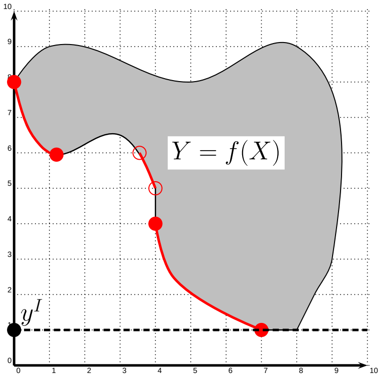

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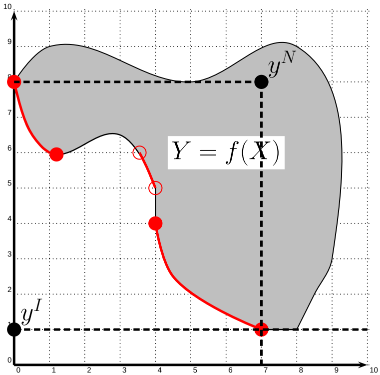

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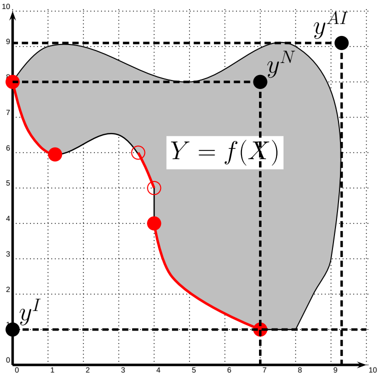

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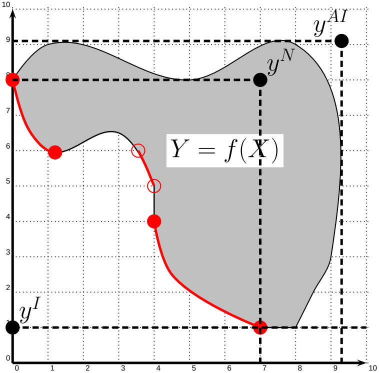
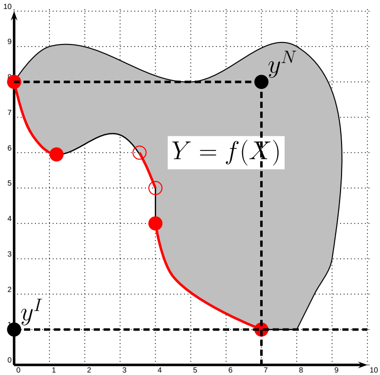


General Assumptions

- X_E is non-empty
- $y^I \neq y^N$



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Principle of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations

- Correctness: Optimal solutions are (weakly, properly) efficient
- Completeness: All (weakly, properly) efficient solutions can be found

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- Aggregate objectives
- Convert objectives to constraints
- Minimize distance to ideal point

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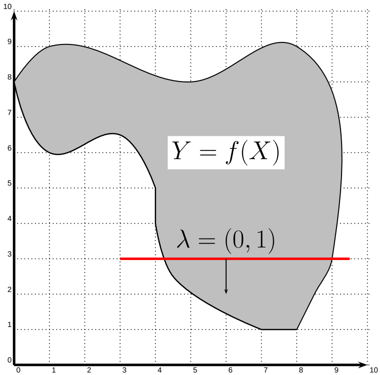
The Weighted Sum Method

Let $\lambda \geq 0$

$$\min \left\{ \sum_{k=1}^{p} \lambda_k f_k(x) : x \in X \right\} \quad (1)$$



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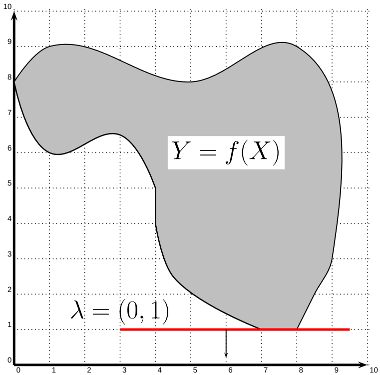

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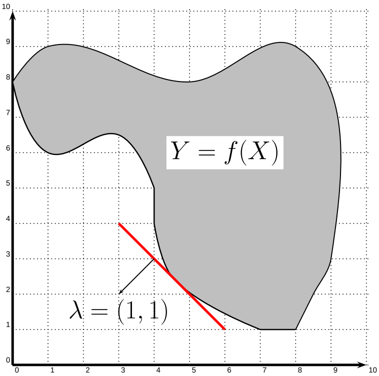

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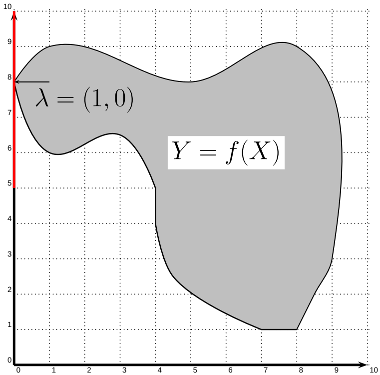

The Weighted Sum Method: Results

Theorem

Let $\hat{x}$ be an optimal solution of (1).

1. *If $\lambda \geq 0$ then $\hat{x} \in X_{wE}$.*
2. *If $\lambda \geq 0$ and $f(\hat{x})$ is unique then $\hat{x} \in X_E$.*
3. *If $\lambda > 0$ then $\hat{x} \in X_{pE}$.*

Proof.

1. By contradiction
2. By contradiction
3. Construct M so that larger tradeoff would contradict optimality of $\hat{x}$

□

The Weighted Sum Method: Results

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The Weighted Sum Method: Results

Theorem (Geoffrion 1968)

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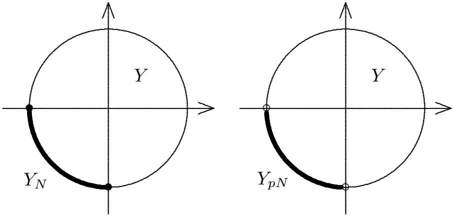
Nondominated and Properly Nondominated Points



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Theorem

Assume that $Y + \mathbb{R}^p_{\geqq}$ is closed and convex. Then

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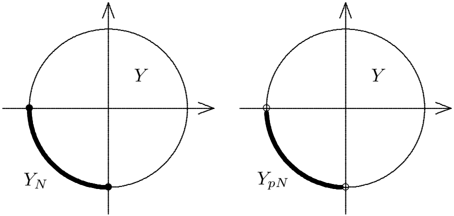

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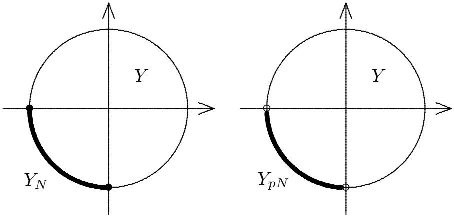

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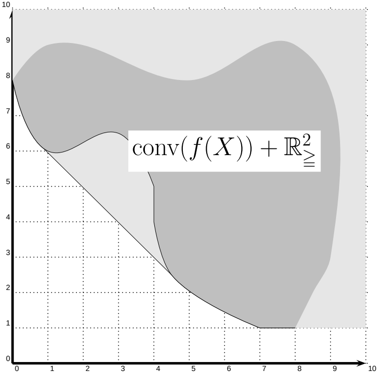
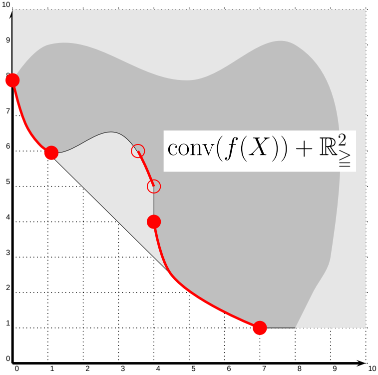
Supported Efficient Solutions

Supported efficient solutions are efficient solutions with $f(x)$ on the convex hull of Y

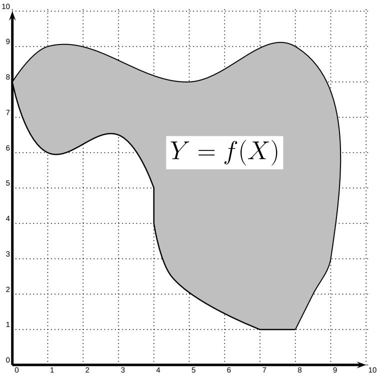




The ε-constraint Method

Let $\varepsilon \in \mathbb{R}^p$

$$\begin{aligned} \min\ & f_l(x) & & & \\ \text{s.t.}\ & f_k(x) & \leqq & \ \varepsilon_k & k \neq l \\ & g_j(x) & \leqq & \ 0 & j = 1, \ldots, m \end{aligned} \qquad (2)$$



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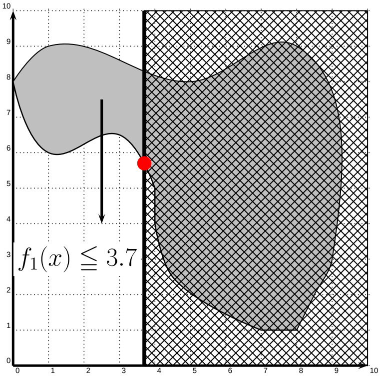

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Let $\lambda \in \mathbb{R}^p_{\geq}$ and $\varepsilon \in \mathbb{R}^p$

$$\min \sum_{k=1}^{p} \lambda_k f_k(x) \qquad (3)$$

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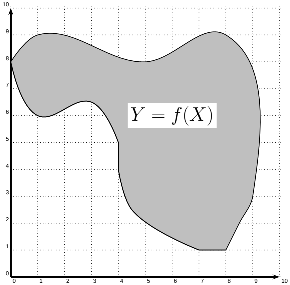

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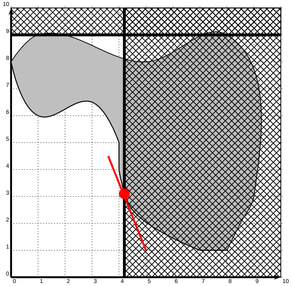

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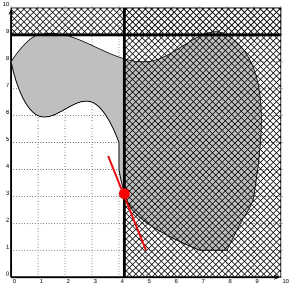

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Compromise Solutions

Let $\lambda \in \mathbb{R}^p_{\geq}$ and $1 \leq q < \infty$

$$\min_{x \in X} \left(\sum_{k=1}^{p} \lambda_k (f_k(x) - y^I_k)^q \right)^{\frac{1}{q}} \qquad (4)$$

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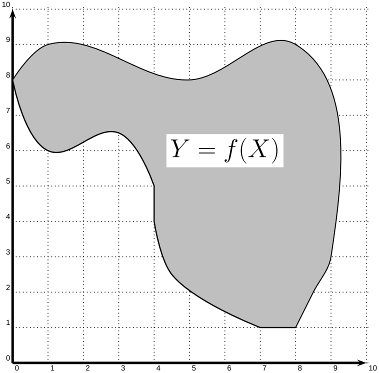

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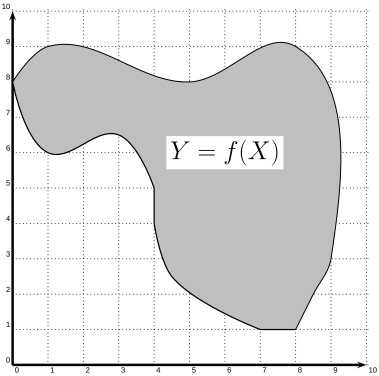

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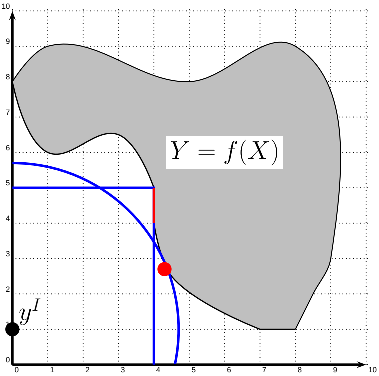

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- For $q = 1$ (4) is the weighted sum scalarization
- If y^I is replaced by y^U in (4) stronger results follow
  Solutions obtained are properly efficient, and Y_N is contained in the closure of the set of all solutions obtained (Sawaragi et al. 1985)
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More General Concepts

- l_q norms can be replaced by more general distance functions
- Ideal point can be replaced by a reference point and the distance function by a ((strictly, strongly) increasing) achievement function $\mathbb{R}^p \to \mathbb{R}$ (Wierzbicki 1986)

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Overview

1. Introduction
   - Problem Formulation and Definitions of Optimality
2. Finding Efficient Solutions – Scalarization
   - The Idea of Scalarization
   - Scalarization Techniques and Their Properties
3. Multiobjective Linear Programming
   - Formulation and the Fundamental Theorem
   - Solving MOLPs in Decision and Objective Space
4. Multiobjective Combinatorial Optimization
   - Definitions Revisited and Characteristics
   - Solution Methods
5. Applications
6. Commercials

MOLP Formulation

- $f(x) = Cx$ where $C \in \mathbb{R}^{p \times n}$
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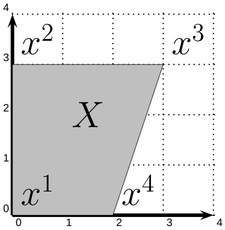

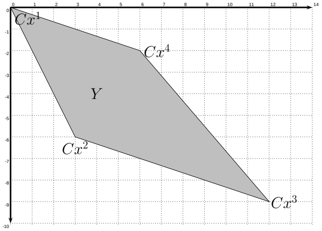

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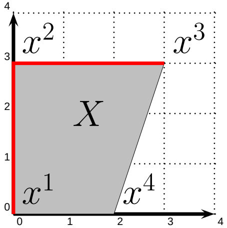

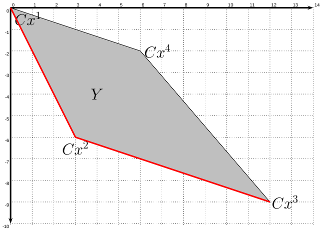

Theorem (Isermann 1974)

A feasible solution $\hat{x} \in X$ is efficient if and only if there is $\lambda > 0$ such that $\lambda^T C\hat{x} \leqq \lambda^T x$ for all $x \in X$.

Proof.

- If $\hat{x}$ is efficient, $\max\{e^T z : Ax = b, Cx + Iz = C\hat{x}; x, z \geqq 0\}$ has optimal solution $\hat{z} = 0$
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- By duality $\min\{u^T b + w^T C\hat{x} : u^T A = w^T C \geqq 0 : w \geqq e\}$ has optimal solution $(\hat{u}, \hat{w})$ with $\hat{u}^T b = -\hat{w}^T C\hat{x}$
- $\hat{u}$ is optimal solution of $\min\{u^T b : u^T A \geqq -\hat{w}^T C\}$
- By duality an optimal solution of $\max\{-\hat{w}^T Cx : Ax = b, x \geqq 0\}$ exists
- $\hat{x}$ is an optimal solution of this LP

□

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1. Introduction
   - Problem Formulation and Definitions of Optimality
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Multiobjective Simplex Algorithm

- Phase I: Feasibility
  MOLP is feasible if and only if
  $\min\{e^T z : Ax + Iz = b; x, z \geqq 0\}$ has optimal value 0
  Let $(x^0, \hat{z})$ be optimal solution
- Phase II: First efficient solution
  If $\min\{u^T b + w^T Cx^0 : u^T A = w^T C \geqq 0; w \geqq e\}$
  is infeasible then $X_E = \emptyset$
  Let $\hat{w}$ be optimal solution
  Optimal solution $\hat{x}$ to $\min\{\hat{w}^T Cx : Ax = b, x \geqq 0\}$ is efficient
- Phase III: Explore efficient solutions by identifying entering variables

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Multiobjective Simplex Algorithm

- Reduced cost matrix

$$R := (C - C_{\mathcal{B}} A_{\mathcal{B}}^{-1} A)_{\mathcal{N}}$$

- x_j is efficient nonbasic variable if there is $\lambda > 0$ such that $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$
- At every efficient basis there exists an efficient nonbasic variable and every feasible pivot leads to another efficient basis

Theorem (Evans and Steuer 1973)

Nonbasic variable x_j is efficient if and only if the LP

$$\max\{e^T v : Rz - r^j \delta + Iv = 0, z\delta, v \geqq 0\}$$

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- *The set of all efficient bases is connected by pivots with efficient entering variables.*
- *The set of all efficient extreme points of X_E is connected by efficient edges.*
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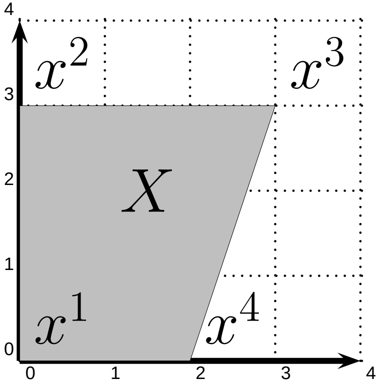

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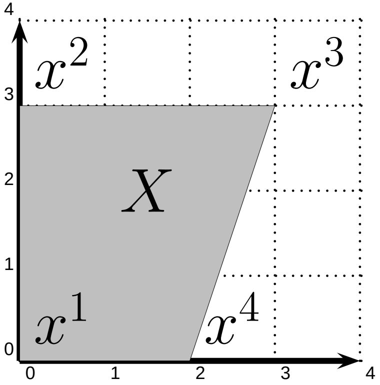

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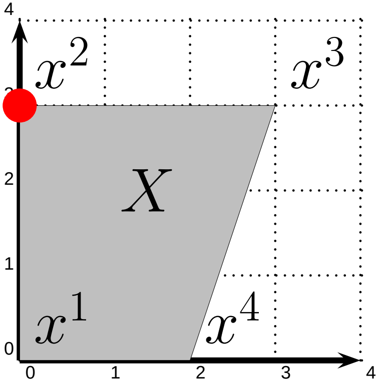

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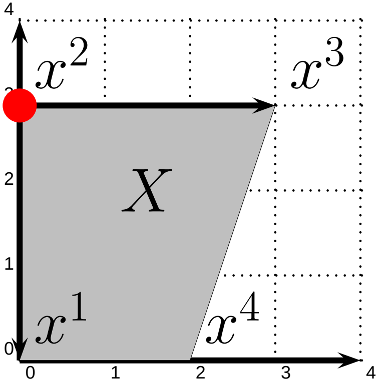

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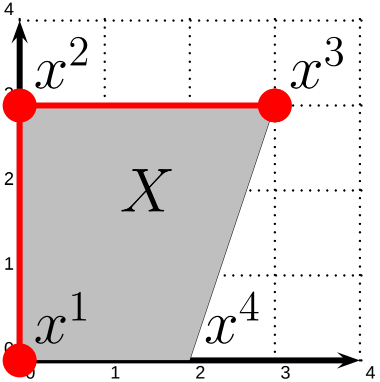

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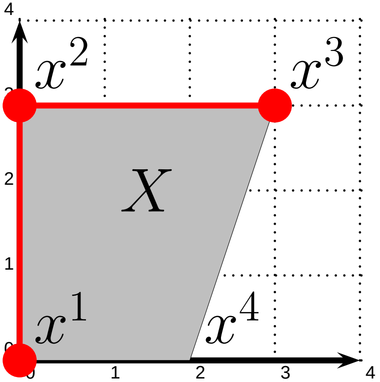

Solving MOLPs in Objective Space

(Benson 1998)

- Degeneracy causes problems for simplex algorithm
- Decisions based on objective function values
- Usually dim $Y \leqq p \ll$ dim X
- Assume X is bounded

Theorem (Benson 1998)

The dimension of $Y + \mathbb{R}^p_{\geqq}$ is p and $(Y + \mathbb{R}^p_{\geqq})_N = Y_N$.

$$Y' := (Y + \mathbb{R}^p_{\geqq}) \cap (y^{AI} - \mathbb{R}^p_{\geqq})$$

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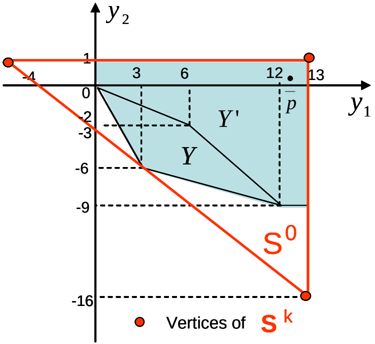
  Let $\hat{y}$ be solution of $\min\{e^T y : y \in Y\}$
  Simplex S^0 such that $Y' \subseteq S^0$ defined by axes-parallel hyperplanes and supporting hyperplane with normal e at $\hat{y}$
- While $S^k \neq Y'$
  - Find vertex y^k of S^{k-1} with $s^k \notin Y'$
  - Find $\alpha^k > 0$ such that $\alpha y^k + (1 - \alpha)\hat{p}$ is on the boundary of Y'
  - Find supporting hyperplane to Y' through boundary point



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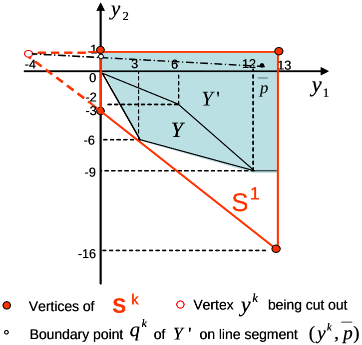

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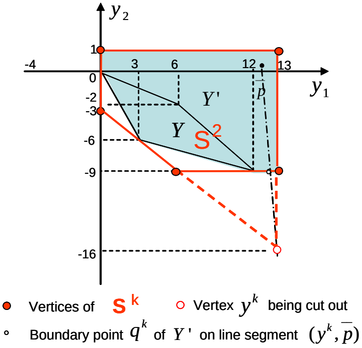

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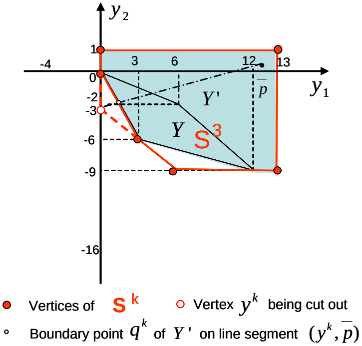

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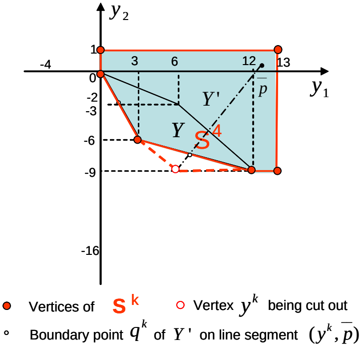

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Mathematical Formulation

$$\begin{aligned} \min z(x) &= Cx \\ \text{subject to } Ax &= b \\ x &\in \{0,1\}^n \end{aligned}$$

$x \in \{0,1\}^n \longrightarrow n$ variables, $i = 1, \ldots, n$

$C \in \mathbb{Z}^{p \times n} \longrightarrow p$ objective functions, $k = 1, \ldots, p$

$A \in \mathbb{Z}^{m \times n} \longrightarrow m$ constraints, $j = 1, \ldots, m$

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- $X = \{x \in \{0,1\}^n : Ax = b\}$
  feasible set in decision space
- $Y = z(X) = \{Cx : x \in X\}$
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- $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$

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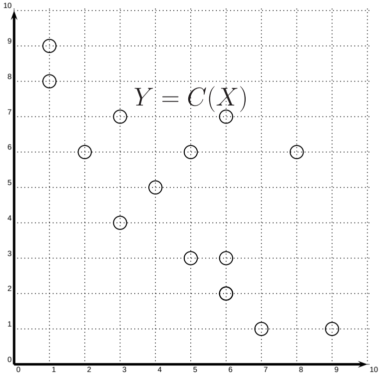

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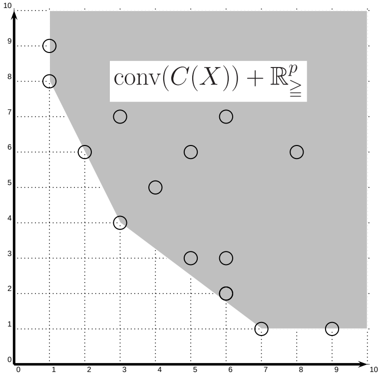
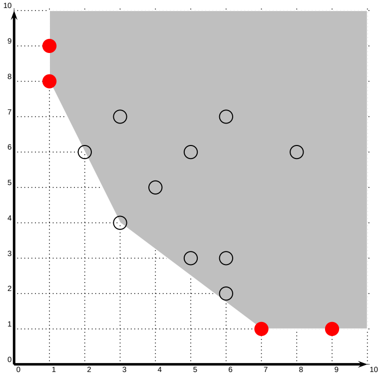


Lexicographic Optimality

- Individual minima
  $z_k(\hat{x}) \leqq z_k(x)$ for all $x \in X$
- Lexicographic optimality (1)
  $z(\hat{x}) \leq_{lex} z(x)$ for all $x \in X$
- Lexicographic optimality (2)
  $z^\pi(\hat{x}) \leq_{lex} z^\pi(x)$ for all $x \in X$ and some permutation z^π of $(z_1, \ldots, z_p)$



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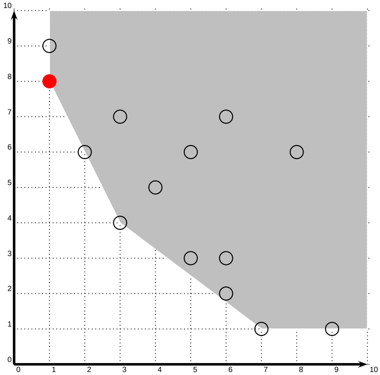

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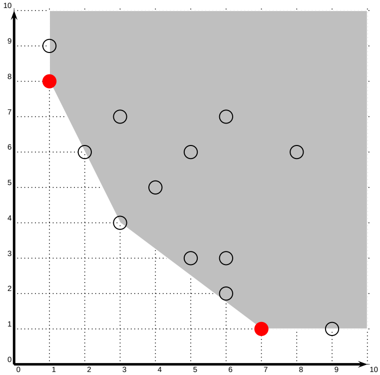

Efficient Solutions

- Weakly efficient solutions X_{wE}
  There is no x with $z(x) < z(\hat{x})$
  $z(\hat{x})$ is weakly nondominated
  $Y_{wN} := z(X_{wN})$
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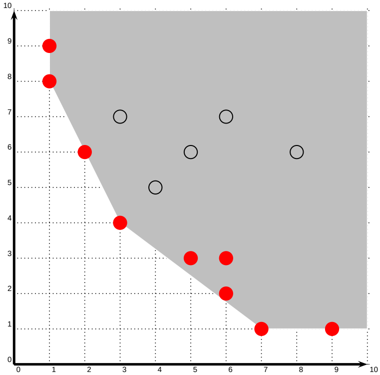

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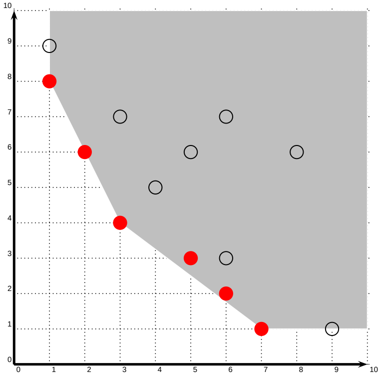

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- Supported efficient solutions X_{sE}: There is $\lambda > 0$ with $\lambda^T C\hat{x} \leqq \lambda^T Cx$ for all $x \in X$
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  - $C\hat{x}$ is in relative interior of face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq} \rightarrow X_{sE2}$
- Nonsupported efficient solutions X_{nE}: $C\hat{x}$ is in interior of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$

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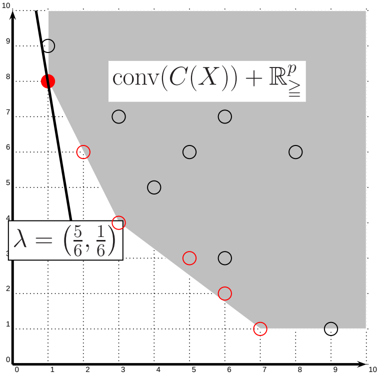

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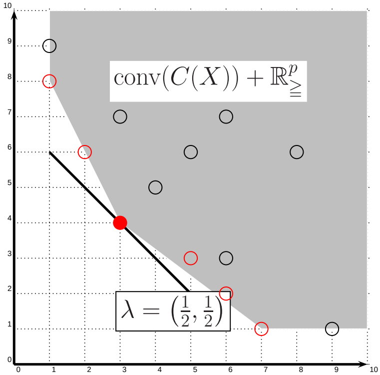

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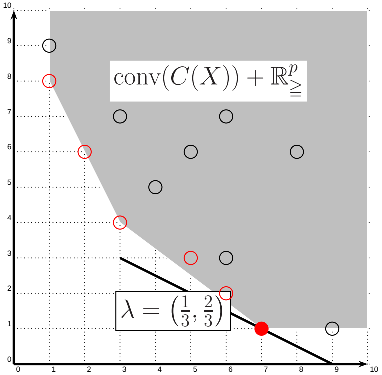

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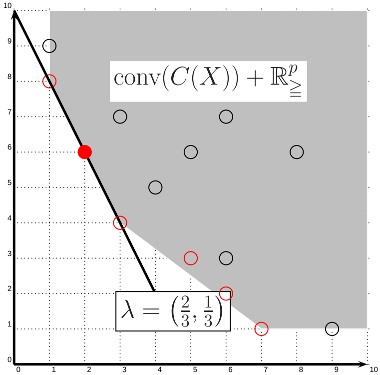

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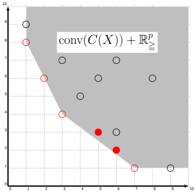

Classification of Efficient Sets

Hansen 1979:

- $x^1, x^2 \in X_E$ are equivalent if $Cx^1 = Cx^2$
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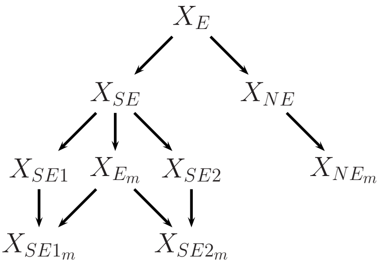

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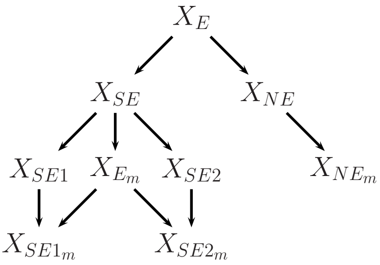

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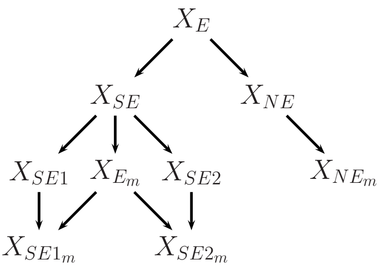

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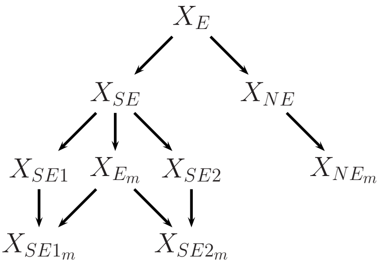

MOCO Problems Are Hard

Theorem

Multiobjective combinatorial optimization problems are NP-hard, #P-complete, and intractable.

Examples:

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Intractable: X_E, even Y_{sN}, can be exponential in the size of the instance

Empirically often

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But this depends on numerical values of C

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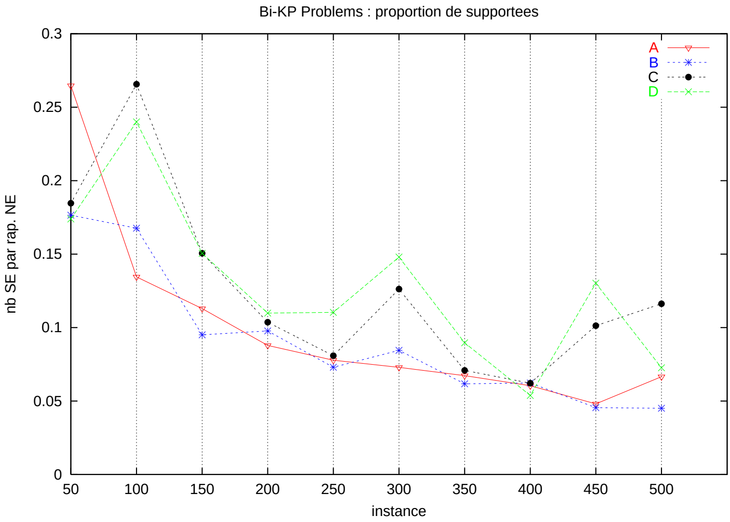
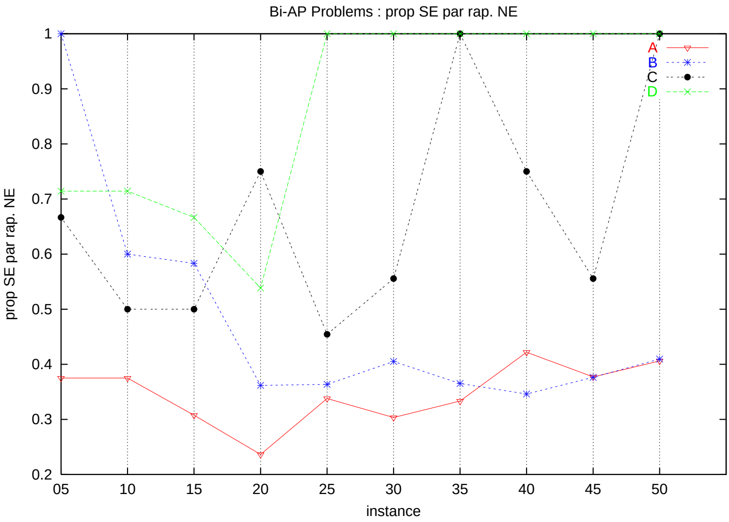

Bi-KP Problems : proportion de supportees
nb SE par rap. NE
instance
0.3
0.25
0.2
0.15
0.1
0.05
0
50
100
150
200
250
300
350
400
450
500
A
B
C
D


Bi-AP Problems : prop SE par rap. NE
prop SE par rap. NE
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
05
10
15
20
25
30
35
40
45
50
instance
A
B
C
D

Overview

1. Introduction
   - Problem Formulation and Definitions of Optimality
2. Finding Efficient Solutions – Scalarization
   - The Idea of Scalarization
   - Scalarization Techniques and Their Properties
3. Multiobjective Linear Programming
   - Formulation and the Fundamental Theorem
   - Solving MOLPs in Decision and Objective Space
4. Multiobjective Combinatorial Optimization
   - Definitions Revisited and Characteristics
   - Solution Methods
5. Applications
6. Commercials

Solving MOCO Problems

- Scalarization
- Single objective problem polynomially solvable and algorithm can be directly extended to multiple objectives
- Single objective problem polynomially solvable and ranking algorithm exists: The 2 Phase Method
- Single objective problem NP-hard: General integer programming methods

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Principle and Properties of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations: (Wierzbicki 1984)

- Correctness: Optimal solutions are (weakly) efficient
- Completeness: All efficient solutions can be found
- Computability: Scalarization is not harder than single objective version of problem (theory and practice)
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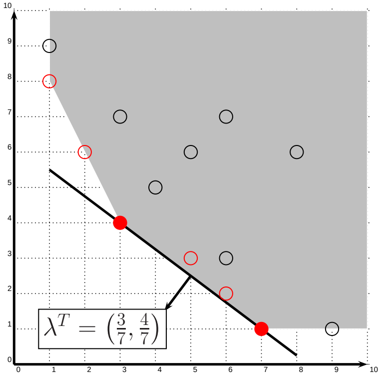
Scalarization Methods

- Weighted sum:
  $$\min_{x \in X} \left\{ \lambda^T z(x) \right\}$$
- ε-constraint:
  $$\min_{x \in X} \{ z_l(x) : z_k(x) \le \varepsilon_k, k \ne l \}$$
- Weighted Chebychev:
  $$\min_{x \in X} \left\{ \max_{k=1,\dots,p} \nu_k (z_k(x) - y_k^I) \right\}$$



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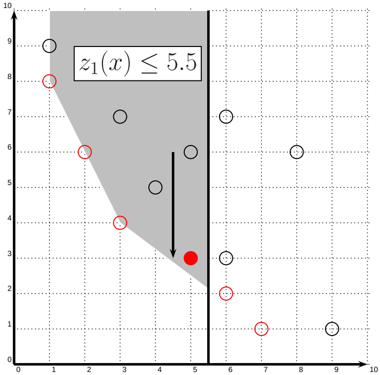

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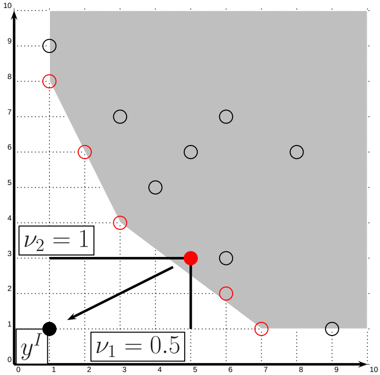

General Formulation

$$\min_{x \in X} \left\{ \max_{k=1}^{p} \left[\nu_k (c_k x - \rho_k) \right] + \sum_{k=1}^{p} \left[\lambda_k (c_k x - \rho_k) \right] \right\}$$

$$\text{subject to} \quad c_k x \leq \varepsilon_k \qquad k = 1, \ldots, p$$

Includes	Correct	Complete	Computable	Linear
Weighted sum	+	-	+	+
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General Formulation

Theorem (Ehrgott 2005)

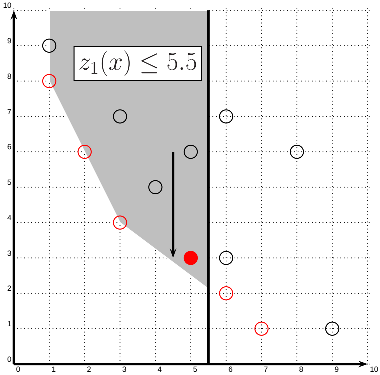
1. *The general scalarization is NP-hard.*
2. *An optimal solution of the Lagrangian dual of the linearized general scalarization is a supported efficient solution.*

Method of Elastic Constraints

$$\min_{x \in X} c_l x + \sum_{k \neq l} \mu_k w_k$$

$$\text{s.t.} \quad c_k x + v_k - w_k \leq \varepsilon_k \quad k \neq l$$

$$v_k, w_k \geq 0 \quad k \neq l$$



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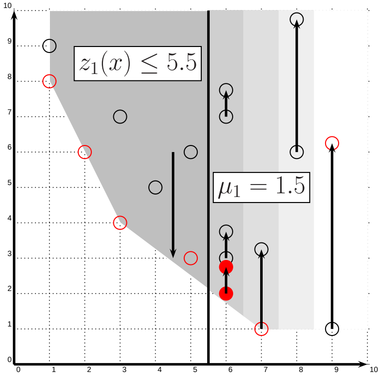

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Theorem (Ehrgott and Ryan 2002)

The method of elastic constraints

- *is correct and complete,*
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Integer Programming Duality

Theorem (Klamroth et al. 2004)

- $\hat{x} \in X_E$ *if and only if there is* $\hat{F} \in \mathcal{F} := \{F : \mathbb{R}^{m+p-1} \to \mathbb{R}$ *nondecreasing*$\}$ *such that* $\hat{x}$ *is an optimal solution to*

$$\max\left\{c_j x - \hat{F}\left((c_k x)_{k \neq j}, b\right) : Ax \leqq b, x \geqq 0, x \text{ integer}\right\}.$$

- $\hat{F}$ *can be chosen as an optimal solution of the IP dual* $\min\left\{F(-e, b) : F((-c_k x)_{k \neq j}, Ax) \geq c_j x \ \forall x \in \mathbb{Z}^n_{\geqq}, F \in \mathcal{F}\right\}$ *of* $\max\{c_l x : c_k x \geq \varepsilon_k, k \neq l, Ax = b, x \in \mathbb{Z}^n_{\geqq}\}$
- *The level curve of the objective function of the composite IP at level 0 defines an upper bound on* Y_N*.*

Direct Application of Single Objective Method

- The Shortest Path Problem
  - Shortest path from node s to node t in a directed graph
  - Labels are vectors, each node has set of labels
  - New labels deleted if dominated by another label
  - Labels dominated by new label dominated
- More general: Dynamic Programming
- The Spanning Tree Problem
  - Generalizations of Prim's and Kruskal's algorithms

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The Two Phase Method

- Phase 1: Compute X_{sE}
  1. Find lexicographic solutions
  2. Recursively:
     Calculate λ
     Solve $\min_{x \in X} \lambda^T Cx$
- Phase 2: Compute X_{nE}
  1. Solve by triangle
  2. Use neighborhood (wrong)
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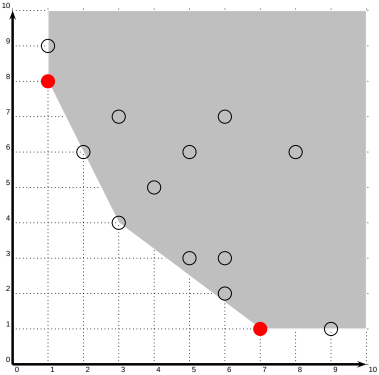

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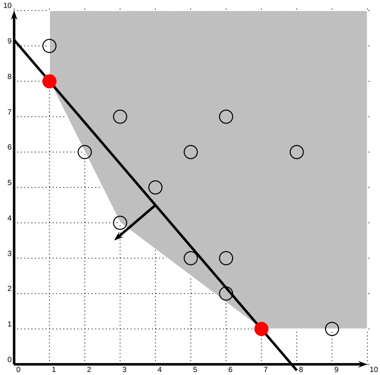

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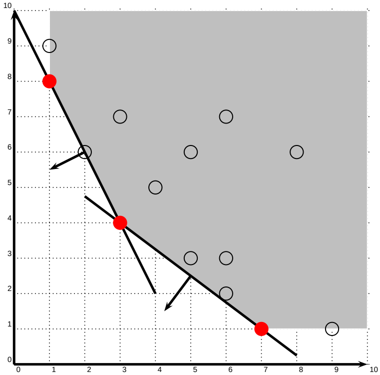

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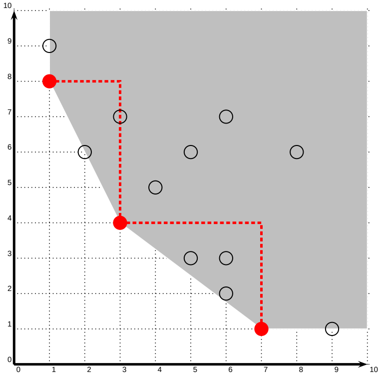

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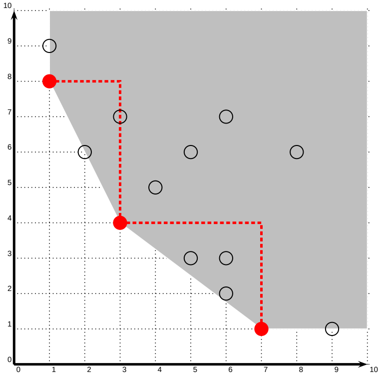

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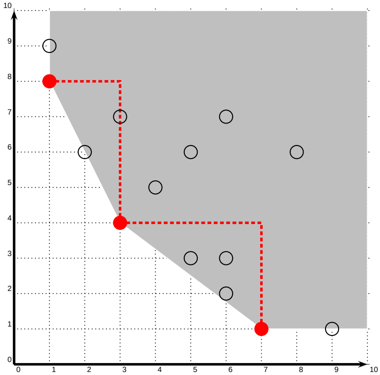

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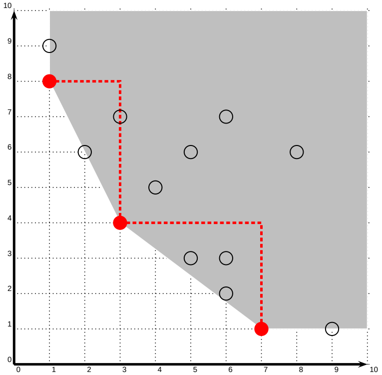

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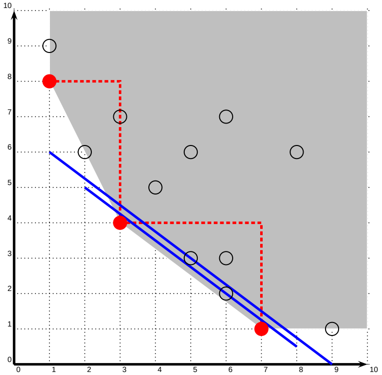

Enumeration Problems

- Finding maximal complete set:
  - Enumeration to find all optimal solutions of $\min_{x \in X} \lambda^T Cx$
  - Enumeration to find all $x \in X_{nE}$ with $Cx = y \in Y_{nD}$
- Finding minimal complete set:
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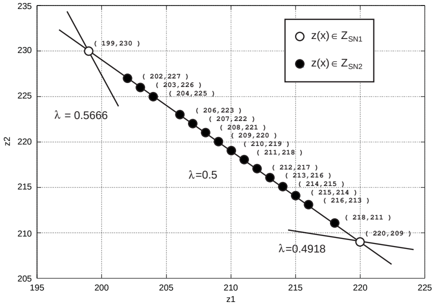

2 Phase Algorithm for Biobjective Assignment

(Przybylski et al. 2004)

- Hungarian Method for $\min_{x \in X} \lambda^T Cx$
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- Ranking of (non-optimal) solutions of $\min_{x \in X} \lambda^T Cx$ (Chegireddy and Hamacher 1987)

Results for 100×100:

Range	Variable Fixing	Seek & Cut	Ranking
20	14049.17	2755.75	220.07
40	$\times$	17441.35	225.06
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The 2 Phase Method for 3 Objectives

- Phase 1:
  - Dichotomic search impossible since normal defined by three nondominated extreme points need not define positive weights
  - $y^1 = (11, 11, 14), y^2 = (15, 9, 17), y^3 = (19, 14, 10)$ are three nondominated extreme points, normal is $(-1, 40, 28)$
  - Nondominated extreme point $y^4 = (13, 16, 11)$ not found
- Phase 2:
  - Search by triangle impossible due to lack of natural order of points
  - $y^1 = (22, 42, 25), y^2 = (38, 33, 27), y^3 = (39, 31, 30)$ are three nondominated extreme points
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Weight Set Decomposition

$$W^0 := \left\{ \lambda > 0 : \lambda_p = 1 - \sum_{k=1}^{p} \lambda_k \right\}$$

$$W^0(y) := \left\{ \lambda \in W^0 : \lambda^T y = \min\{\lambda^T y : y \in Y\} \right\}$$

Theorem

- *If y is a nondominated extreme point of Y then dim $W^0(y) = p - 1$.*
- $W^0(y) = \bigcup_{y \in Y_{sN1}} W^0(y)$.
- *dim $W^0(y)$ + dim $F(y) = p - 1$ for all $y \in Y_{sN}$, where $F(y)$ is the maximal nondominated face of Y containing y.*

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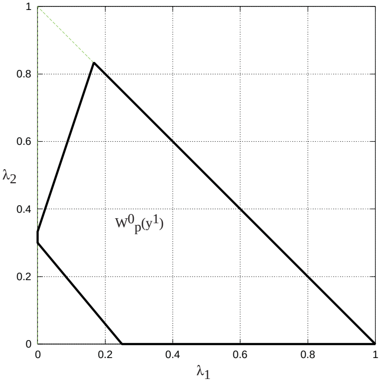
Finding Nondominated Extreme Points

- For $p = 1, \ldots, p$ find y^k minimizing the k-th objective
- $S := \{y^1, \ldots, y^p\}$ and $W_p^0(y^k) = \{\lambda \in W^0 : \lambda^T y = \min\{\lambda^T y : y \in S\}\}$
- Facets of $W_p^0(y^k)$ define biobjective problems
- Solve biobjective problems for all facets for all y^k to find new nondominated extreme points added to S
- Stop if $W_p^0(y) = W^0(y)$ for all $y \in S$



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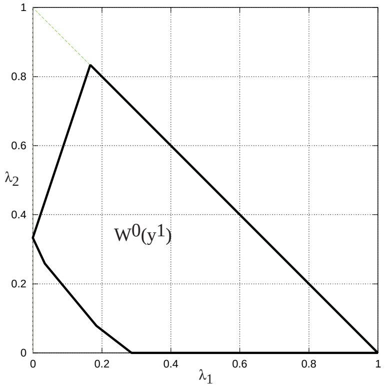

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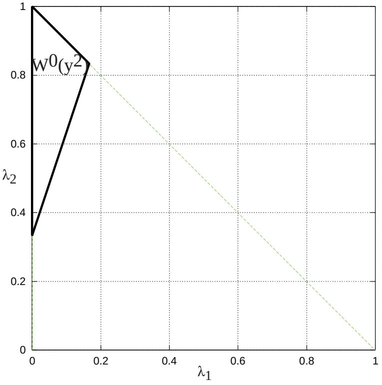

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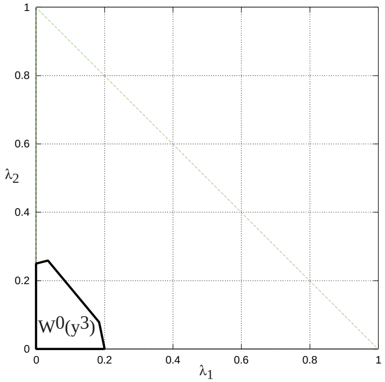

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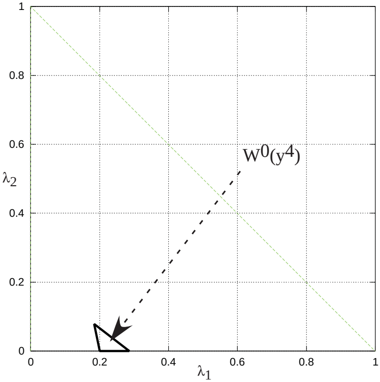

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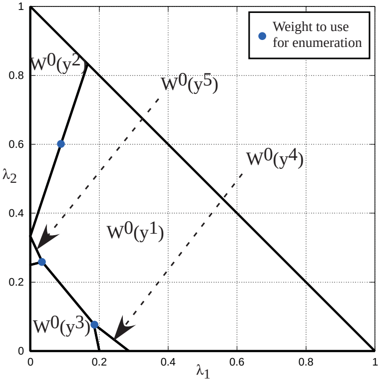
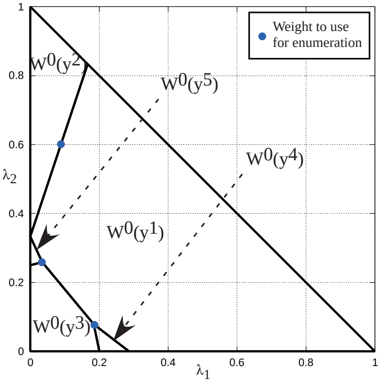


Finding Supported Nondominated Points

- Relevant weights
  - Intersection points of at least three sets $W^0(y)$
  - Points in the interior of faces where two sets $W^0(y)$ intersect
- Enumerate all optimal solutions of weighted sum problems



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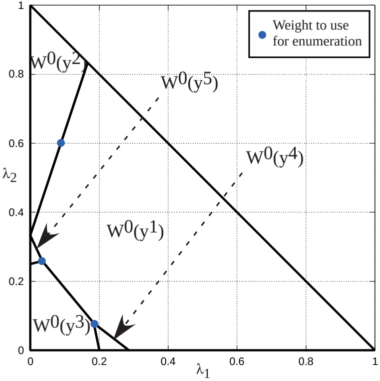

Finding Nonsupported Nondominated Points

The search area

$$A = \left((\text{conv } Y_{sN})_N + \mathbb{R}^p_{\geqq}\right) \setminus \left(Y_{sN} + \mathbb{R}^p_{\geqq}\right)$$

$$= \left((\text{conv } Y_{sN})_N + \mathbb{R}^p_{\geqq}\right) \cap \left(\cup_{u \in D(Y_{sN})} u - \mathbb{R}^p_{\geqq}\right)$$

- Procedure to calculate $D(Y_{sN})$
- For each $u \in D(Y_{sN})$ find closest nondomiated facet of Y
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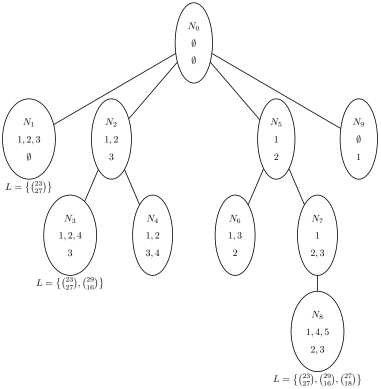
Results for Three-Objective Assignment Problem

n	$\lvert Y_N \rvert$	S/C 2004	T-P 2003	L et al. (2005)	P et al. 2007
5	12	0.15	0.04	0.15	0.00
10	221	99865.00	97.30	41.70	0.08
15	483	×	544.53	172.29	0.36
20	1942	×	×	1607.92	4.51
25	3750	×	×	5218.00	30.13
30	5195	×	×	15579.00	55.87
35	10498	×	×	101751.00	109.96
40	14733	×	×	×	229.05
45	23941	×	×	×	471.60
50	29193	×	×	×	802.68

Multicriteria Branch and Bound

- Ulungu and Teghem 1997, Mavrotas and Diakoulaki 2002
- Branching: As in single objective case
- Bounding: Ideal point of problem at node is dominated by efficient solution
- Branching may be very ineffective
- Use lower and upper bound sets



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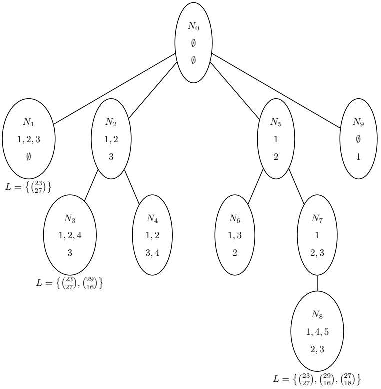

Multicriteria Branch and Bound

- Ulungu and Teghem 1997, Mavrotas and Diakoulaki 2002
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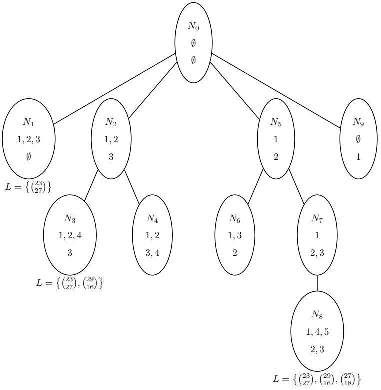

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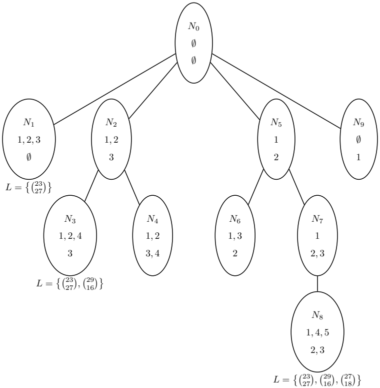

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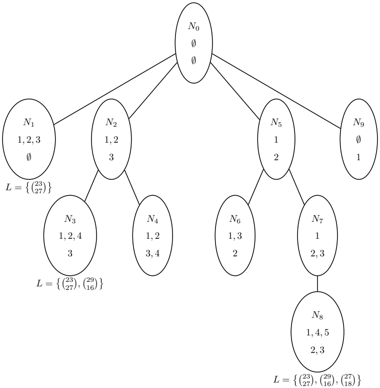

Bound Sets

Ehrgott and Gandibleux 2005:

1. Lower bound set L
   - is $\mathbb{R}^p_{\geqq}$-closed
   - is $\mathbb{R}^p_{\geqq}$-bounded
   - $Y_N \subset L + \mathbb{R}^p_{\geqq}$
   - $L \subset \left(L + \mathbb{R}^p_{\geqq}\right)_N$
2. Upper bound set U
   - is $\mathbb{R}^p_{\geqq}$-closed
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   - $Y_N \in \mathrm{cl}\left[\left(U + \mathbb{R}^p_{\geqq}\right)^c\right]$
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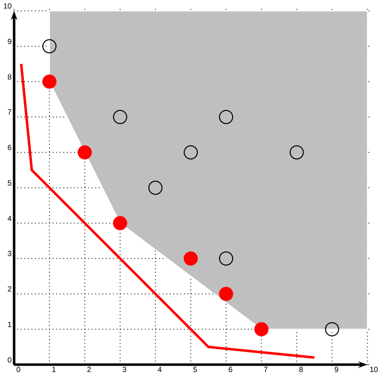

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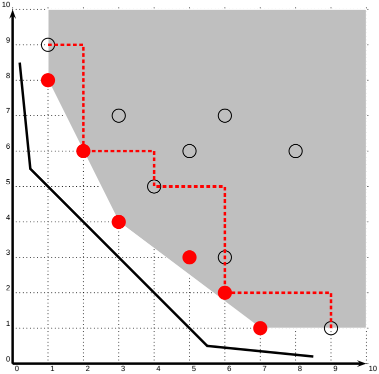
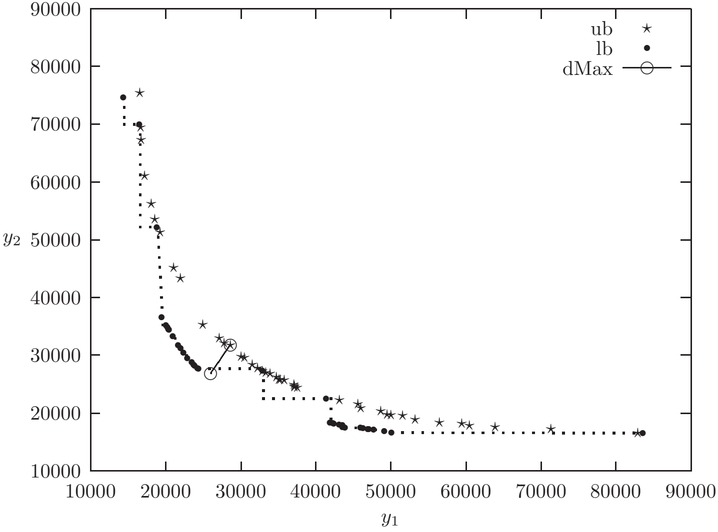
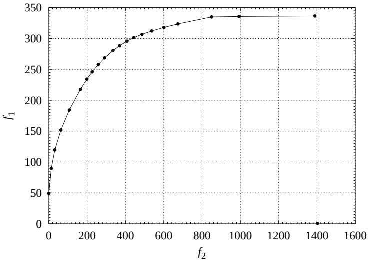



90000
80000
70000
60000
50000
40000
30000
20000
10000
y_2
10000
20000
30000
40000
50000
60000
70000
80000
90000
y_1
ub
lb
dMax

Portfolio Selection

Markowitz 1952 with cardinality constraint, e.g. Chang et al. 2000

$$\begin{aligned} \max z_1(x) &= \mu^T x \\ \min z_2(x) &= x^T \sigma x \\ \text{subject to } e^T x &= 1 \\ x_i &\leqq u_i y_i \\ x_i &\geqq l_i y_i \\ e^T y &= k \\ y &\in \{0,1\}^n \end{aligned}$$



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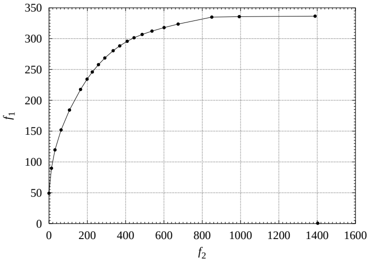
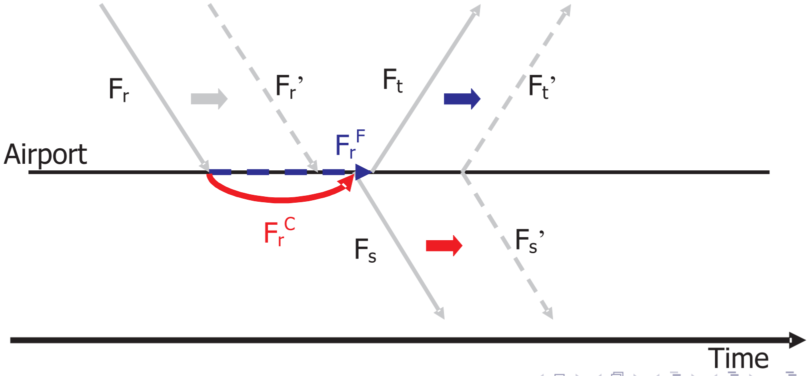


Airline Crew Scheduling

Partition flights into set of pairings, but minimizing cost can cause delays ...

and be very expensive



Airline Crew Scheduling

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BBC NEWS

Sunday, 4 August, 2002, 20:29 GMT 21:29 UK

Delays as Easyjet cancels 19 flights

Passengers with low-cost airline Easyjet are suffering delays after 19 flights in and out of Britain were cancelled.

The company blamed the move - which comes a week after passengers staged a protest sit-in at Nice airport - on crewing problems stemming from technical hitches with aircraft.

Crews caught up in the delays worked up to their maximum hours and then had to be allowed home to rest.

Mobilising replacement crews has been a problem as it takes time to bring people to airports from home. Standby crews were already being used and other staff are on holiday.

Airline Crew Scheduling

Model 1: Minimize cost and minimize non-robustness (Ehrgott and Ryan 2002)

$$a_{ij} = \begin{cases} 1 & \text{pairing } j \text{ includes flight } i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \min z_1(x) &= c^T x \\ \min z_2(x) &= r^T x \\ \text{subject to } Ax &= e \\ Mx &= b \\ x &\in \{0,1\}^n \end{aligned}$$

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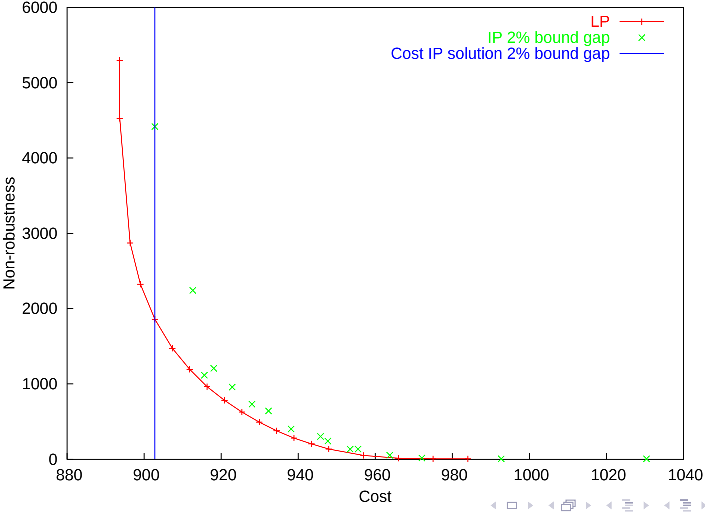
LP
IP 2% bound gap
Cost IP solution 2% bound gap
Non-robustness
6000
5000
4000
3000
2000
1000
0
880
900
920
940
960
980
1000
1020
1040
Cost

Radiotherapy Treatment Design

Choose beam directions and intensities to destroy tumour and spare healthy organs (e.g. Holder 2004)

$$
\begin{aligned}
\min(z_{\mathcal{T}}, z_{\mathcal{S}}, z_{\mathcal{N}}) & & \\
\text{subject to } A_{\mathcal{T}}x + z_{\mathcal{T}}e & \geq & l_{\mathcal{T}} \\
A_{\mathcal{T}}x & \leq & u_{\mathcal{T}} \\
A_{\mathcal{S}}x - z_{\mathcal{S}}e & \leq & u_{\mathcal{S}} \\
A_{\mathcal{N}}x - z_{\mathcal{N}}e & \leq & u_{\mathcal{N}} \\
z_{\mathcal{S}} & \geq & -u_{\mathcal{S}} \\
z_{\mathcal{N}} & \geq & 0 \\
x & \geq & 0 \\
x & \leq & Mye \\
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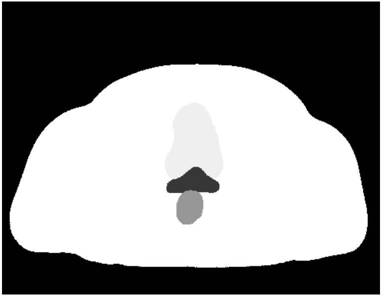

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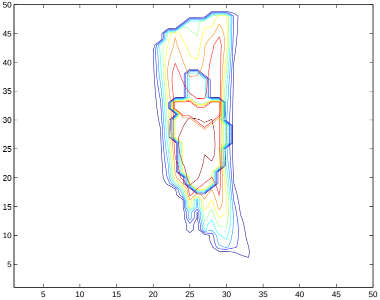


Multicriteria
Optimization


MCDM
19th International Conference on Multiple Criteria Decision Making
MCDM for Sustainable Energy and Transportation Systems
7th – 12th January 2008
The University of Auckland, Auckland, New Zealand
THE UNIVERSITY OF AUCKLAND
FACULTY OF ENGINEERING
Deadline for abstract submission September 30, 2007
Deadline for early registration October 15, 2007
www.esc.auckland.ac.nz/mcdm2008
mcdm2008@esc.auckland.ac.nz