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Discrete Solutions of Electric Power Systems Based on a Differentiation Matrix and a Newton Method

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***Abstract*— A time-domain approach based on a discrete representation of the differentiation operation is presented in this paper to compute periodic steady-state solutions of electric power systems. The finite-dimensional representation of the derivative operator reproduces the exact derivative of a trigonometric polynomial. The time-domain representation of the electric network in terms of ordinary differential equations is transformed into a nonlinear algebraic formulation and solved using a Newton algorithm, where the unknowns of the algebraic equations are the samples of the state variables. Besides, the incorporation of sparse techniques improves the efficiency of the discrette-time solution in terms of storage and computational effort. Test cases incorporating nonlinear devices such as transformers, electric arc furnaces and STATCOMs are presented to illustrate the effectiveness of this method. Comparative results are reported using the well-known finite-difference method.**

***Index Terms*— Periodic solution, Newton method, differentiation matrix, finite-difference method, sparse techniques.**

I. Introduction

LOOKING for solutions to nonlinear problems in the power systems field is an important and permanent pursuit. Althought Brute Force (BF) methods [1] can be used to obtain the periodic steady-state of an electric circuit through straightforward integration of ordinary differential equations (ODEs) till the initial transient response dies-out, this method has been questioned because of potential drawbacks such as slow convergence due to poor system damping [2], harmonics [3] and other characteristics of the system [4]. Therefore, shooting methods [5], extrapolation methods [6] and Newton methods based on the Poincaré map [7] have been proposed for the fast time domain computation of the periodic steady-state solution of nonlinear networks.

In addition, a variety of methods have been developed to determine the steady-state response of nonlinear systems by approximating the derivative operator. A discrete-time method to compute the steady-state of nonlinear autonomous systems is presented in [8], which is based on a Gear method and a polynomial fitting procedure. An extension of this method has been proposed in [9] to determine the steady-state response of switched nonlinear circuits. Besides, a discrete-time representation of the derivative has been introduced in [10] to locate limit cycles of non-autonomous systems. However, no analyses associated to important aspects such as convergence properties, storage requirements and computational effort have been provided for this method in the latter contribution.

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Continuing on these developments, the aim of this paper is to present a Discrete-Time Solution (DTS) method to obtain numerical approximations to the steady-state of electric power networks based on the use of a differentiation matrix and a Newton method. A direct method for solving sparse systems of equations is incorporated to the Newton-Raphson (NR) method in order to improve the efficiency of the DTS approach in terms of time and storage. Results obtained with the DTS method are compared with the well-known finite-difference method.

II. Discrete-Time Solution

Considering a trigonometric polynomial of degree r

$$p\left(t\right)=a_0+\sum_{q=1}^{r}a_q\cos\left(qt\right)+\sum_{q=1}^{r}b_q\sin\left(qt\right) \tag{1}$$

which includes $N=2r+1$ coefficients, the computation of these coefficients for a set of N data points defined as $x\left(t_j\right)$ can be perform with a Lagrange formulation for polynomial interpolation,

$$p\left(t\right)=\sum_{j=1}^{N}x\left(t_j\right)\prod_{m=1,m\neq j}^{N}\frac{\sin\frac{1}{2}\left(t-t_m\right)}{\sin\frac{1}{2}\left(t_j-t_m\right)} \tag{2}$$

Assuming that the trigonometric polynomial $p\left(t\right)$ is periodic with period 2π, then it can be evaluated at the discrete points $-\pi<t_1<t_2<\cdots<t_N\leq\pi$. The differentiation matrix for the trigonometric polynomial (2) can be obtained as,

$$\dot{p}\left(t_i\right)=\sum_{j=1}^{N}D_{ij}x\left(t_j\right) \tag{3}$$

for $i=1,2,\cdots,N$. It has been demonstrated in [10] that the differentiation matrix $\mathbf{D}$ with dimensions $N\times N$ has the entries,

$$D_{ij}=\begin{cases}\frac{1}{2}\quad\sum\limits_{m=1}^{N}\cot\frac{t_i-t_m}{2}, & i=j,\, i\neq m\\ \frac{1}{2}\quad\dot{q}\left(t_i\right)\csc\frac{t_i-t_j}{2}, & i\neq j\end{cases} \tag{4}$$

where

$$q\left(t\right)=\prod_{m=1}^{N}\frac{\sin\frac{1}{2}\left(t-t_m\right)}{\sin\frac{1}{2}\left(t_j-t_m\right)} \tag{5}$$

Given a set of equally spaced discrete-time samples $t_i=-\pi+\frac{2\pi i}{N}$, the entries of matrix $\mathbf{D}$ are defined as,

$$D_{ij} = \begin{cases} 0, & i = j \\ \frac{1}{2}\frac{(-1)^{i+j}}{\sin\frac{\pi}{N}(i-j)}, & i \neq j \end{cases} \tag{6}$$

A. Newton-Raphson method

Considering a system of dimension n defined as,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{7}$$

which has periodic properties

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t + T) \tag{8}$$

with $\mathbf{x} = [x_1\, x_2 \dots x_n]^T$ and $\mathbf{f} = [f_1\, f_2 \dots f_n]^T$. Let $\tilde{\mathbf{x}}$ be a column vector of unknowns $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_1\, \tilde{\mathbf{x}}_2 \dots \tilde{\mathbf{x}}_n]^T$ of order nN, where $\tilde{\mathbf{x}}_1\, \tilde{\mathbf{x}}_2 \dots \tilde{\mathbf{x}}_n$ are vectors of dimension N containing a sequence of points that represent the samples of the state variables. The sequence of points are equidistant and located at time instants $\mathbf{t}_N = [h,\, 2h, \cdots Nh]$, where $h = \frac{T}{N}$ is the time step and N is an odd integer. Hence, (7) can be transformed into a set of nonlinear algebraic equations using (6) as follows,

$$\begin{bmatrix} \mathbf{D} & 0 & \cdots & 0 \\ 0 & \mathbf{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},\, \mathbf{t}_N) \\ \tilde{\mathbf{f}}_2(\tilde{\mathbf{x}},\, \mathbf{t}_N) \\ \vdots \\ \tilde{\mathbf{f}}_n(\tilde{\mathbf{x}},\, \mathbf{t}_N) \end{bmatrix} \tag{9}$$

or in compact form,

$$\mathbf{H}\tilde{\mathbf{x}} = \tilde{\mathbf{f}} \tag{10}$$

where $\tilde{\mathbf{f}} = \left[\tilde{\mathbf{f}}_1\, \tilde{\mathbf{f}}_2 \dots \tilde{\mathbf{f}}_n\right]^T$ and $\tilde{\mathbf{f}}_1\, \tilde{\mathbf{f}}_2 \dots \tilde{\mathbf{f}}_n$ are column vectors of dimension N, which are obtained by evaluating the functions $f_1\, f_2 \dots f_n$ at the discrete points $\tilde{\mathbf{x}}$. This system represents a set of nN nonlinear equations with nN unknowns.

Substracting the left-hand side of (10) on both sides, then the resulting system

$$\mathbf{G}\left[\tilde{\mathbf{x}}\right] = 0 \tag{11}$$

can be solved using a Newton-Raphson (NR) algorithm. The computation of the Jacobian matrix by blocks for the nonlinear problem (11) can be defined with the notation,

$$\mathbf{J}^m = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \cdots & \mathbf{J}_{1n} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \cdots & \mathbf{J}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_{n1} & \mathbf{J}_{n2} & \cdots & \mathbf{J}_{nn} \end{bmatrix} \tag{12}$$

where each entry is a $N \times N$ matrix described as,

$$\mathbf{J}_{kl} = \begin{cases} -diag\left(\frac{\partial \mathbf{f}_k}{\partial \tilde{\mathbf{x}}_l}\right) + \mathbf{D} & for\ k = l \\ -diag\left(\frac{\partial \mathbf{f}_k}{\partial \tilde{\mathbf{x}}_l}\right) & for\ k \neq l \end{cases} \tag{13}$$

for $k = 1 \dots n$ and $l = 1 \dots n$.

The term $\triangle\tilde{\mathbf{x}}$ involved in the Newton-Raphson method is represented as,

$$\triangle\tilde{\mathbf{x}}^m = -\mathbf{H}\tilde{\mathbf{x}}^m + \tilde{\mathbf{f}}^m \tag{14}$$

and the Newton procedure applied to (11) has the form,

$$\left[\tilde{\mathbf{x}}^{m+1}\right] = \left[\tilde{\mathbf{x}}^m\right] + \left[\mathbf{J}^m\right]^{-1}\left[\triangle\tilde{\mathbf{x}}^m\right] \tag{15}$$

where m represents the number of Newton's iterations for $m = 0,\, 1,\, 2, \dots$. The initial guess for the Newton method $\tilde{\mathbf{x}}^0 = [\tilde{\mathbf{x}}_1\, \tilde{\mathbf{x}}_2 \dots \tilde{\mathbf{x}}_n]^T$ is determined after solving (7) for an initial number of cycles with a conventional integration method.

B. Sparse technique

The Newton-Raphson method implemented in this work for the DTS approach produces a sparse matrix equation. The application of a direct method for sparse systems to the solution of the set of nonlinear algebraic equations provides an efficient procedure in both storage and time. Taking into account that the term nN may be several hundred for the Jacobian matrix of size $nN \times nN$, the incorporation of sparsity techniques is crucial for the feasibility of the DTS method for solving practical high-dimensional systems. Therefore, a basic sparse matrix storage routine is implemented in this work, where the sparse matrix is stored as a concatenation of the sparse vectors representing its columns. An integer array of row indices and a floating point array of nonzero elements are defined for each sparse vector. Furthermore, a third array saves the index of the leading element for each one of the sparse vectors [11]. For the nonlinear formulation $\mathbf{J}\Delta\tilde{\mathbf{x}}' = \Delta\tilde{\mathbf{x}}$, the vector $\Delta\tilde{\mathbf{x}}$ is not sparse but the Jacobian matrix $\mathbf{J}$ is normally highly sparse. Due to the fact that the jacobian $\mathbf{J}$ is square, a LU decomposition and a forward-backward substitution with $\Delta\tilde{\mathbf{x}}$ are implemented.

III. Test Cases

Three test cases are presented in this section to show the applicability of the DTS method, where the tolerance error to locate the steady-state is set to $1.0e^{-10}$. The elapsed times are measured in a computer at 1.67 GHz and 1GB RAM memory. Furthermore, results obtained with the well-known finite-difference method (FD) are compared with those obtained with the DTS method. The FD method iterates a sequence of points $\{x_1, \dots, x_N\}$ until $x_k \approx \phi_h(x_{k-1},\, t_{k-1})$ for $k = 1, \dots, N$ [1]. In this work, the FD method is implemented with a Trapezoidal rule algorithm and the resulting set of nonlinear algebraic equations are solved with a Newton method.

A. Electric arc furnace

An electric arc furnace is fed with a 1.0 pu voltage source through a step-down transformer represented with an inductance of 0.52 pu. The model of the electric arc furnace used in this work is a general dynamic model in the form of a differential equation based on the principle of conservation of energy [12]. The differential equation for the arc is defined as,

$$K_1 r^{n_1} + K_2 r\frac{dr}{dt} = \frac{K_3}{r^{m_1+2}}i^2 \tag{16}$$

The arc voltage is given by,

$$v = \frac{K_3}{r^{m_1+2}}i \tag{17}$$

TABLE I

MISMATCHES DURING CONVERGENCE OF THE ELECTRIC ARC FURNACE.

m	DTS method		
	63	127	255
1	1.868e-02	6.820e-02	2.757e-02
2	2.075e-02	2.813e-03	9.046e-04
3	2.950e-03	1.869e-05	7.096e-06
4	8.208e-06	1.764e-10	1.504e-10
5	8.158e-11	4.829e-15	4.329e-15
	FD method		
1	1.318e-01	3.242e-02	2.840e-02
2	2.910e-02	5.284e-03	2.708e-04
3	2.074e-03	4.354e-05	7.062e-07
4	7.620e-06	2.039e-08	9.130e-13
5	3.926e-10	6.661e-16	
6	5.551e-16		

TABLE II

MISMATCHES DURING CONVERGENCE OF THE POWER NETWORK.

m	DTS method			
	7	15	31	63
1	4.330e-01	5.508e-01	5.466e-01	7.264e-01
2	5.391e-04	2.904e-04	2.094e-04	1.940e-04
3	6.180e-07	2.129e-07	4.959e-08	5.393e-08
4	1.369e-12	6.131e-14	1.221e-14	5.828e-15
	FD method			
1	4.367e-01	5.858e-01	5.574e-01	7.274e-01
2	2.425e-02	3.800e-03	1.300e-03	3.518e-04
3	1.640e-03	1.352e-05	1.046e-06	1.224e-07
4	7.932e-06	2.878e-10	1.054e-12	2.415e-14
5	1.998e-10	3.206e-15		
6	4.041e-14			

where the constants K_1, K_2, K_3, n_1 and m_1 define the operating point of the electric arc furnace. The parameters used in this test case are reported in the Appendix.

Table I summarizes the number of Newton-Raphson iterations (m) required to achieve the periodic steady-state. It can be observed that the DTS method needed 5 iterations, whilst the FD method required 6, 5 and 4 iterations. The elapsed times needed by the DTS and FD method are {0.18, 0.62, 2.48} and {0.52, 0.88, 1.57} seconds, respectively, for a number of samples {63, 127, 255}. Fig. 1 shows the steady-state solution of the arc voltage obtained with the DTS method and N=127.

B. Single-phase electric network

A single-phase electric power network containing a transmission line system and a transformer at its receiving-end terminal is presented in this section. Considering the differential length of a single-phase transmission line, with per unit length electric parameters r, l, g and c distributed perfectly along the line, then the expressions for the voltages and currents at any point of the line are [13],

$$\frac{\partial v}{\partial \chi} = ri + l\frac{\partial i}{\partial t} \tag{18}$$

$$\frac{\partial i}{\partial \chi} = gv + c\frac{\partial v}{\partial t} \tag{19}$$

The partial derivatives equations (PDEs) can be transformed into a set of ODEs by discretizing the line with respect to the distance χ. The equivalent lumped parameter circuit for the

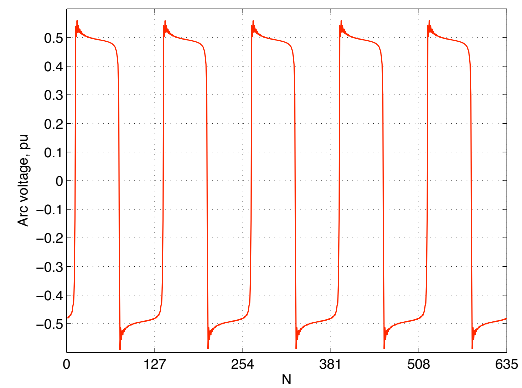


Fig. 1. Discrete-time solution for the arc voltage.

j-th section of the transmission line is defined by the set of ODEs,

$$\dot{i}_j = (v_j - Ri_j - v_{j+1})/L \tag{20}$$

$$\dot{v}_{j+1} = (i_j - Gv_{j+1} - i_{j+1})/C \tag{21}$$

where the data for the lumped parameters are listed in the Appendix.

On the other hand, the equivalent circuit for the unloaded single-phase transformer is described as,

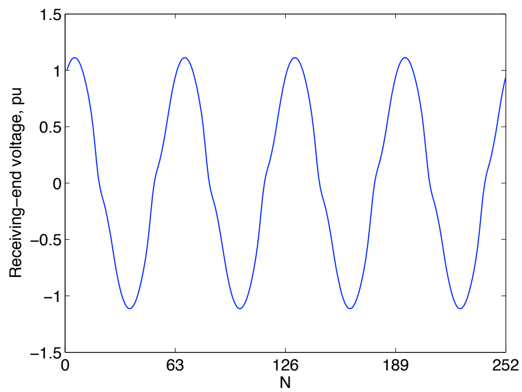
$$\begin{array}{l} \dot{i}_p = (v_{j+1} - (R_p + R_c)\, i_p + R_c i_m)/L_p \\ \dot{\lambda}_m = R_c\,(i_p - i_m) \end{array} \tag{22}$$

where R_p and L_p are the resistance and inductance of the primary side of the transformer and R_c represents the core losses. The saturation characteristic of the single-phase transformer is represented by means of a polynomial defined as,

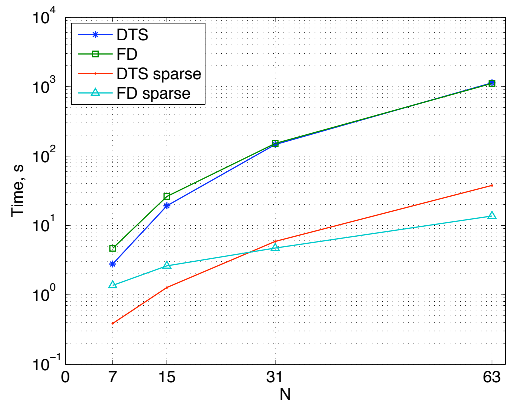
$$i_m = 0.001\lambda_m + 0.0006\lambda_m^{19}\ \text{pu} \tag{23}$$

Table II shows the number of iterations of the NR method and the maximum mismatches during convergence for a transmission system with 63 sections and a transformer at its receiving-end terminal. Four iterations of the NR method are needed using the DTS method, whilst the FD method required 6, 5 and 4 iterations with a number of samples $N = \{7, 15, 31, 63\}$. The NR method shows quadratic convergence for the DTS method, while the convergence of the FD method degradates with $N < 31$.

Fig. 2(a) shows the time domain solution of the voltage at the receiving-end terminal of the transmission line. It can be observed that the sending-end voltage is free of distortion, whilst the receiving-end voltage is distorted due to the operation of the transformer at its nonlinear region. Furthermore, the densities of the Jacobian matrix using N={7,15,31,63} are $D_{DTS} = \{0.010, 0.0088, 0.0083, 0.0081\}$ and $D_{FD} = \{0.0066, 0.0031, 0.0015, 0.00074\}$ for the DTS and FD method, respectively. Figure 2(b) summarizes the elapsed times needed by the DTS and FD methods with and without using sparsity techniques. It can be appreciated that speedup factors of nearly 30 and 85 are achieved with the DTS



(a)



(b)

Fig. 2. Discrete solution for a).- the receiving-end voltage and b). - elapsed times to calculate the steady-state solution.

and FD methods including sparsity techniques and $N = 63$. Meanwhile, speedup gains of 7 and 3 are measured with $N = 7$ for the DTS and FD methods, respectively. Therefore, the DTS method is faster than the FD method for $N < 31$ with and without sparsity techniques.

C. Three-phase STATCOM

A simple three-phase system including an equivalent transmission line and a synchronous, three-phase, STATic COMpensator (STATCOM) is presented in this section. The STATCOM model comprises the Voltage Source Converter (VSC) based on a square-wave switching scheme and the coupling transformer, which is modelled as a linear impedance (see Fig. 3). The objective of the STATCOM is to either supply or absorb active power to maintain constant the DC capacitor voltage and exchange reactive power with the point of common coupling (PCC). The STATCOM control system (not shown) maintains the DC capacitor voltage at 2.0 pu and the voltage at the point PCC at 1.0 pu.

The VSC used in the STATCOM model is a three-phase six-pulse bridge. Neglecting losses in the semiconductor switches, the VSC model can be represented with the following voltage and current relationships [12] ,

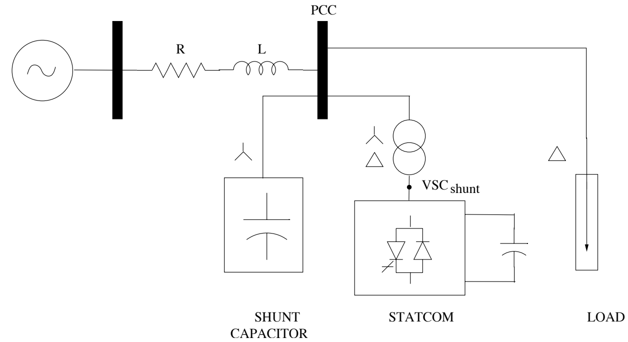


Fig. 3. Block diagram of STATCOM.

$$\begin{bmatrix} e_{ab} \\ e_{bc} \\ e_{ca} \end{bmatrix} = \begin{bmatrix} s_a \\ s_b \\ s_c \end{bmatrix} v_{dc} \tag{24}$$

and

$$i_{dc} = \begin{bmatrix} i_a & i_b & i_c \end{bmatrix} \begin{bmatrix} s_a \\ s_b \\ s_c \end{bmatrix} \tag{25}$$

where

s_a, s_b, s_c switching functions that govern the VSC

e_{ab}, e_{bc}, e_{ca} line voltages on the secondary side of the transformer

i_a, i_b, i_c AC currents of the VSC

v_{DC} DC capacitor voltage

The ordinary differential equation of the DC capacitor is,

$$\frac{dv_{dc}}{dt} = \frac{1}{C} i_{dc} \tag{26}$$

where C is the capacitance associated to the DC side.

Table III summarizes the total number of NR iterations needed to compute the steady-state solution. It can be appreciated that only two iterations are needed by the DTS and FD method using a set of samples {33, 69, 123, 189}. Figs. 4(a) show the discrete-time solution obtained with N=69 for the current through the STATCOM coupling transformer. As expected, the AC currents are highly distorted due to the VSC switching pattern. On the other hand, Fig. 4(b) summarizes the elapsed times needed to calculate the steady-state solution for the DTS and FD methods using the sparse matrix techniques. It can be appreciated that the DTS approach is faster than the FD method for N<63 with and without using sparse techniques.

IV. Conclusions

A discrete-time domain method based on a finite dimensional representation of the derivative and a Newton method has been presented in this paper to determine the periodic steady-state solution of electric power networks. Sparse matrix techniques are applied to improve the DTS method efficiency in time and storage when solving practical problems. It must be brought into attention the fact that the DTS approach demands important memory resources to allocate the arrays involved in this methodology. However, this problem is overcome with the application of sparsity techniques. The DTS method

TABLE III

MISMATCHES DURING CONVERGENCE OF THE THREE-PHASE STATCOM.

m	DTS method			
	33	69	123	189
1	7.868e-01	6.460e-01	5.649e-01	5.175e-01
2	3.153e-14	1.172e-13	1.292e-13	5.773e-14
	FD method			
1	2.518e+00	2.529e+00	2.532e+00	2.531e+00
2	1.161e-13	1.207e-13	6.248e-13	1.072e-12

shows better convergence properties and speed up factors than the FD method for a number of samples up to 31 and 63 for a single-phase network and a three-phase system with a STATCOM, respectively. Both methods require a trajectory to be specified by a sequence of points and a Newton-Raphson to iterate till the sequence satisfies a set of nonlinear algebraic equations. Nevertheless, the FD method requires to satisfy an integration formula, whilst the DTS method satisfies the set of differential equations transformed into a set of nonlinear algebraic equations.

APPENDIX

The parameters used to solve the test cases presented in this work are listed in the per unit system.

Electric arc furnace

$K_1 = 0.08$, $K_2 = 0.005$, $K_3 = 3$, $n_1 = 2$ and $m_1 = 0$.

Transmission line lumped parameters

$R = 0.03$ pu, $G = 1.0e-09$ pu, $L = 0.001$ pu and $C = 1.0e-08$ pu.

Transformer

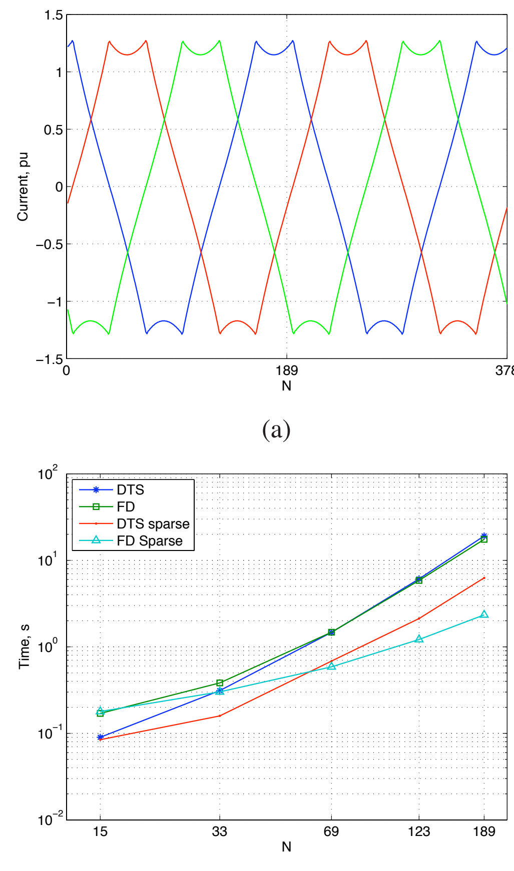
$R_p = 0.005$ pu, $R_c = 100$ pu and $L_p = 0.075$ pu.

Three-phase STATCOM

$R = 0.01$, $L = 0.01$

$C_{pcc} = 0.1$, $R_{load} = 1.0$, $L_{load} = 0.01$

$R_{statcom} = 0.01$, $L_{statcom} = 0.15$, $C_{statcom} = 100$



(a)

(b)

Fig. 4. Discrete-time solution for a).- the currents on the AC side of the VSC and b).- the elapsed times to determine de steady-state solution.

REFERENCES

[1] T.S. Parker and L.O. Chua, *Practical Numerical Algorithms for Chaotic Systems*, Springer-Verlag, 1989.

[2] M.S. Nakhla and J. Vlach, "A Piecewice Harmonic Balance Technique for Determination of Periodic Response of Nonlinear Systems", *IEEE Trans. Circuits and Systems*, Vol. 23, No.2, pp. 85-91, Feb. 1976.

[3] A. Semlyen, E. Acha and J. Arrillaga, "Harmonic Norton Equivalent for the Magnetising Branch of a Transformer", *IEE Proceedings Gener. Trans. and Distrib.*, Part C, vol. 134, no. 2, pp. 162-169, March 1987.

[4] H.W. Dommel et al., "Harmonics from Transformer Saturation", *IEEE Trans. Power Systems*, Vol. 1, No. 2, pp. 209-214, Apr. 1986.

[5] T.J. Aprille and T.N. Trick, "Steady-State Analysis of Nonlinear Circuits with Periodic Inputs", *Proceedings of the IEEE*, Vol. 60, No. 1, pp. 108-114, Jan. 1972.

[6] S. Skelboe, "Computation of the Periodic Steady-State Response of Nonlinear Networks by Extrapolation Methods", *IEEE Trans. Circuits and Systems*, Vol. 27, No. 3, pp. 161-175, Mar. 1980.

[7] A. Semlyen and A. Medina, "Computation of the Periodic Steady State in Systems with Nonlinear Components Using a Hybrid Time and Frequency Domain Methodology", *IEEE Trans. Power Systems*, Vol. 10, No. 3, pp. 1498-1504, Aug. 1995.

[8] J. B. Dalmau and P. P. Schönwälder "A Discrete-Time Approach to the Steady-State and Stability Analysis of Distributed Nonlinear Autonomous Circuits", *IEEE Trans. on Circuits and Systems I:Fundamental Theory and Applications*, Vol. 47, No. 2, pp. 231-236, Feb. 2000.

[9] F.A. López et al.,"A Discrete-Time Technique for the Steady State Analysis of Nonlinear Switched Circuits with Inconsistent Initial Conditions", *IEEE Symposium on Circuits and Systems*, Vol. 3, pp. 6-9, May 2001.

[10] R.G. Campos and L.O. Pimentel, "A finite-dimensional representation of the quantum angular momentum operator", *Nuovo Cimento* B, vol. 116, no. 1, pp. 31-45, Jan. 2001.

[11] S.C. Eisenstat et al., "Yale Sparse Matrix Package I" Int. J. Numer. Methods in Engin., vol. 18, no. 8, pp. 1145-1151, Aug. 1982.

[12] E. Acha and M. Madrigal, *Power Systems Harmonics: Computer Modelling and Analysis*, Wiley, 2001.

[13] M. Salih and M. Koksal, "Solution of Eigenproblems for State-Space Transient Analysis of Transmission Lines", Elsevier Electric Power Systems Research, vol. 55, no. 1, pp. 7-14, July 2000.