Equilibrium Pricing of Weather Derivatives in a Multi-period Trading Environment

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Abstract-- **The prevalence of commercial activities whose profit and cost are correlated with weather risk makes weather derivatives valuable financial instruments that enable hedging of price or volumetric (quantity) risk in many industries. This paper proposes a multi-period equilibrium pricing model for weather derivative. In our stylized economy representative agents of weather-sensitive industries optimizes their hedging portfolios that drive the supply and demand for weather derivative which are dynamically determined based on a utility indifference pricing framework. At equilibrium the weather derivative market will be cleared and their market price can be obtained. Numerical examples illustrate the equilibrium prices and optimal choices for the weather derivative as function of the correlation between weather indices and demand for the underlying commodity. We also demonstrate the benefit of multiple trading opportunities which allows rebalancing of the hedging portfolio prior to the commodity delivery date, as compared to a single shot framework.**

Index Terms—**Hedging, Energy Risk, Weather Derivatives, Volumetric Risk, Equilibrium Pricing Model, Indifference Pricing**

I. Introduction

Weather derivatives are contingent claims whose payoff depends on the value of an underlying weather index such as degree days over the contract periods. Many weather-sensitive industries and in particularly the energy industry, confront two types of risk; price risk and volumetric (or quantity) risk, which are both correlated with weather. Existing commodity derivatives cannot be used to fully hedge price and volumetric risks because the two risks are not perfectly correlated and volume instruments are not traded. Therefore, weather-sensitive industries may wish to diversify their portfolios by introducing financial instruments driven by weather indices to mitigate risk which cannot be covered by commodity derivatives. One of the main buyers of weather derivatives is the energy industry because it needs to stabilize profits as the market becomes deregulated and more competitive and because temperature is the most significant factor which affects price and demand of power and natural gas.

Weather derivatives were initially introduced in the over-the-counter (OTC) markets of the US in 1996. In order to meet rapidly growing demand and increase liquidity and accessibility, the Chicago Mercantile Exchange (CME) launched the first electronic market place for standardized weather derivatives in 1999. Since 2007 18 US cities' weekly, monthly, and seasonal weather derivatives, 6 European, 2 Asia-Pacific, 6 Canadian cities' monthly and seasonal weather derivatives are being traded at CME. Also Hurricane weeklies were newly listed in 2007. Despite the advantages and increased use of weather derivatives, there are no effective pricing models for these instruments because the underlying weather index is not a tradable commodity or equity share and the market is incomplete. Previous studies on pricing weather derivatives can be classified as actuarial pricing [1][2][3], risk-neutral pricing [5], indifference pricing [6][7][8][9], and equilibrium pricing [10][11][12][13]. Unlike other pricing approaches an equilibrium pricing model can explain the market dynamics more realistically because the market price of weather derivatives will settle at the equilibrium price which is between a bid and ask price spread resulting from the various pricing methods. Moreover, in typical weather derivatives markets underwriters have only limited control over prices and the ultimate price of the weather derivatives is determined by supply and demand resulting from hedging activities of market participants exposed to weather risk who are optimizing their hedging portfolios.

In this paper we propose an equilibrium pricing model in a multi-period setting under an exponential utility preference function. The paper extends previous work by the authors [12] on a single-period equilibrium pricing model in a multi-commodity setting. We break up the planning horizon of market participants in a discrete manner. In each period agents can rebalance their portfolios, employing all available information, so as to achieve higher expected utility level of terminal wealth. In order to obtain an equilibrium price of weather derivatives we recursively derive supply and demand functions in each period by combining the indifference pricing method and the expected utility maximization problems of buyers and sellers using a dynamic programming algorithm. An equilibrium price of the weather derivatives in each period is determined from the derived supply and demand functions and a market clearing condition.

The outline of the paper is as follows. In section 2 we first

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study a single-period indifference pricing problem faced by buyers and sellers and then introduce the multi-period formulation. Numerical examples are used to illustrate the results and extract insights in section 3. We conclude in section 4.

II. MODEL

A. Assumptions and notation

Our model assumes a frictionless economy and we consider a multi-period portfolio optimization problem where each representative market participant can rebalance her portfolio at the beginning of each period given the updated information. We assume that there are three types of market participants; weather derivatives buyers with a liquid commodity derivatives market (type i buyers), buyers dealing with a commodity for which there is no liquid commodity derivatives market (type j buyers), and an issuer (underwriter, denoted by m) of weather derivatives who is a pure financial entity and does not engage in sales of physical commodities or trading of commodity derivatives. All agents can trade on either side, i.e., both types of buyers can engage in short sales whereas the issuer can buy weather derivatives from the market if necessary. We assume that all agents maximize the expected utility of terminal wealth subject to a self-financing constraint. The income function for each agent has the form of (regulated retail price - wholesale spot price)$\times$demand, which assumes that each buyer has an obligation to meet random demand at a fixed price either due to a regulatory constraint or competitive pressure. This type of the income function is common in energy industries (e.g. electricity) where the supplier has an obligation to meet variable load at fixed retail prices fixed by regulators or through a service contract for a load slice. Given the income function, each buyer faces not only spot price risk but also volumetric risk, whereas, spot price, demand, and temperature are all correlated. The commodity portfolio consists of a risk free bond and commodity derivatives which include forward contracts and European call and put options of all strikes. The underlying asset price of the commodity derivatives is its own spot price. Each buyer can trade these financial assets from time 0 to time N-1 so as to hedge the two types of risks and maximize expected utility. In addition each buyer is willing to include weather derivatives in her portfolio at or below the indifference price. In other words, each buyer is willing to pay a certain amount for the weather derivative to get utility gains by reducing uncertainty but does not want to be worse off in expected utility terms than without the weather derivative. By applying the indifference pricing to valuation of the weather derivative in the portfolio optimization problem we can determine supply and demand functions for the weather derivative in each period which in turn determine the equilibrium price through a market clearing condition.

We assume that all agents are exponential utility maximizers. The exponential utility function has the form of $U(x) = -\frac{1}{a}\exp(-ax)$ where a denotes the risk aversion coefficient and $U(\cdot)$ is smooth, increasing and strictly concave on R, and twice continuously differentiable on R. In our analysis we define states, controls, randomness, and the value functions of the stochastic dynamic programming problems corresponding to type i and j buyers and the issuer. We then derive the equilibrium price under the exponential utility function with a single and with multiple trading opportunity. In each period the derived equilibrium price and optimal choices of the weather derivative provide the value functions for type i and j buyers and the issuer recursively.

B. Single-period Indifference Price

In this section we define states, controls, randomness, and the value functions of the stochastic dynamic programming problems corresponding to type i and j buyers and the issuer. We then derive the equilibrium price under the exponential utility function with a single trading opportunity. First we describe the dynamic system equations for type i buyers as

$$S_{i,n+1} = z_{i,n}(S_{i,n}, x_{i,n+1}, Y_{i,n+1}) \quad \forall n = 0,1,\dots,N-1 \tag{1}$$

where $S_{i,n}$ denotes a state of the dynamic system at time n and has the vector form $(V_{i,n}, P_{i,n}, W_n)$. $V_{i,n}$ is the type i buyer's portfolio value at time n and $z_{i,n}(\cdot)$ denotes some deterministic function which maps the previous state to the next state. The randomness $Y_{i,n+1}$ can be expressed as $(P_{i,n+1}, W_{n+1})$. The last state $S_{i,N}$ has the additional randomness because terminal wealth of the type i buyers is dependent on the random demand $D_{i,N}$ and temperature T_N, therefore the randomness at N $Y_{i,N}$ has the form $(P_{i,N}, D_{i,N}, T_N)$. In addition our control $x_{i,n+1}(P_{i,n+1})$ at time n represents the payoff function of the commodity derivatives portfolio which consists of a risk-free bond and commodity derivatives. Note that $x_{i,n+1}(P_{n+1})$ is predictable, i.e. it is $\mathcal{F}_n$-measurable. The type i buyer's profit function (or terminal wealth) at time N can be defined as:

$$\Pi_{i,N} = I_i(D_{i,N}, P_{i,N}) + x_{i,N}(P_{i,N}) + \alpha_{i,N} W_N = I_i(D_i, P_{i,N}) + V_{i,N} \tag{2}$$

where $I_i(D_{i,N}, P_{i,N})$ denotes the income of type i buyers from the retail business. $\alpha_{i,N}$ and W_N are the quantity purchased at time N-1 and the payoff (or price) at time N of the weather derivative, respectively. Then the corresponding value function of type i buyers at time n can be defined as

$$\begin{aligned} J(V_{i,n} - \alpha_{n+1} W_n, \alpha_{n+1}) &= \max_{\{x_{i,n+1}(P_{n+1})\}} E_n[U_i(\Pi_{i,N})] \\ s.t.\ E_n^{\mathbb{Q}}\left[\frac{x_{i,n+1}(P_{i,n+1})}{1+r_n}\right] &+ \alpha_{i,n+1} W_n - V_{i,n} = 0 \end{aligned} \tag{3}$$

where r_n denotes the interest rate at time n. In the above self-financing trading strategy constraint the expected value of the discounted portfolio payoff under a risk-neutral probability measure $\mathbb{Q}$ is the price of optimal commodity portfolio at time n. For each realization p of the random price $P_{i,n+1}$ we will find the optimal payoff function $x_{i,n+1}(p)$ of problem (3).

In [14] it was shown that any twice continuously differentiable function, $f(S)$, of the terminal stock price S can be replicated by a unique initial position of $f(S_0)-f'(S_0)S_0$ unit discount bonds, $f'(S_0)$ shares, and $f''(K)dK$ out-of-the-money options of all strikes K:

$$f(S)=[f(S_0)-f'(S_0)S_0]+f'(S_0)S+\int_0^{S_0}f''(K)(K-S)^+dK \\ +\int_{S_0}^{\infty}f''(K)(S-K)^+dK \tag{4}$$

However, the replication of the optimal payoff function in an incomplete market is out of scope of this paper and we will not elaborate on this issue any further. We will just specify the closed form of the optimal payoff function in each period.

To solve the stochastic dynamic programming problem (3), we need to work backwards. In addition the problem in each period is a convex programming problem because the concavity of the objective function and the convexity of the feasible set. Therefore we formulate the original constrained problem as a Lagrangian relaxation problem and the optimal solutions of the relaxed problem will have no duality gap. At time N we have

$$J(V_{i,N})=E_N[U_i(\Pi_{i,N})]=U_i(\Pi_{i,N})=U_i(I_i(D_i,P_{i,N})+V_{i,N}) \tag{5}$$

At time N-1 the buyers' maximization problem is the same as in a single period problem. The corresponding single-period problem is:

$$J(V_{i,N-1}-\alpha_{i,N}W_{N-1},\alpha_{i,N})=\max_{\{x_{i,N}(P_{i,N})\}}E_{N-1}[U_i(\Pi_{i,N})] \\ s.t.\quad E^{\mathbb{Q}}_{N-1}[\frac{x_{i,N}(P_{i,N})}{1+r_{N-1}}]+\alpha_{i,N}W_{N-1}-V_{i,N-1}=0 \tag{6}$$

Proposition 1 The optimal payoff function $x_{i,N}(P_{i,N})$ of problem (6) is :

$$x^{w*}_{i,N}(P_{i,N})=\frac{1}{a_i}(\ln E_{N-1}[\exp(-a_i(I_i+\alpha_{i,N}W_N))\mid P_{i,N}] \\ -E^{\mathbb{Q}}_{N-1}[\ln E_{N-1}[\exp(-a_i(I_i+\alpha_{i,N}W_N))\mid P_{i,N}]] \\ -(\ln\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}-E^{\mathbb{Q}}_{N-1}[\ln\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}]) \\ +a_i(1+r_{N-1})(V_{i,N-1}-\alpha_{i,N}W_{N-1})) \tag{7}$$

where $f_{i,N}(P_{i,N})$ and $g_{i,N}(P_{i,N})$ are probability density functions of the type i commodity spot price under the real world probability measure $\mathbb{P}$ and a risk neutral probability measure $\mathbb{Q}$, respectively and r_{N-1} is the interest rate at time N-1. The proof follows directly from first order optimality conditions and is omitted due to space limitation.

To get the indifference price for type i buyers, we need to solve the expected utility maximization problem without the weather derivative.

$$J(V_{i,N-1},0)=\max_{\{x_{i,N}(P_{i,N})\}}E_{N-1}[U_i(I_i+x_{i,N}(P_{i,N}))] \\ s.t.\quad E^{\mathbb{Q}}_{N-1}[\frac{x_{i,N}(P_{i,N})}{1+r_{N-1}}]-V_{i,N-1}=0 \tag{8}$$

Proposition 1 implies the following optimal payoff function in case of no weather derivative.

$$x^{n*}_{i,N}(P_{i,N}))=\frac{1}{a_i}(\ln E_{N-1}[\exp(-a_iI_i)\mid P_{i,N}]-E^{\mathbb{Q}}_{N-1}[\ln E_{N-1}[\exp(-a_iI_i)\mid P_{i,N}]] \\ -(\ln\frac{g(P_{i,N})}{f(P_{i,N})}-E^{\mathbb{Q}}_{N-1}[\ln\frac{g(P_{i,N})}{f(P_{i,N})}])+a_i(1+r_{N-1})V_{i,N-1}) \tag{9}$$

Now we have the maximized expected utility function with and without weather derivative and therefore, the indifference price can be obtained from the equation.

$$J(V_{i,N-1}-\alpha_{i,N}W_{N-1},\alpha_{i,N})=J(V_{i,N-1},0) \tag{10}$$

As a result, the indifference price of type i buyers has the form:

$$W_{N-1}=\frac{1}{a_i(1+r_{N-1})\alpha_{i,N}}\ln\frac{\Delta_{i,N-1}}{\Lambda_{i,N-1}}=h_{i,N-1}(\alpha_{i,N}) \tag{11}$$

where the Greeks are given by

$$\Delta_{i,N-1}=E_{N-1}[\frac{\exp(-a_iI_i)\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}}{E_{N-1}[\exp(-a_iI_i)\mid P_{i,N}]}] \\ \times\exp(E^{\mathbb{Q}}[\ln E_{N-1}[\exp(-a_iI_i)\mid P_{i,N}]]) \tag{12}$$

$$\Lambda_{i,N-1}=E_{N-1}[\frac{\exp(-a_i(I_i+\alpha_NW_N))\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}}{E_{N-1}[\exp(-a_i(I_i+\alpha_NW_N))\mid P_{i,N}]}] \\ \times\exp(E^{\mathbb{Q}}[\ln E_{N-1}[\exp(-a_i(I_i+\alpha_NW_N))\mid P_{i,N}]]) \tag{13}$$

Note that the indifference price of the weather derivative is independent of the portfolio value $V_{i,N-1}$ due to the Constant Absolute Risk Aversion (CARA) property of the exponential utility function.

Next we define the dynamic system equations of the type j buyers as

$$S_{j,n+1}=z_{j,n}(S_{j,n},\beta_{j,n+1},Y_{j,n+1})\quad\forall n=0,1,\ldots,N-1 \tag{14}$$

where a state $S_{j,n}$ is the vector form of $(V_{j,n},W_n)$, the control $\beta_{j,n+1}$ is the risk-free bond position, and the randomness $Y_{j,n+1}$ has the single term W_{n+1}. The randomness of the last state $S_{j,N}$ includes the additional random variables because the income function of terminal wealth of type j buyers has the random price and the random demand of type j commodity. So $Y_{j,N}$ has the vector form $(P_{j,N},D_{j,N},T_N)$. The profit function (or terminal wealth) of the type j buyers is

$$\Pi_{j,N}=I_j(D_{j,N},P_{j,N})+\alpha_{j,N}W_N+\beta_{j,N}B_N \tag{15}$$

The type j buyers at time n have the following utility maximization problem if the weather derivative is included in their portfolios.

$$J(V_{j,n}-\alpha_{j,n+1}W_n,\alpha_{j,n+1})=\max_{\beta_{j,n+1}}E_n[U_j(\Pi_{j,N})]$$

$$s.t.\ \alpha_{j,n+1}W_n + \beta_{j,n+1}B_n - V_{j,n} = 0 \quad (16)$$

At time N-1 the single-period problem of the type j buyers can be simplified as an unconstrained problem if we plug $\beta_{j,N}$ obtained from the self-financing constraint into the objective function. Then, the reformulated problem is

$$J(V_{j,N-1} - \alpha_{j,N}W_{N-1}, \alpha_{j,N})$$
$$= E_{N-1}[U_j(I_j + \alpha_{j,N}W_N + (V_{j,N-1} - \alpha_{j,N}W_{N-1})(1+r_{N-1}))] \quad (17)$$

To get the indifference price we need to consider the problem without the weather derivative in the portfolio.

$$J(V_{j,N-1}, 0) = E_{N-1}[U_j(I_j + V_{j,N-1}(1+r_{N-1}))] \quad (18)$$

Equating the right hand sides of equations (17) and (18) gives the indifference price of type j buyers as

$$W_{N-1} = \frac{1}{a_j(1+r_{N-1})\alpha_{j,N}} \ln \frac{E_{N-1}[\exp(-a_j I_j)]}{E_{N-1}[\exp(-a_j(I_j + \alpha_{j,N}W_N))]} = h_{j,N-1}(\alpha_{j,N}) \quad (19)$$

So far we have studied the buyers' problems and in this paragraph we will explore the issuer's problem. We assume that the issuer is a pure financial firm, which can trade only risk-free bonds and weather derivatives but not commodity derivatives. The issuer is assumed to be able to sell or buy back outstanding weather derivative in any period. When the weather derivatives are sold in the market the issuer will receive the selling price and at maturity settle the payoff of the weather derivative. The issuer's system equations are similar to type j buyers except at the terminal time since the issuer does not have the income function which contains two random variables, price and quantity for the commodity. Then the issuer's dynamic system equations can be expressed as:

$$S_{m,n+1} = z_{m,n}(S_{m,n}, \beta_{m,n+1}, Y_{m,n+1}) \quad \forall n = 0,1,\ldots,N-1 \quad (20)$$

where a state $S_{m,n}$ is characterized by the vector $(V_{m,n}, W_n)$ and the randomness $Y_{m,n+1}$ of the dynamic system equations is the weather derivative price W_{n+1}. The issuer's value function is then,

$$J(V_{m,n} + \alpha_{m,n+1}W_n, -\alpha_{m,n+1}) = \max_{\beta_{m,n+1}} E_n[U_j(-\alpha_{m,N}W_N + \beta_{m,N}B_N)]$$
$$s.t.\quad \alpha_{m,n+1}W_n + V_{m,n} = \beta_{m,n+1}B_n \quad (21)$$

The issuer's indifference price at N-1, therefore, can be calculated from the following equation.

$$J(V_{m,N-1} + \alpha_{m,N}W_{N-1}, -\alpha_{m,N})$$
$$= E_{N-1}[U_m((V_{m,N-1} + \alpha_{m,N-1}W_{N-1})(1+r_{N-1}) - \alpha_{m,N-1}W_N)]$$
$$= E_{N-1}[U_m(V_{m,N-1}(1+r_{N-1}))] = J(V_{m,N-1}, 0) \quad (22)$$

After simplifying the above equation we have the following indifference price for the issuer.

$$W_{N-1} = \frac{\ln E_{N-1}[\exp(a_m \alpha_{m,N} W_N)]}{a_m(1+r_{N-1})\alpha_{m,N}} = h_{m,N-1}(\alpha_{m,N}) \quad (23)$$

Now we have the supply and demand functions for the weather derivative at N-1. If the supply and demand quantities are a function of price then we can directly apply the market clearing condition i.e. a zero net supply equation. In our case the price, however, is a function of quantities and the inverse functions of the derived supply and demand are hard to get, so we need to solve the following system of equations numerically. Then we can determine the equilibrium price W^*_{N-1} and the optimal choices $\alpha^*_{k,N}$ for all k in $\{i, j, m\}$ which clear the market.

$$W^*_{N-1} = h_{i,N-1}(\alpha^*_{i,N}) = h_{j,N-1}(\alpha^*_{j,N}) = h_{m,N-1}(\alpha^*_{m,N}) \forall i, j \quad (24)$$

$$\sum_i \alpha^*_{i,N} + \sum_j \alpha^*_{j,N} = \alpha^*_{m,N} \quad (25)$$

By substituting the derived optimal portfolio positions and the equilibrium price into the objective function we can find the value function of the type i buyers as

$$J(V_{i,N-1} - \alpha^*_N W^*_{N-1}, \alpha^*_N) = J^*_{i,N-1}(V_{i,N-1})$$
$$= -\frac{1}{a_i}\exp(-a_i(1+r_{N-1})V_{i,N-1})\Theta_{i,N-1} \quad (26)$$

where $\Theta_{i,N-1}$ is an $\mathcal{F}_{N-1}$ -measurable random variable , independent on $V_{i,N-1}$, and has the form:

$$\Theta_{i,N-1} = \exp(a_i(1+r_{N-1})\alpha^*_{i,N}W^*_{N-1}$$
$$+ E^{\mathbb{Q}}_{N-1}[\ln E_{N-1}[\exp(-a_i(I_i + \alpha^*_{i,N}W_N)) \mid P_{i,N}] - \ln\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}])$$
$$\times E_{N-1}\left[\frac{\exp(-a_i(I_i + \alpha^*_{i,N}W_N))\frac{g_{i,N}(P_{i,N})}{f_{i,N}(P_{i,N})}}{E_{N-1}[\exp(-a_i(I_i + \alpha^*_{i,N}W_N)) \mid P_{i,N})]}\right] \quad (27)$$

In equation (26) we use $J^*_{i,N-1}(V_{i,N-1})$ as short hand notation for $J(V_{i,N-1} - \alpha^*_N W^*_{N-1}, \alpha^*_N)$. From now on we will also use this short hand notation for the type j buyers and the issuer. Similarly, the value function of type j buyers will be

$$J(V_{j,N-1} - \alpha^*_N W^*_{N-1}, \alpha^*_N) = J^*_{j,N-1}(V_{j,N-1})$$
$$= -\frac{1}{a_j}\exp(-a_j(1+r_{N-1})V_{j,N-1})\Theta_{j,N-1} \quad (28)$$

where Θ_{j,N_1} is again independent of $V_{j,N-1}$, $\mathcal{F}_{N-1}$ -measurable, and has the form:

$$\Theta_{j,N_1} = \exp(a_j(1+r_{N-1})\alpha^*_{j,N}W^*_{N-1})E_{N-1}[\exp(-a_j(I_j + \alpha^*_{j,N}W_N))] \quad (29)$$

Finally, the value function for the issuer will be

$$J(V_{m,N-1} + \alpha^*_N W^*_{N-1}, -\alpha^*_N) = J^*_{m,N-1}(V_{m,N-1})$$
$$= -\frac{1}{a_m}\exp(-a_m(1+r_{N-1})V_{m,N-1})\Theta_{m,N-1}$$
$$= -\frac{1}{a_m}\exp(-a_m(1+r_{N-1})V_{m,N-1}) \quad (30)$$

The last equality comes from the indifference pricing equality because $\Theta_{m,N-1}$ has the form:

$$\Theta_{m,N-1} = \exp(-a_m(1+r_{N-1})\alpha^*_{m,N}W^*_{N-1})E_{N-1}[\exp(a_m\alpha^*_{m,N}W_N)] = 1 \quad (31)$$

Note that all value functions are again exponential forms multiplied by $\mathcal{F}_{N-1}$ -measurable random variables and therefore, we can see the preservation for the dynamic programming algorithm.

C. Multi-period Indifference Price

Now we sketch the solution for a multi-period indifference pricing problem. Consider the indifference pricing model for type i buyers, first. In the previous section we already solved a single-period problem at N-1, so in this section we will start from time N-2 and generalize the result for N periods. By applying the Bellman's *Principle of Optimality* we get:

$$\max_{\{x_{i,N-1},\lambda_{i,N-1}\}} E_{N-2}[U_i(\Pi_{i,N})]-\lambda_{i,N-1}C_{i,N-2}$$
$$=\max_{\{x_{i,N-1},\lambda_{i,N-1}\}} E_{N-2}[\max_{\{x_{i,N},\lambda_{i,N}\}} U_i(\Pi_{i,N})-\lambda_{i,N}C_{i,N-1}]-\lambda_{i,N-1}C_{,N-2} \quad (32)$$

where $C_{i,n}$ denotes the self-financing constraint at time n. If we use the law of iterated expectations, then we can find the recursive form known as the Bellman equation.

$$J(V_{i,N-2}-\alpha_{i,N-1}W_{N-2},\alpha_{i,N-1})$$
$$=\max_{\{x_{i,N-1},\lambda_{i,N-1}\}} E_{N-2}[\max_{\{x_{i,N},\lambda_{i,N}\}} E_{N-1}[U_i(\Pi_{i,N})]-\lambda_{i,N}C_{i,N-1}]-\lambda_{i,N-1}C_{i,N-2}$$
$$=\max_{\{x_{i,N-1},\lambda_{i,N-1}\}} E_{N-2}[J^*_{i,N-1}(V_{i,N-1})]-\lambda_{i,N-1}C_{i,N-2} \quad (33)$$

Define R_n as $\prod_{i=n}^{N-1}(1+r_i)$. By solving the problem (33) we derive the following optimal solution at time N-2.

Proposition 2 *The optimal payoff function* $x_{i,N-1}(P_{i,N-1})$ *at N-2 is*

$$x^{w*}_{i,N-1}(P_{i,N-1})=\frac{1}{a_iR_{N-1}}(\ln E_{N-2}[\exp(-a_iR_{N-1}\alpha_{i,N-1}W_{N-1})\Theta_{i,N-1}\mid P_{i,N-1}]$$
$$-E^{\mathbb{Q}}_{N-2}[\ln E_{N-2}[\exp(-a_iR_{N-1}\alpha_{i,N-1}W_{N-1})\Theta_{i,N-1}\mid P_{i,N-1}]]$$
$$-(\ln\frac{g_{i,N-1}(P_{i,N-1})}{f_{i,N-1}(P_{i,N-1})}-E^{\mathbb{Q}}_{N-2}[\ln\frac{g_{i,N-1}(P_{i,N-1})}{f_{i,N-1}(P_{i,N-1})}]) \quad (34)$$
$$+a_iR_{N-2}(V_{i,N-2}-\alpha_{i,N-1}W_{N-2}))$$

We omit the proof which can be found in [13] due to space limitation.

Again by setting

$$J(V_{i,N-2}-\alpha_{i,N-1}W_{N-2},\alpha_{i,N-1})=J(V_{i,N-2},0) \quad (35)$$

we can obtain the indifference price W_{N-2} for the type i buyers. Based on the pattern of the optimality condition at time N-1 and N-2, inductive reasoning implies that the Bellman equation can be written as

$$J^*_{i,n}(V_{i,n})=\max_{\{x_{i,n+1},\lambda_{i,n+1}\}} E_n[J^*_{i,n+1}(V_{i,n+1})]-\lambda_{i,n+1}C_{i,n}$$
$$\forall n=0,1,\ldots,N-1 \quad (36)$$

and the optimality conditions at time n are:

$$\frac{\partial L_{i,n}}{\partial x_{i,n+1}(p)}=E_n[J^{*'}_{i,n+1}(V_{i,n+1})\frac{\partial V_{i,n+1}}{\partial x_{i,n+1}(p)}\mid p]f_{i,n+1}(p)-\lambda_{i,n+1}\frac{g_{i,n+1}(p)}{1+r_n}=0 \quad (37)$$

$$\frac{\partial L_{i,n}}{\partial \lambda_{i,n+1}}=V_{i,n}-(E^{\mathbb{Q}}_n[\frac{x_{i,n+1}(P_{i,n+1})}{1+r_n}]+\alpha_{i,n+1}W_n)=0 \quad (38)$$

By solving the above optimality conditions or by induction we can obtain the following optimal payoff function.

Proposition 3 *The optimal payoff function* $x^{w*}_{i,n+1}(P_{i,n+1})$ *at n can be expressed as*

$$x^{w*}_{i,n+1}(P_{i,n+1})=\frac{1}{a_iR_{n+1}}(\ln E_n[\exp(-a_iR_{n+1}\alpha_{i,n+1}W_{n+1})\Theta_{i,n+1}\mid P_{i,n+1}]$$
$$-E^{\mathbb{Q}}_n[\ln E_n[\exp(-a_iR_{n+1}\alpha_{i,n+1}W_{n+1})\Theta_{i,n+1}\mid P_{i,n+1}]]$$
$$-(\ln\frac{g_{i,n+1}(P_{i,n+1})}{f_{i,n+1}(P_{i,n+1})}-E^{\mathbb{Q}}_n[\ln\frac{g_{i,n+1}(P_{i,n+1})}{f_{i,n+1}(P_{i,n+1})}]) \quad (39)$$
$$+a_iR_n(V_{i,n}-\alpha_{i,n+1}W_n))\quad \forall n=0,1,\ldots,N-2$$

Moreover the indifference price of type i buyers at time n will be

$$W_n=\frac{1}{a_iR_n\alpha_{i,n+1}}\ln\frac{\Delta_{i,n}}{\Lambda_{i,n}}=h_{i,n}(\alpha_{i,n+1}) \quad (40)$$

where

$$\Delta_{i,n}=E_n[\frac{\Theta_{i,n+1}\frac{g_{i,n+1}(P_{i,n+1})}{f_{i,n+1}(P_{i,n+1})}}{E_n[\Theta_{i,n+1}\mid P_{i,n+1}]}]\times\exp(E^{\mathbb{Q}}_n[\ln E_n[\Theta_{i,n+1}\mid P_{i,n+1}]]) \quad (41)$$

$$\Lambda_{i,n}=E_n[\frac{\exp(-a_iR_{n+1}\alpha_{i,n+1}W_{n+1})\Theta_{i,n+1}\frac{g_{i,n+1}(P_{i,n+1})}{f_{i,n+1}(P_{i,n+1})}}{E_n[\exp(-a_iR_{n+1}\alpha_{i,n+1}W_{n+1})\Theta_{i,n+1}\mid P_{i,n+1}]}]$$
$$\times\exp(E^{\mathbb{Q}}_n[\ln E_n[\exp(-a_iR_{n+1}\alpha_{i,n+1}W_{n+1})\Theta_{i,n+1}\mid P_{i,n+1}]]) \quad (42)$$

Following a similar procedure to the above we obtain an expression for the indifference price at time n for customer type j as:

$$W_n=\frac{1}{a_jR_n\alpha_{j,n+1}}\ln(\frac{E_n[\Theta_{j,n+1}]}{E_n[\exp(-a_jR_{n+1}\alpha_{j,n+1}W_{n+1})\Theta_{j,n+1}]})$$
$$=h_{j,n}(\alpha_{j,n+1})\quad \forall n=0,1,\ldots,N-2 \quad (43)$$

Lastly we consider the issuer's selling price at which she is indifferent between not selling the weather derivative versus selling the weather derivative at the selling price today and settling the claim at maturity. The issuer's problem is similar to type j buyers' problem in the sense that she has only two tradable assets, the risk-free bond and the weather derivative but the incoming and outgoing cash flows are opposite to the type j buyers. Therefore, the issuer's portfolio value at n will be

$$V_{m,n}=(V_{m,n-1}+\alpha_{m,n}W_{n-1})(1+r_{n-1})-\alpha_{m,n}W_n \quad (44)$$

Solving a dynamic programming problem for the issuer in a similar way to that described above leads to the following expression for the indifference selling price at time n:

$$W_n=\frac{1}{a_mR_n\alpha_{m,n+1}}\ln(E_n[\exp(a_mR_{n+1}\alpha_{m,n+1}W_{n+1})])$$
$$=h_{m,n}(\alpha_{m,n+1})\quad \forall n=0,1,\ldots,N-2 \quad (45)$$

We have found the demand and supply functions for the weather derivative at time n based on the indifference pricing model. The equilibrium price W^*_n and the optimal choices $\alpha^*_{k,n+1}$ of the buyers and the issuer can be recursively determined from the market clearing conditions.

$$W^*_n=h_{i,n}(\alpha^*_{i,n+1})=h_{j,n}(\alpha^*_{j,n+1})=h_{m,n}(\alpha^*_{m,n+1})\quad \forall i,j \quad (46)$$

$$\sum_i\alpha^*_{i,n+1}+\sum_j\alpha^*_{j,n+1}=\alpha^*_{m,n+1} \quad (47)$$

In each period the derived equilibrium price and optimal choices of the weather derivative provide the value functions for type i and j buyers and the issuer recursively.

III. NUMERICAL EXAMPLE

The results are illustrated by means of a numerical example employing Monte-Carlo simulations. The example illustrates the formation of the equilibrium prices and the optimal choices of the weather call options in a single and multi-period setting under various correlations between volumetric and weather risks. The value of multiple trading opportunities is also demonstrated. We also show how the weather derivative improves risk hedging capability by reducing variance of terminal wealth, especially in situations where commodity derivatives are not available.

Our example assumes, for simplicity, a two-period planning horizon and prices a plain-vanilla weather call option with a strike of $70^\circ F$. The underlying weather index is one day average temperature. However, the pricing model can be easily extended to Cooling Degree Days (CDD)/Heating Degree Days (HDD) if we specify the stochastic processes of temperature during the contract period. In our economy there are 3 market participants; type i buyer which has a liquid commodity derivatives market, type j buyer which does not, and the issuer of the weather call option. All commodity prices and demand of the buyers are assumed to be positively correlated with temperature. In addition the temperature process follows Brownian motion as it is often assumes in the literature after removing seasonality effects. Moreover, we model the commodity price and demand of the type i and j buyers as geometric Brownian motions with a drift term.

The correlated temperature, demand, and price processes can be defined as

$$T_N = T_0 + \mu_T N + W_N^1$$

$$D_{k,N} = D_{k,0}\exp((\mu_{D_k} - \frac{1}{2}\sigma_{D_k}^2)N + \sigma_{D_k} W_{k,N}^2)$$

$$P_{k,n} = P_{k,0}\exp((\mu_{P_k} - \frac{1}{2}\sigma_{P_k}^2)n + \sigma_{P_k} W_{k,n}^3)$$

$$\forall k \in \{i,j\} \quad and \quad n \in \{n_0 = 0, n_1, \ldots, n_N = N\}$$

where $W_N^1, W_{i,N}^2, W_{j,N}^2, W_{i,n}^3$, and $W_{i,n}^3$ are correlated Brownian motions representing the correlation among temperature, demand, and price of the type i and j commodities. Since we have a two-period planning horizon, Brownian motions will be discretized by combining correlated standard normal distributions and a Markovian property. In other words we observe the realization of Brownian motions at $n_0 = 0$, $n_1 = \Delta$, and $n_2 = 2\Delta$.

The correlation coefficients among the Brownian motions are listed in Table 1. Other parameters in this numerical example are specified in Table 2. In addition we assume that the real world probability measure $\mathbb{P}$ is equal to a risk-neutral probability measure $\mathbb{Q}$ in each commodity market. This assumption has been justified in the Nordic power market by [15].

TABLE I
CORRELATION MATRIX FOR NUMERICAL EXAMPLE

	T_N	$D_{i,N}$	$P_{i,N}$	$D_{j,N}$	$P_{j,N}$
T_N	**1**	**0.8**	**0.3**	**0.6**	**0.3**
$D_{i,N}$		**1**	**0.5**	**0.15**	**0.1**
$P_{i,N}$			**1**	**0.1**	**0.1**
$D_{j,N}$				**1**	**0.5**
$P_{j,N}$					**1**

TABLE 2
PARAMETERS

	Parameters
Temperature	T(0)=65, $\mu_T = 1$, $\sigma_T = 5$
Type i & j	$\mu_{D_k} = \mu_{P_k} = 0.1$ $\sigma_{D_k} = 0.3$ $\sigma_{P_k} = 0.1$, $P_k^R = 11$
Risk aversion.	0.1
Δ	0.5

In our single-period indifference pricing formula (11) of type i buyers we need to calculate conditional expectation $E_{N-1}[\exp(-a_i(I_i + \alpha_N W_N)) \mid P_{i,N}]$ and this can be obtained from the following conditional probability density functions.

$$\ln D_{i,N} \mid P_{i,N} \sim N(\mu_1 + \rho_{D_{i,N},P_{i,N}}(\ln(P_{i,N}) - \mu_2)\frac{\sigma_{D_i}}{\sigma_{P_i}}, (1-\rho_{D_{i,N},P_{i,N}}^2)\sigma_{D_i}^2 n_1) \quad (48)$$

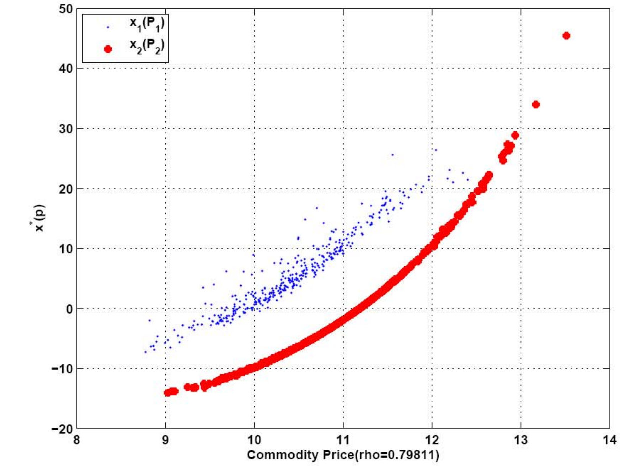
$$T_N \mid P_{i,N} \sim N(\mu_3 + \rho_{T,P_{i,N}}(\ln(P_{i,N}) - \mu_2)\frac{\sigma_T}{\sigma_{P_i}}, (1-\rho_{D_{i,N},P_{i,N}}^2)\sigma_T^2 n_1) \quad (49)$$

where

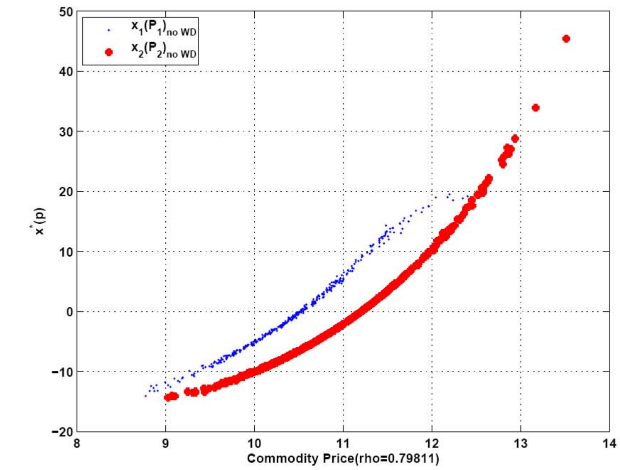
$$\mu_1 = \ln(D_{i,0}) + (\mu_{D_i} - \frac{1}{2}\sigma_{D_i}^2)(2\Delta)$$
$$(66)$$
$$\mu_2 = \ln(P_{i,n_1}) + (\mu_{P_i} - \frac{1}{2}\sigma_{P_i}^2)\Delta \quad (50)$$
$$\mu_3 = T_0 + \mu_T(2\Delta)$$

Figure 1 shows the optimal payoff functions of the commodity portfolios at time 0 and 1 with and without the use of weather, derivative (for customer type i who can use such hedging instruments).

Figure 2 illustrates the probability density functions (p.d.f) of terminal wealth for the type i buyer in three cases. If the type i buyer is not hedged and exposed to all price and volumetric risks, the p.d.f of terminal wealth is widely spread. However, if the commodity derivatives are included in the portfolio then the risk is greatly reduced. Compared to the portfolio with commodity derivatives only, the portfolio with commodity derivatives and the weather call option shows very similar probability density functions. This can be explained by the fact that our portfolio of commodity derivatives includes a continuum of strikes whereas the weather call option is just one derivative. However, in Figure 3 type j buyer is better off in terms of risk reduction when the weather call option is purchased. Therefore, we conclude that the weather call option plays a relatively important role in the portfolio of type j buyer but is insignificant for type i buyer who can use commodity derivatives to hedge both price and volumetric risk.



(a) With Weather Call Option



(b) Without Weather Call Option

Fig. 1. Payoff function of commodity derivative portfolio

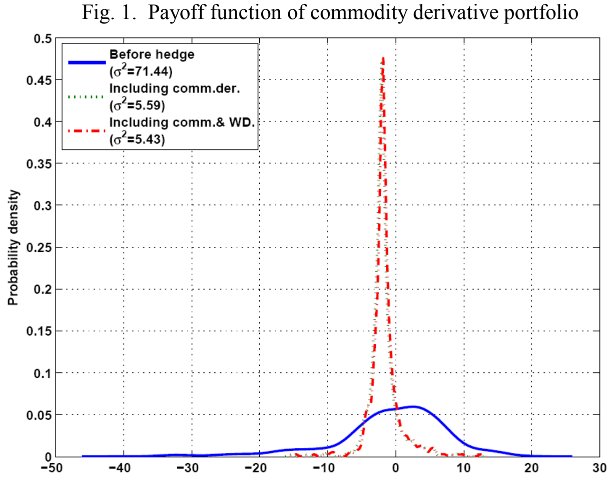


Figure 2: Terminal wealth distribution for Type i customer with and without hedging

Figure 4 compares the certain equivalent of terminal wealth for customer i and j with a single and two trading periods as we vary the correlation coefficient between temperature *(T)* and demand for the underlying commodity *(D)*. We observe that adding a trading period significantly increases the certain equivalent for both buyer types. Furthermore adding a second trading opportunity reduces the sensitivity of the certain equivalent to the correlation between temperature and demand.

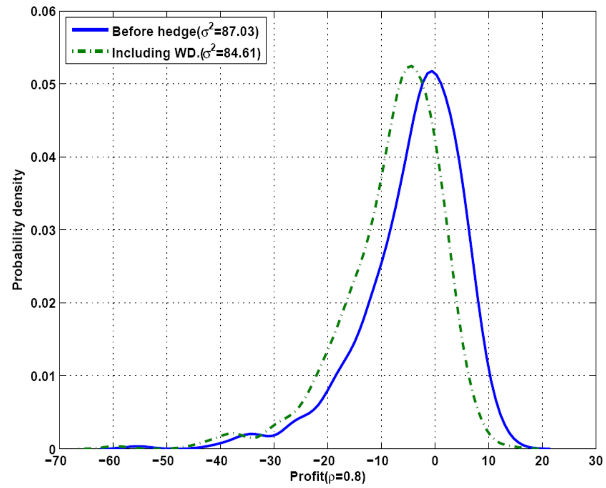
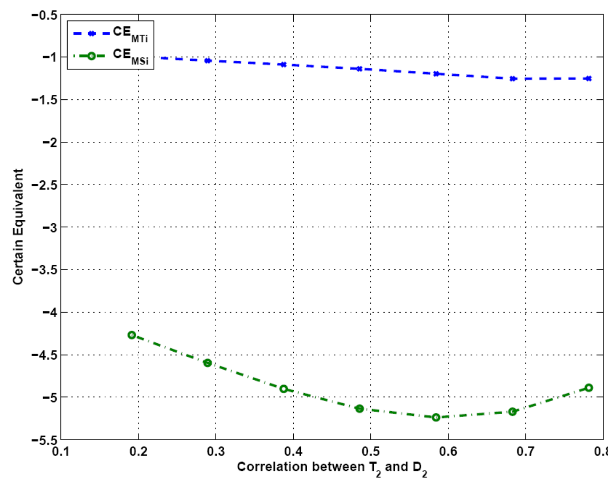
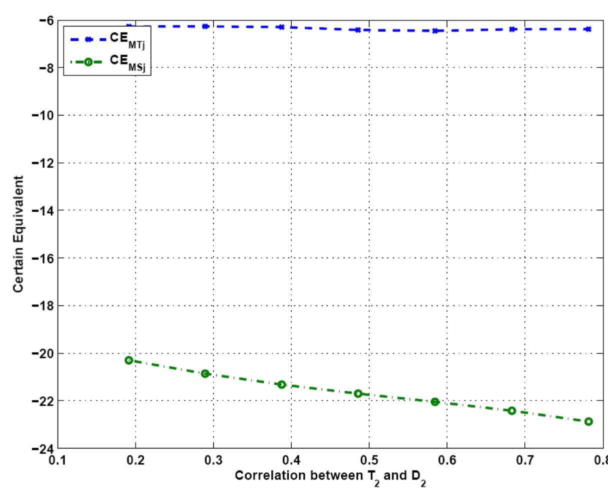


Figure 3: Terminal wealth distribution for Typej customer with and without hedging



(a) CE of Two & Single-period of type i buyer



(b) CE of Two & Single Period of type j buyer

Figure 4: Certain equivalent with one and two trading opportunities

IV. CONCLUSION

Many weather-sensitive industries such as energy, insurance, agriculture, and leisure are exposed to price and volumetric risks coming from the stochastic aspect of cost (or wholesale price) and demand in their profit functions. In addition these price and volumetric risks are all correlated with weather. Commodity derivatives can mitigate price risk but volumetric risk typically associated with weather changes can only be *partially* hedged via commodity derivatives. Therefore, new financial instruments are needed and weather derivatives represent an effective means for hedging volume risk because demand is strongly correlated with weather. However, pricing weather derivative is not a trivial task because of market incompleteness and it becomes even more challenging in a dynamic setting.

In this paper we investigate a multi-period equilibrium pricing model for weather derivative pricing within a framework of a stylized economy. Three types of market participants are considered, buyers with and without a liquid commodity derivatives market and an issuer. All market participants are assumed to maximize expected utility of terminal wealth subject to self-financing trading constraints and are able to rebalance their portfolios in each period. We use dynamic programming and indifference pricing to recursively derive the supply and demand function for the weather derivative in each period. We then apply a market clearing condition to determine the equilibrium prices of the weather derivative in each period.

A numerical example employing Monte-Carlo simulations illustrates the formation of the equilibrium prices and the optimal choices of the weather call options in a single and multi-period setting. We show how the weather derivative improves risk hedging capability by reducing variance of terminal wealth, especially in situations where commodity derivatives are not available. We also demonstrate the advantage of additional trading opportunities.

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VI. BIOGRAPHIES

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