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Exact and Approximate Causal Time-Domain Green's functions for Linear with Frequency Attenuation

Vaughn E. Holmes
Department of Electrical and Computer Engineering
Michigan State University, East Lansing, USA

Robert J. McGough
Department of Electrical and Computer Engineering
Michigan State University, East Lansing, USA

***Abstract*—Exact and approximate analytical time-domain Green's functions that describe linear with frequency attenuation are presented for the Chen-Holm space-fractional wave equation. These exact and approximate time-domain Green's functions contain a Cauchy density multiplied by a Heaviside function, where the Heaviside function guarantees that the response is causal. To demonstrate the distinguishing features of these analytical expressions, exact and approximate time-domain Green's functions are numerically evaluated and compared for linear with frequency attenuation. Numerical results are evaluated for an attenuation of 0.5 dB/cm/MHz, and the results demonstrate excellent agreement between the exact and approximate solutions. At this and at all other distances, the exact and approximate solutions overlap almost perfectly. Comparisons of the peak values evaluated and the full width at half maximum (FWHM) evaluated as a function of time at different distances for the exact and approximate expressions also agree very closely. Thus, the approximate expressions are excellent approximations to the exact time-domain Green's functions for the Chen-Holm space-fractional wave equation, which is the only known fractional wave equation that yields causal analytical time-domain Green's functions describing linear with frequency attenuation.**

I. Introduction

Several fractional wave equations describe power law attenuation for medical ultrasound, including the Szabo [7], Caputo [2], power law [6], Chen-Holm [3], and Treeby-Cox [8] wave equations. For the power law exponent $y = 1$, which is an important value in medical ultrasound, all of these fractional wave equations break down except for the Chen-Holm space-fractional wave equation. The Chen-Holm wave equation also admits exact analytical time-domain Green's functions for $y = 1$, where the resulting expressions are advantageous for subsequent analytical manipulations and numerical evaluations. This motivates further examination of the Chen-Holm space-fractional wave equation for the special case $y = 1$. Exact and approximate Green's functions are derived for the Chen-Holm space-fractional wave equation when $y = 1$. These are represented by Cauchy densities multiplied by a Heaviside function, which guarantees that the response is causal. To demonstrate the effect of attenuation in the time-domain, these exact and approximate time-domain Green's functions are computed and displayed for different distances. The results demonstrate decay and and waveform spreading as a function of distance, which suggests that these analytical time-domain Green's functions will provide valuable insight when applied to numerical and analytical models of medical ultrasound.

II. Exact and approximate time-domain Green's functions for the Chen-Holm space-fractional wave equation

For the special case $y = 1$, a space-fractional model for the power law attenuation experienced by medical ultrasound propagating in soft tissue is given by [3]

$$\nabla^2 g - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} g - \tau \frac{\partial}{\partial t}\left(-\nabla^2\right)^{1/2} g = 0. \quad (1)$$

In Eq. 1, c_0 is the speed of sound constant, t is the time, $\tau = 2\alpha_0$ is the fractional relaxation time when $y = 1$, α_0 is the attenuation constant, $\left(-\nabla^2\right)^{1/2}$ Eq. 1 is the fractional Laplacian operator for the power law exponent $y = 1$, and $g(r, t)$ is a solution to the Chen-Holm space-fractional wave equation, where r represents the distance from the origin. To obtain the attenuation and the phase velocity, Eq. 1 is Fourier transformed in space and in time, which yields

$$k^2 + j\omega\tau k - \frac{\omega^2}{c^2} = 0 \quad (2)$$

for the dispersion relation. The root of Eq. 2 with the positive real part is given by

$$k = -\frac{j\omega\tau}{2} + \omega\sqrt{\frac{1}{c_0^2} - \frac{\tau^2}{4}}, \quad (3)$$

where the real and imaginary parts both depend on the angular frequency. Letting $\tau = 2\alpha_0$ and taking the negative of the real part of Eq. 3 provides the exact expression for the attenuation as a function of frequency $\alpha(\omega)$ described by the Chen-Holm wave equation when $y = 1$, which is

$$\alpha(\omega) = \alpha_0\omega, \quad (4)$$

The phase velocity $c(\omega)$ for Chen-Holm when $y = 1$ is obtained from Eq. 3 after dividing by the angular frequency ω, substituting $\tau = 2\alpha_0$, and then taking the reciprocal of the result, which yields

$$c(\omega) = \frac{c_0}{\sqrt{1 - \alpha_0^2 c_0^2}}. \quad (5)$$

The expression in Eq. 5 is well-approximated by Eq. 10 from [10] after substituting $\tau = 2\alpha_0$ and letting $y = 1$, which is

$$1/c(\omega) \approx 1/c_0 - 1/2\alpha_0^2 c_0. \quad (6)$$

To obtain exact and approximate time-domain Green's functions for the Chen-Holm wave equation, a impulsive input in time and space is applied to Eq. 1,

$$\nabla^2 g - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} g - \tau \frac{\partial}{\partial t}\left(-\nabla^2\right)^{1/2} g = -\delta(t)\delta(\boldsymbol{R}), \quad (7)$$

and then Eq. 7 is Laplace transformed in time and Fourier transformed in space, which gives

$$k^2\nabla^2 \hat{G} + \frac{s^2}{c_0^2}\frac{\partial^2}{\partial t^2}\hat{G} + \tau s k \hat{G} = 1. \quad (8)$$

In Eq. 8, $\hat{G}(k, s)$ describes the Laplace and Fourier transform of the Green's function solution $g(r, t)$, where upper case indicates the Laplace transform and the hat indicates the Fourier transform. Solving for $\hat{G}(k, s)$, multiplying by c_0^2/c_0^2, and reorganizing yields

$$\hat{G}(k, s) = \frac{c_0^2}{\left(s + \tau k c_0^2/2\right)^2 + k^2 c_0^2 - \left(\tau k c_0^2/2\right)^2}. \quad (9)$$

The inverse Laplace transform in time is then evaluated with $\mathcal{L}\left\{\mathrm{e}^{-at}\sin(\omega t)\right\} = \omega/\left[(s+a)^2 + \omega^2\right]$ to obtain

$$\hat{g}(k, t) = \frac{c_0 H(t)\,\mathrm{e}^{-\tau k c_0^2 t/2}}{k\sqrt{1 - \tau^2 c_0^2/4}} \sin\left(k c_0 t\sqrt{1 - \tau^2 c_0^2/4}\right). \quad (10)$$

Next, Eq. 10 is inserted into the spherically symmetric inverse three-dimensional (3D) Fourier transform [1],

$$g(r,t) = \frac{4\pi}{(2\pi)^3 r}\int_0^\infty \hat{g}(k,t)\sin(kr)\,k\,dk, \quad (11)$$

and then the resulting expression is rewritten with the product formula $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ and evaluated using $\int_0^\infty \mathrm{e}^{-ax}\cos(bx)\,dx = a/\left(a^2+b^2\right)$ with $\tau = 2\alpha_0$. The exact time-domain Green's function for the Chen-Holm space-fractional wave equation with $y = 1$ is then given by

$$g(r,t) = H(t)\frac{\alpha_0 c_0^3 t/\left(4\pi^2 r\sqrt{1-\alpha_0^2 c_0^2}\right)}{\left(r - c_0 t\sqrt{1-\alpha_0^2 c_0^2}\right)^2 + \alpha_0^2 c_0^4 t^2} \quad (12a)$$

$$-H(t)\frac{\alpha_0 c_0^3 t/\left(4\pi^2 r\sqrt{1-\alpha_0^2 c_0^2}\right)}{\left(r + c_0 t\sqrt{1-\alpha_0^2 c_0^2}\right)^2 + \alpha_0^2 c_0^4 t^2} \quad (12b)$$

The expression on the right hand side of Eq. 12a describes an outbound linear with frequency attenuated spherical wave, and the expression in Eq. 12b describes an inbound linear with frequency attenuated spherical wave. The term in Eq. 12b is negligible, which yields the approximate expression

$$g(r,t) \approx H(t)\frac{\alpha_0 c_0^3 t/\left(4\pi^2 r\sqrt{1-\alpha_0^2 c_0^2}\right)}{\left(r - c_0 t\sqrt{1-\alpha_0^2 c_0^2}\right)^2 + \alpha_0^2 c_0^4 t^2} \quad (13)$$

for the time-domain Green's function of the Chen-Holm space fractional wave equation when $y = 1$. The expressions in Eqs. 12 and 13 are both causal due to the Heaviside function $H(t)$.

III. Methods

The exact and approximate time-domain Green's functions are computed and plotted in Matlab. The computed Green's functions are each scaled by $4\pi r$ to emphasize the properties of the Cauchy density that describe linear with frequency attenuation. The scaled Green's functions are evaluated at $\alpha_0 = 0.5$ dB/cm/MHz. For numerical calculations, the units required for α_0 are Np/m/(rad/sec), which requires division by $20\log_{10}\mathrm{e}$, multiplication by 100, division by $2\pi \times 10^6$, for the conversions to Np, cm, and rad/sec, respectively.

IV. Results

The exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ are computed with Eqs. 12 and 13, respectively, at distances of $r = 1$, 2, 3, 4, and 5 cm. The time-domain Green's functions in each figure are normalized to the peak value at $r = 1$ cm. Figs. 1-5 demonstrate that the peak decays rapidly as the time-domain Green's function propagates from 1 to 5 cm. Similarly, the temporal width increases as the propagation distance increases.

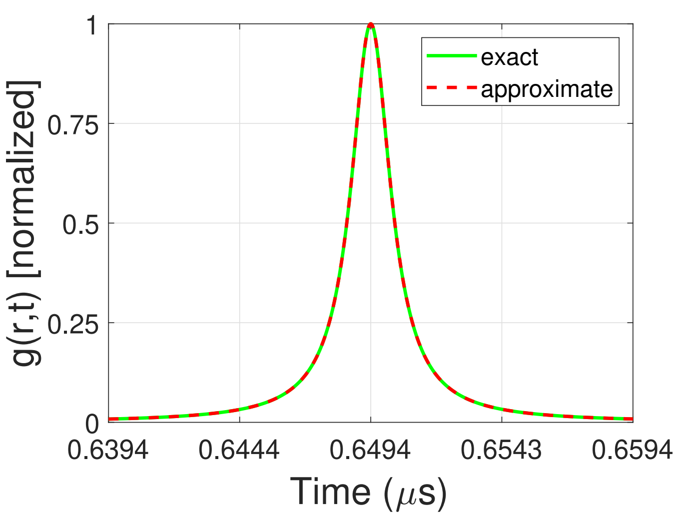

Fig. 1. Exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ evaluated at $r = 1$ cm.

V. Conclusion

As demonstrated in Section II, the Chen-Holm space fractional wave equation exactly describes linear with frequency attenuation when $y = 1$. The Chen-Holm space-fractional wave equation is also dispersionless in the sense that the phase velocity is equal to a constant value when the power law exponent y is equal to 1. Exact and approximate Green's functions are also derived for the Chen-Holm wave equation when $y = 1$. These are evaluated and compared in Figs. 1-5, which demonstrate that the exact and approximate expressions

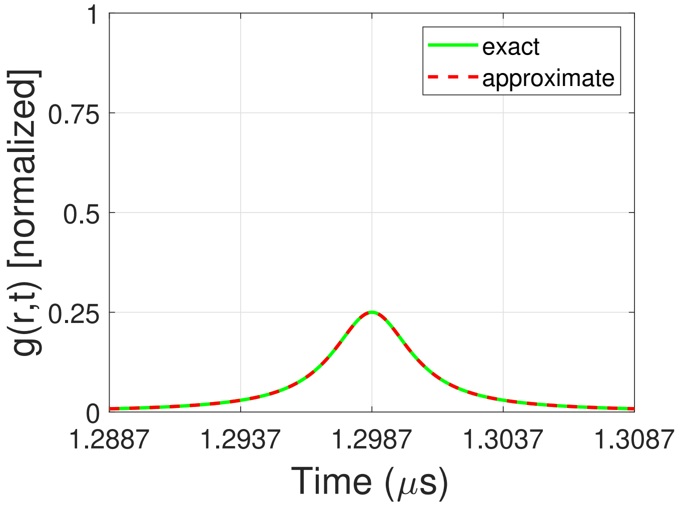

Fig. 2. Exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ evaluated at $r = 2$ cm.

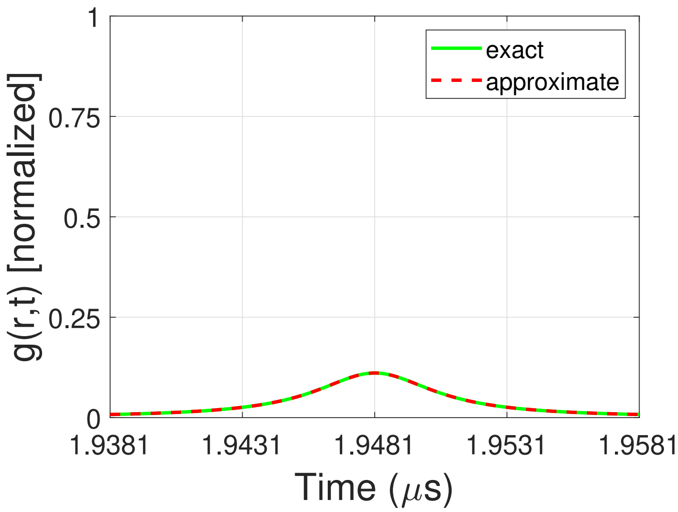

Fig. 3. Exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ evaluated at $r = 3$ cm.

are nearly identical when evaluated at distances of 1-5 cm. Furthermore, the exact and approximate expressions provided in Eqs. 12 and 13, respectively, are causal. The Chen-Holm wave equation, which is the only known fractional wave equation that is causal for $y = 1$, provides an additional advantage in that exact and approximate analytical time-domain Green's function are available for numerical computations in the time-domain. The main difference between this and other time-domain Green's functions developed for medical ultrasound [4], [5], [6], [9], [10] is that the Chen-Holm space-fractional wave equation is nondispersive.

Acknowledgments

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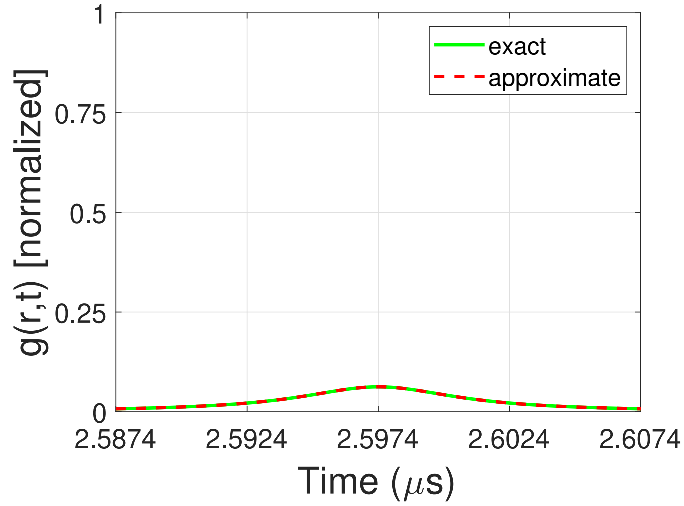

Fig. 4. Exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ evaluated at $r = 4$ cm.

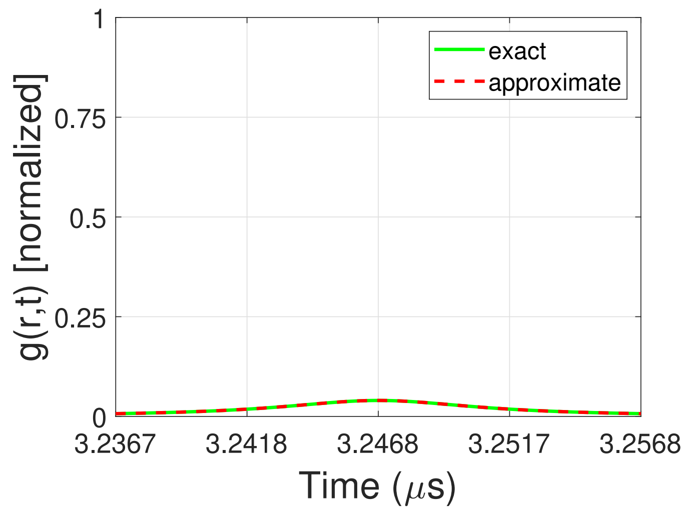

Fig. 5. Exact and approximate time-domain Green's functions for the Chen-Holm wave equation with $y = 1$ evaluated at $r = 5$ cm.

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