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On the theory of smooth topographic waveguides for Rayleigh waves

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Abstract—**'In the present paper, it is demonstrated that the existence of guided modes of Rayleigh waves on some types of smooth solid surfaces, often called 'smooth topographic waveguides', can take place under the condition of total internal reflection of Rayleigh waves from the 'external' areas of surfaces surrounding the 'internal' areas of wave localisation. In the framework of the geometrical acoustics approximation, the possibility of total internal reflection of Rayleigh waves in smooth topographic structures of complex geometry is linked to the presence of internal areas on the surfaces characterised by the geometry-modified angular-dependent local phase velocities of Rayleigh waves that are smaller in the direction of guided wave propagation than their velocities in the surrounding external areas. The above-mentioned condition of wave localisation is illustrated by theoretical calculations of the dispersion curves of guided waves for several examples of guided wave propagation. The obtained results for the dispersion curves of localised waves are compared with the known solutions, where available.**

Keywords - Rayleigh waves; smooth topographic waveguides; total internal reflection.

I. INTRODUCTION

In the present paper, a brief overview of the current status of the theory of smooth topographic waveguides for Rayleigh waves is presented. It is demonstrated that the existence of localised modes of Rayleigh waves in smooth topographic waveguides can be understood using the classical definition of open waveguides utilizing the condition of total internal reflection of Rayleigh waves from the areas surrounding the area of wave localisation. The possibility of total internal reflection in structures of complex geometry is often linked to the presence of internal areas on the surfaces characterised by the geometry-dependent local phase velocities of Rayleigh waves that are smaller in the direction of guided wave propagation than their values in the surrounding areas. This is similar to the well-known case of guided wave propagation in atmospheric or underwater Acoustics.

The above-mentioned condition of wave localisation is illustrated in this paper by theoretical calculations of frequency-dependent phase velocities of guided modes for Rayleigh waves propagating along solid noncircular cylinders. The obtained results for dispersion curves of localised waves are compared with the known solutions, where available.

The approach used in this paper is based on the geometrical acoustics approximation, which is an asymptotic high-frequency solution (sometimes called ray-tracing solution) to the differential equations and boundary conditions describing wave propagation in inhomogeneous media. Geometrical acoustics (GA) is used widely in underwater or atmospheric acoustics [1]. However, in the acoustics of solids its use is not so frequent, which can be partly explained by the complexity of real inhomogeneous solid structures. In the same time, the use of GA to describe wave propagation in such inhomogeneous solids is very efficient, and it can provide a clear physical interpretation of the waveguiding properties of different topographic structures [2-4].

In what follows, the geometrical acoustics theory of Rayleigh wave propagation along arbitrary curved surfaces is initially considered. The results are then applied to the description of localised waves propagating in the smooth ridge-type topographic waveguides for Rayleigh waves. Such waveguides are often present in nature, for example for seismic waves resulting from earthquakes and propagating along mountain ridges. They also appear in a variety of engineering applications, such as propagation of traffic-induced ground vibrations along real curved surfaces.

It should be noted that localised wave propagation in smooth topographic waveguides can be also considered by other methods, largely based on direct approaches (see, e.g. [5, 6]). For example, in the paper [6], the authors consider asymptotically and numerically Rayleigh wave propagation in a topographic waveguide formed by a smooth ridge-type elevation over a flat surface. Although the factual results obtained by the authors are correct, the associated physical interpretation of the Rayleigh wave localisation phenomenon does not look very convincing. In contrast to that, the approach based on geometrical acoustics approximation described in the present paper provides a clear physical explanation of the guiding properties of surface topography for all structures of this type and gives numerical results for phase velocities of guided surface modes that are in good agreement with numerical calculations and with experiments. Part of the results described in this paper have been presented in the conference paper [7].

II. Propagation of Rayleigh waves on curved surfaces

In this section, a brief discussion of the developments of the geometrical acoustics theory for Rayleigh waves in solids of complex topography is given, based mainly on the original results of the present author [2-4]. Initially, the propagation of Rayleigh waves along arbitrary curved surfaces is considered. The obtained results are then applied to the description of localised modes in the so-called smooth topographic waveguides for surface waves.

When Rayleigh waves propagate along curved surfaces (Figure 1) their velocities change due to the effect of curvature [2-4]. Generally, curved surfaces represent anisotropic and inhomogeneous media for propagating Rayleigh waves. Only a spherical surface is both isotropic and homogeneous. A circular cylindrical surface is anisotropic, but homogeneous, and a non-circular cylindrical surface is both anisotropic and inhomogeneous (in the direction perpendicular to the cylindrical axis). Thus, if the surface of a solid body is curved, then there are two main features associated with propagation of high-frequency Rayleigh waves on this surface: these are the anisotropy of the local wave velocity and the inhomogenity of the medium due to a variable surface curvature.

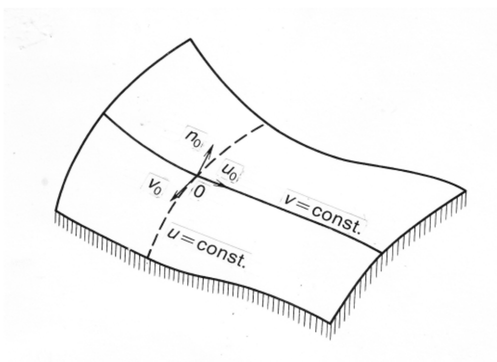

Figure 1. Geometry of the problem; *u* and *v* are geodesic surface coordinates.

The starting point for the geometrical acoustics theory of Rayleigh waves on curved surfaces of arbitrary form is the high-frequency asymptotic expression for the local Rayleigh wave velocity as a function of two local radii of surface curvature [2-4]:

$$c = c_0\left(1 + a_u \frac{1}{k_0 \rho_u} + a_v \frac{1}{k_0 \rho_v}\right). \qquad (1)$$

Here c_0 is the Rayleigh wave velocity on a flat surface, $k_0 = \omega/c_0$ is the corresponding wavenumber, ρ_u and ρ_v are the radii of the surface curvature in the direction of wave propagation and in the direction perpendicular to it respectively (see Figure 1), a_u and a_v are the non-dimensional coefficients that depend on Poisson's ratio of the medium.

The approximate expression (1) is valid under the conditions $k_0\rho_u >> 1$ and $k_0\rho_v >> 1$, i.e. if both radii of curvature are much larger than Rayleigh wavelength. The expression (1) has been established by several researchers, including the present author (see Ref 4 for details). The values of a_u and a_v for all values of Poisson's ratio can be found in Ref 4. The important fact to be mentioned here is that a_u is always positive, whereas a_v is always negative. The latter feature is paramount for Rayleigh wave localisation, and, as will be discussed in the next section, it is responsible for guided wave propagation in smooth topographic waveguides for Rayleigh waves.

If the two radii of curvature ρ_u and ρ_v are known as functions of surface coordinates, the usual formalism of geometrical acoustics (in scalar approximation) can be applied to describe either vertical or horizontal component of a propagating Rayleigh wave in the arbitrary point of the curved surface. In order to do so, one should initially establish the trajectories of all possible rays that can be traced from a chosen point of Rayleigh wave excitation. After the ray trajectories have been established, the solution for a wave propagating along any particular trajectory, using a surface coordinate s measured along the trajectory, can be written in the form [2, 3]

$$u = A(s)e^{i\int k(s)ds}, \qquad (2)$$

where $A(s)$ and $k(s) = \omega/c$ are slowly varying functions of s describing the amplitude and the local wavenumber of the Rayleigh wave.

To calculate ray trajectories of Rayleigh waves propagating over surfaces of variable curvature it is convenient to use the Hamiltonian approach [4]. For example, in the case of a smooth noncircular cylinder shown in Figure 2(a), it can be demonstrated that Rayleigh waves propagating at oblique angles from the top area experience total internal reflection starting from a certain angle and thus become captured within the top area of maximum curvature corresponding to the minimum of phase velocity in the x-direction. This constitutes a waveguide effect of such inhomogeneous surfaces.

III. Smooth topographic waveguides

As was mentioned above, non-circular cylindrical surfaces can support guided Rayleigh waves. If the waveguiding properties are attributed to the influence of surface geometry, as in the case considered, then the associated waveguides are often called 'topographic waveguides'. The need to take into account waveguiding properties of surfaces appears in seismology and in different applications of ultrasonic non-destructive testing.

A rigorous analysis of topographic waveguides is rather difficult. As a rule, it cannot be done analytically and requires numerical calculations. However, there are several important cases that can be considered using approximate analytical approaches. Among these cases are smooth topographic waveguides, two types of which are shown in Figure 2. For smooth topographic waveguides, the minimum radius of surface curvature is greater than the Rayleigh wavelength. Therefore, such waveguides can be considered in geometrical acoustics approximation on the basis of the asymptotic expression (1) for the local phase velocity of Rayleigh waves.

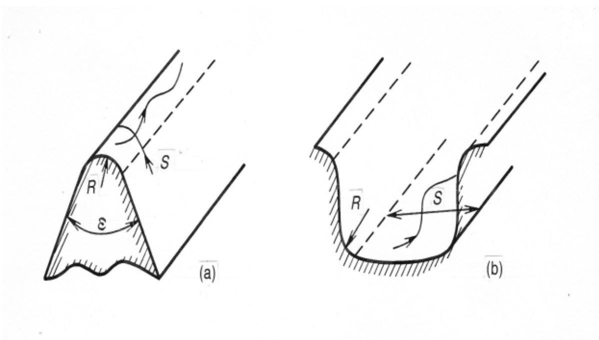

Figure 2. Ridge and groove types of smooth topographic waveguides [2].

The first geometrical acoustics consideration of smooth topographic waveguides, including physical interpretation of the Rayleigh wave localisation in such topographic structures, has been given by the present author [2, 3] (see also the monograph [4]). The geometrical acoustics theory of smooth topographic waveguides provides a clear and physically explicit explanation of the reason for the presence of waveguide effect in such structures. Namely, it demonstrates that the existence of propagating localised modes of Rayleigh waves in smooth topographic waveguides can be explained by the presence of an 'internal' area on the surface with the curvature-dependent local phase velocity of Rayleigh waves that is smaller in the direction of guided wave propagation than its values for adjacent surfaces with smaller curvature or flat surfaces (with zero curvature). The consequence of this is the possibility of total internal reflection of the curvature-modified Rayleigh waves from the surrounding areas of very small, zero or negative surface curvature [2-4].

Let us briefly consider derivation of the dispersion equation for symmetric guided modes of the smooth topographic ridge-type waveguide shown in Figure 2(a). This structure is formed by a part of a circular cylinder of radius R and by two flat surfaces positioned at the angle ε. Applying formula (1) to a Rayleigh wave propagating along a circular cylindrical surface at an angle Φ in respect of the element of cylinder and using Euler's formulas for the radii of curvature ρ_u and ρ_v in the case of circular cylinder of radius R, one can obtain the following expression for Rayleigh wave velocity on the curved part of the structure [2]:

$$c = c_0\left\{1 + \frac{1}{k_0 R}\left[(a_u - a_v)\sin^2\Phi + a_v\right]\right\}. \quad (3)$$

The dependence of the velocity c on the angle Φ in (3) describes the anisotropy of Rayleigh wave velocity on circular cylindrical surfaces, even if the material is isotropic. We remind the reader that formula (3) is valid for $k_0R >> 1$.

The above-described smooth ridge-type structure, which has clearly defined boundaries between the curved part of the structure (on the top) and the two flat parts (on the sides), can be considered as a three-layered plane medium in respect of Rayleigh wave propagation, where the internal (curved) area is characterised by the Rayleigh wave velocity c described by formula (3) and the two side (flat) areas are characterised by Rayleigh wave velocity on the flat surface c_0. Therefore, in order to analyse guided Rayleigh wave propagation along such a noncircular cylinder, one can use the standard dispersion equation for a three-layered scalar medium, in which one should take into account the anisotropy of Rayleigh wave velocity c in the internal area. For example, for the m-th symmetric guided mode this equation has the form [2, 4]:

$$\frac{a}{2}\left[k_c^{\,2}(\Phi) - \gamma^2\right]^{1/2} = m\frac{\pi}{2} + \tan^{-1}\left(\frac{\gamma^2 - k_0^{\,2}}{k_c^{\,2}(\Phi) - \gamma^2}\right)^{1/2}, \quad (4)$$

where $\gamma = k_c \cos\Phi$ is the constant (the wavenumber) describing wave propagation in a waveguide, $a = (\pi - \varepsilon)R$ is thickness of the internal layer (see Figure 2(a)), and $k_c(\Phi) = \omega/c(\Phi)$ is the wavenumber of a Rayleigh wave on the curved surface, where Rayleigh wave velocity $c(\Phi)$ is defined by formula (3).

It should be noted that the use of the dispersion equation (4) assumes that the Fresnel formula for acoustic wave reflection from a boundary between two media, which is used in derivation of the standard dispersion equation for a three-layered scalar medium, remains valid also in the case of Rayleigh waves incident at oblique angles α on the boundary between curved and flat surfaces. It has been shown [4] that this is really the case for small glancing angles $\Phi = \pi/2 - \alpha$ typical for guided wave propagation, so that the use of equation (4) is justified.

Combining the expressions (3) and (4) and resolving them in respect of γ, one can obtain, after some simple rearrangements, that the expression for γ for the lowest order mode ($m = 0$) takes the form:

$$\gamma = k_0\left(1 + \frac{\beta_1}{2k_0 R} - \frac{\beta_1^{1/2}}{\pi - \varepsilon}\frac{1}{(k_0 R)^{3/2}}\right), \quad (5)$$

where $\beta_1 = -2a_v$. As expected, the expression (5) describes the waveguide propagation of Rayleigh waves at the velocity $v = \omega/\gamma$ that is slightly lower than the velocity of Rayleigh waves on a flat surface c_0.

The expression (5) coincides in form with the first terms of the expansion for γ earlier obtained by a direct method of solving the corresponding boundary problem [5] (see also other references in the monograph [4]). However, it is important to emphasise that the geometrical acoustics approach described in this section is physically explicit and incomparably simpler. The amplitude distributions of guided modes in the lateral direction can be easily constructed using the expression (5) (the details are not shown here for brevity). It is useful to mention though that the wave amplitudes decay exponentially away from the curved area.

The groove type smooth topographic waveguide shown in Figure 2(b) can be considered in the same way as the ridge-type waveguide, i.e. as a three-layered plane medium having the lowest Rayleigh wave velocity on the bottom part.

In a general case of smooth topographic waveguides formed by noncircular cylinders without clearly defined boundaries between areas of different curvature, one should first determine the ray turning points for incident Rayleigh waves, that define their total internal reflection, and then apply the general geometrical acoustics dispersion equation in integral form, which is similar to Bohr-Sommerfeld quantization condition [8].

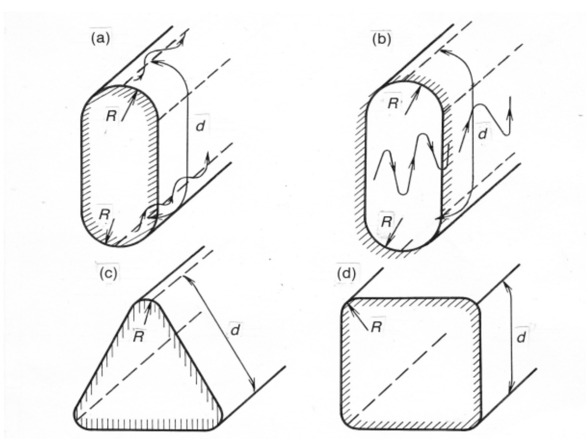

Figure 3. Some complex topographic structures [2].

It is instructive to make a few comments on guided Rayleigh wave propagation along smooth ridge type waveguides that are curved in a vertical plane. Such structures can form saddle-type surfaces (in the case of a negative (concave) surface curvature in a vertical plane) or hill-type surfaces (in the case of a positive (convex) surface curvature in a vertical plane). According to (1), it is quite obvious that in the former case the negative curvature will decrease the already low Rayleigh wave velocity in the x-direction in the top area even more, thus amplifying the waveguide effect in such a structure. However, in the latter case, the positive curvature will increase the Rayleigh wave velocity in the x-direction of the top area. This will result in weakening of the waveguide effect, and, depending on the relationship between the two radii of curvature, may cause its complete elimination.

Finally, some types of complex topographic structures (see Figure 3) can be considered as systems of coupled topographic waveguides of more simple geometry. Such systems can be analysed using the well-established theory of coupled waveguides [2, 4]. For the structures shown in Figure 3, these will be systems of two ridge-type (a) or groove-type (b) coupled waveguides, and systems of three (c) and four (d) ridge-type coupled waveguides.

IV. CONCLUSIONS

In the present paper, propagation of localised Rayleigh waves in smooth topographic waveguides has been considered theoretically using geometrical acoustics approach. Derivations of the expressions for dispersion curves of guided waves have been discussed and the results compared with the known solutions, where available.

It has been demonstrated that the possibility of localisation of Rayleigh waves can be explained by the existence of total internal reflection on surfaces of variable curvature due to the presence of areas characterised by the geometry-modified local phase velocities of Rayleigh waves that are smaller in the direction of guided wave propagation than their values in the surrounding areas. This is similar to the condition of guided wave propagation in open waveguides studied in atmospheric or underwater Acoustics.

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