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DIRECT STATISTICAL MATHIMATICAL MODEL TO CALCULATE THE FULL ENERGY PEAK EFFICIENCY OF HPGe DETECTOR

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Abstract

A direct statistical mathematical model was implemented to calculate the full energy peak efficiency (FEPE) of HPGe detectors over gamma ray energy range of 20 keV to 3 MeV. This mathematical model can be applied at any height from the detector face of an axial point source. The idea of the model depends of on tracking of successive interactions of gamma ray photons in the energy range under consideration and uses the physics of these interactions with the geometry information to calculate the photo peak attenuation coefficient μ_p, and consequently the photo peak efficiency ε_p.

All calculations were carried out for different cylindrical detector sizes over distances of 0 to 25 cm from the detector face of the axial point source. The relative efficiencies of different sizes of HPGe detectors to NaI detectors that were calculated by the present model have an excellent agreement with published work.

The calculations of FEPE carried out by the present model were compared with results from other methods such as experimental, semi empirical and Monte Carlo calculations. The results of the present model are in excellent agreement with published FEPE results from these methods, and provide the best match of experimental results than other theoretical methods.

(1) Introduction

Gamma spectrometry is one of the tools commonly used for the measurement of various environmental radionuclides. Where, the absolute activity of different gamma peaks in a wide energy range can be determined, Gamma ray radioactive isotopes can be identified, measuring the absorbed doses and determination of the interaction cross sections.

To calculate the absolute activity, the sample to detector full-energy peak efficiency should be known. The calculation of sample to detector full-energy peak efficiency using semi-empirical, Monte Carlo approaches and experimental measurements have been treated by several authors. The semi-empirical method has been used by Hoste [1], Moens et al. [2] , Lippert [3] ,Mihaljevic et al. [4], Wang et al.,[5] and Wang et al., [6] . In this method, the full-energy peak efficiency is calculated by combining point source measurements with computer calculations using empirical formulae. It involves some approximations and simplifications so the application is restricted to limited source–detector configurations. While Wainio and Knoll [7], and Overwater et al. [8] and B.Lal et al. [9] used Monte Carlo method in which, the history of each individual photon, is simulated in an analog step-by-step process in the detector active zone. There are no approximations or limitations to the source–detector configurations, but it is computationally time consuming more discussions about these two theoretical methods are found in Ref. [1,…17]. On the other hand the experimental method has been used to calculate FEPE by T.Paradellis et al. [18] and A.Owens [19]. The relative advantages and disadvantages of each of this methods are discussed in [1,…19].

The aim of this work is introducing a new direct mathematical Model to calculate full energy peak efficiency of cylindrical HPGe detector for an axial point sources emitting photon of energies up to 3 MeV. To calculate the FEPE (ε_p) the photopeak coefficient (μ_p) must be calculated accurately that could be achieved through the direct statistical calculation and using spherical trigonometry technique. This technique based on the determination of the average path length $(\bar{d})$ covered by a photon inside the detector active medium and the geometrical solid angle (Ω) which represent the angle subtended by the detector at the source point.

The arrangement of this paper is as follows. Sections (2) and (3) present direct statistical mathematical formulae for the total efficiency and peak efficiency for axial point source. Sections (4) and (5) cover the definitions and calculations of full energy peak attenuation coefficient (μ_p) and partial attenuation factor (f_m). Sections (6) and (7) enclose the method of calculations of both average scattered energies and the average cosine angle of scattering respectively. Section (8) contains the longitudinal and lateral limitation of the scattered ray in the finite detector medium. Section (9) presents an example of calculating full energy peak efficiency, comparisons between the calculated peak efficiency using the formulae derived in this work and the published results that proving the validity of the present mathematical formulae. Finally, the Conclusion is presented in Section (10).

(2) The direct statistical mathematical method

The direct statistical mathematical approach, present work, provides one of the simplest, potentially and most accurate methods for predicting the FEPE. Selim and Abbas [20,21,22] derived mathematical expressions to be used directly by the substitution of geometrical parameters of the source-detector system and the total attenuation coefficient (μ) of the incident photon corresponding to its energy (E_γ) to find the total efficiency, for any source shape, for any cylindrical detector and at any geometrical locations. The idea of this method is illustrated in Ref. [20], where the total attenuation coefficient (μ) is the summation of all the coefficients that attribute to consideration efficiency $\mu=\tau+\sigma+\kappa$. Selim and Abbas formula (1) is used for different source shape at different places with respect to cylindrical detector. In this work, we interested in full energy peak efficiency for the axial point source so that the total attenuation coefficient (μ) will be replaced with the photopeak coefficient (μ_p) which calculated accurately through new technique discussed in some details in the following sections.

(3) Axial Point Source

By using spherical coordinate, the integration limits change in steps in accordance to the values of the effective traversed distance ($\bar{d}$) and the polar angle (θ). The geometry of coaxial point source to the cylindrical detector (2R×L) is given in figure (1).

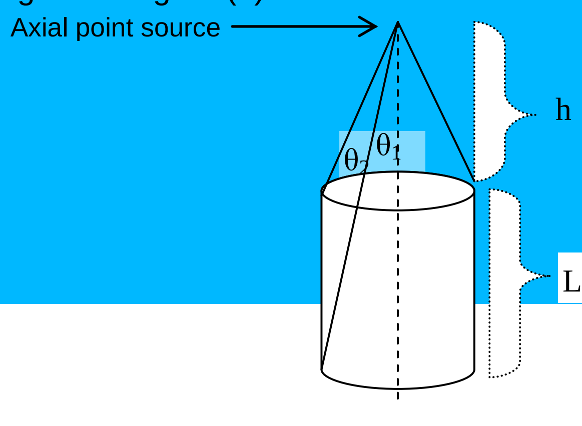


Figure (1): The geometry of coaxial point source to the cylindrical detector (2R×L)

Therefore the total efficiency (ε_T) can be expressed as

$$\therefore \varepsilon_T = \frac{1}{2}\left[\int_0^{\theta_1}\left(1-e^{\frac{-\mu L}{\cos\theta}}\right)\sin\theta\, d\theta + \int_{\theta_1}^{\theta_2}\left(1-e^{\frac{-\mu R}{\sin\theta}+\frac{\mu h}{\cos\theta}}\right)\sin\theta\, d\theta\right] \quad (1)$$

Where;

$$\left.\begin{aligned}\theta_1 &= \tan^{-1}\frac{R}{h+L}\\ \theta_2 &= \tan^{-1}\frac{R}{h}\end{aligned}\right\} \quad (2)$$

Equation (1) is applied to calculate the total efficiency due to an axial point source to a bare detector. The most germanium detectors use a thin cap window so the attenuation of this window must to be taken under consideration. For capped detector the equation (1) multiplied by $e^{\frac{-\mu_t t}{\cos\theta}}$ and becomes

$$\therefore \varepsilon_T = \frac{1}{2}\left[\int_0^{\theta_1} e^{\frac{-\mu_t t}{\cos\theta}}\left(1-e^{\frac{-\mu L}{\cos\theta}}\right)\sin\theta\, d\theta + \int_{\theta_1}^{\theta_2} e^{\frac{-\mu_t t}{\cos\theta}}\left(1-e^{\frac{-\mu R}{\sin\theta}+\frac{\mu h}{\cos\theta}}\right)\sin\theta\, d\theta\right] \quad (3)$$

Where;

μ: is the total attenuation coefficient of the detector material;

L: is the detector height

R: is the radius of detector face

h: is the distance between the source and the detector face

t: is the window thickness, and

$$e^{\frac{-\mu_{t1} t_1}{\cos\theta}} \times e^{\frac{-\mu_{t2} t_2}{\cos\theta}} \times \cdots \times e^{\frac{-\mu_{tn} t_n}{\cos\theta}} \quad (4)$$

(μ_t) is the total attenuation coefficient of the window material which is usually Aluminum unless stated. If there is n-number of caps or shield we multiply the where; (μ_{tn}) is the coefficient of the n$^{\underline{th.}}$ element; And, (t_n) is the thickness of n$^{\underline{th.}}$ element. The dead distance (h_o), the distance between the detector active medium and the window, has to be added to (h) for the source –detector separation distance.

(4) Full Energy Peak Attenuation Coefficient (μ_p)

The full energy peak efficiencies defined as the ratio between the number of photons that are recorded in the detector under certain peak and the number of photons that are emitted from the source with energy relates to this peak."

To calculate the full energy peak efficiencies, we consider only the part of energy deposited in the detector and which only contributes to the peak, say photopeak attenuation coefficient (μ_p) then, the total attenuation coefficient (μ) in equation (3) is replaced by the photo peak coefficient (μ_p) remembering that the photoelectric effect leads always to electrons of maximum energy, we have to consider the incident gamma rays energies is less than 3 MeV; So that the coefficient (μ_p) given by;

$$\mu_p = \tau_o + f_m \sigma_o \tag{5}$$

where, m=1,2,…..

(τ_o) is the photoelectric coefficient,
(σ_o)is the Compton scattering coefficients and

(f_m) is a fraction allowing for successive Compton scattering .

Where, the final scattered photon after Compton scattering is terminated by a photoelectric absorption. This fraction is primarily depends on the detectors dimensions and incident gamma ray energy.

The index (*m)* is integer numbers 1,2, ..Confining ourselves to gamma ray energy less than about 3 MeV; we take only these two terms where the probability of pair production in this energy range is very small. One of the main objects of this work is the determination of the fraction (f_m) with respect to the detector's dimensions, which will maintain in next section.

(5) The Partial Attenuation factor (f_m)

- ***Infinite Size Ge detector:-***

In order to fully understand the exact behavior of the factor (f_m) in the attenuation coefficient, We preferably start with an infinite size detector with interacting gamma-rays energy $h\nu_o$ less than 3 MeV where the major predominant interactions Photoelectric interaction and Compton scattering. *So*, we have only two possible allowances to be followed:-

Photoelectric interaction (τ_0): where, an electron produces in the detector medium with a kinetic energy equal ($h\nu_0$-B.E.), thus, being recorded under the peak energy spectrum.

Compton interaction (σ_o): The result of which is an electron set in motion with a kinetic energy T_o and a scattered photon with energy $h\nu_1$

$$h\nu_1 = h\nu_o - T_o \qquad (6)$$

Consequently; the last scattered photon has one of two fates:-

- *First fate;* a photoelectric effect takes place, thus, an electron with a kinetic energy equal to ($h\nu_1$-B.E.) is set in motion.

Now, this electron and the previous scattered one; take place in rapid succession that they appear as one electron with a total energy $h\nu_o$ that is registered at the maximum peak position of the detector. The relative probability of this to happen is:-

$$f_m = f_1 = \left(\frac{\tau_1}{\mu_1}\right) \qquad (7)$$

μ_1 the total attenuation coefficients of the scattered gamma rays with energy $h\nu_1$.

- *Second fate*; If the previous interaction does not take place, the rest fraction of the original beam σ_1/μ_1 propagates, and will be scattered to an electron with a kinetic energy T_1 and a scattered gamma ray with energy $h\nu_2$. Letting this last scattered photon undergoes a photoelectric effect with a probability $\left(\frac{\tau_2}{\mu_2}\right)$. These three electrons are again emitted promptly and they appear as one with a total kinetic energy equals to original energy of incident ray $h\nu_o$, Since range of the electrons is few microns only up to about a millimeter thus appear again in the photo-peak region.

The relative probability of this to happen is: $\left(\frac{\sigma_1}{\mu_1}\right)\cdot\left(\frac{\tau_2}{\mu_2}\right)$

Of course, only one event of either of the two fates takes place. Then the average probability will be:-

$$f_2 = \frac{1}{2}.\left(\frac{\tau_1}{\mu_1} + \frac{\sigma_1}{\mu_1}.\frac{\tau_2}{\mu_2}\right) \qquad (8)$$

Therefore; for an infinite medium and allowing for further Compton scattering; the average effective fraction of the Compton components is given by:-

$$f_m = \frac{1}{m} \times \left[\frac{\tau_1}{\mu_1} + \frac{\sigma_1}{\mu_1}.\frac{\tau_2}{\mu_2} + \cdots + \frac{\sigma_1}{\mu_1}.\frac{\sigma_2}{\mu_2}.\frac{\sigma_3}{\mu_3}\cdots\frac{\sigma_{m-1}}{\mu_{m-1}}\frac{\tau_m}{\mu_m}\right]; m = 1,2,3,\ldots \qquad (9)$$

Where, *m* is a total number of effective collisions.

Using equations (5,9) and tables in (24) one can calculate the photopeak efficiency for relation (3) after substituting total attenuation coefficient (μ) with photopeak coefficient μ_p then the photopeak efficiency given by for axial point source.

$$\therefore \varepsilon_p = \frac{1}{2}\left[\int_0^{\theta_1} e^{\frac{-\mu_t t}{\cos\theta}}\left(1-e^{\frac{-\mu_p L}{\cos\theta}}\right)\sin\theta\, d\theta + \int_{\theta_1}^{\theta_2} e^{\frac{-\mu_t t}{\cos\theta}}\left(1-e^{\frac{-\mu_p R}{\sin\theta}+\frac{\mu_p h}{\cos\theta}}\right)\sin\theta\, d\theta\right] \quad (10)$$

Now, the problem is calculating the partial coefficient corresponding to the energies of scattered photons τ_1 , τ_2 τ_3….. τ_m and σ_1, σ_2, σ_3, …..σ_{m-1}. So, the average scattered energies should be calculated.

(6) Calculation of Average Scattered Energies

In order to find the partial attenuation fractions, we have to determine the scattered energies of the gamma rays. A statistical process governs this phenomenon, similar to neutron scattering process. Beforehand, the kinematics of the interaction is controlled by the famous Compton's set of equations. According to Compton scattering this interaction, as the photon falls with energy ($h\nu_o$), its momentum is ($h\nu_o/c$). From the conservation of momentum, where the scattering is elastic, the scattered photon must be changed from the incidence direction by angle (θ) and the electron removes with angle (φ) but the three directions, the incidence, the scattered and the remove recoil must lie in the same plane i.e. un-polarized scattered radiation is considered.

From the conservation of energy, the photon loses part of its energy, which the electron acquires as a kinetic energy.

Energy of the scattered photon is given by:

$$h\nu_1 = \frac{h\nu_o}{1+\alpha(1-\cos\theta)} \quad (11)$$

$$\alpha = \frac{h\nu_o}{m_e.c^2} \quad (12)$$

The energy expectation value o nergy for the scattered photon is

$$\langle h\nu_1\rangle = F h\nu_o \quad (13)$$

$$\langle h\nu_1\rangle = \frac{\int h\nu_1 \frac{d\sigma_o}{d\Omega} d\Omega}{\int \frac{d\sigma_o}{d\Omega} d\Omega}$$

(14)

where,

$$d\Omega = 2\pi \sin\theta\, d\theta$$

In order to calculate the average scattered photon energy, it is required to know the angular distribution of the scattered photon to determine the deposited energy. So, The probability of Compton scattering at the angle (θ) can be determined through quantum mechanical calculation of the process. Klein and Nishina [25] derived a formula for the angular distribution to the un-polarized scattered radiation as:-

$$\frac{d\sigma_o}{d\Omega} = \frac{r_o^2}{2} \times \left\{ \frac{1}{[1+\alpha(1-\cos\theta)]^2} \right\} \times \left\{ 1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{[1+\alpha(1-\cos\theta)]} \right\} \quad (15)$$

To calculate the expected scattered photon energy < $h\nu_1$>, we integrate the values of $h\nu_1$ over all the possible angles θ.

$$\langle h\nu_1 \rangle = \frac{\displaystyle\int_0^\pi \frac{h\nu_o\, f(\theta)\, \sin\theta}{[1+\alpha(1-\cos\theta)]^3}\, d\theta}{\displaystyle\int_0^\pi \frac{f(\theta)\, \sin\theta}{[1+\alpha(1-\cos\theta)]^2}\, d\theta} \quad (16)$$

where;

$$f(\theta) = 1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)} \quad (17)$$

Using equations (13,14,15,16) and (17). The reduction factor (F) given by

$$F = \frac{\langle h\nu_1 \rangle}{h\nu_o} = \frac{\displaystyle\int_0^\pi \frac{1}{[1+\alpha(1-\cos\theta)]^3} \times \left[1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)} \right] d(-\cos\theta)}{\displaystyle\int_0^\pi \frac{1}{[1+\alpha(1-\cos\theta)]^2} \times \left[1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)} \right] d(-\cos\theta)} \quad (18)$$

By, Carrying out the integration of the different factors term by term of equation (18) the reduction factor (F) of the original energy leading to averaged energy for the scattered beam or photon in any medium, could be expressed as:-

$$F = \frac{3(1+2\alpha)^3 \ln(1+2\alpha) + 32\alpha^5 + 12\alpha^4 - 36\alpha^3 - 30\alpha^2 - 6\alpha}{3(1+2\alpha)\times\left\{(1+2\alpha)^3[\alpha^2 - 2\alpha - 2]\times\ln(1+2\alpha) + 2\alpha^4 + 18\alpha^3 + 16\alpha^2 + 4\alpha\right\}} \qquad (19)$$

Where, (α) is stated as before in equation (12). Figure (2) indicates the variation of reduction factor (F***)*** with the photon energy (hν). It is important to notice that, for higher energies of the incident gamma rays the sharp reduction of energies is more than that for the incident lower energies.

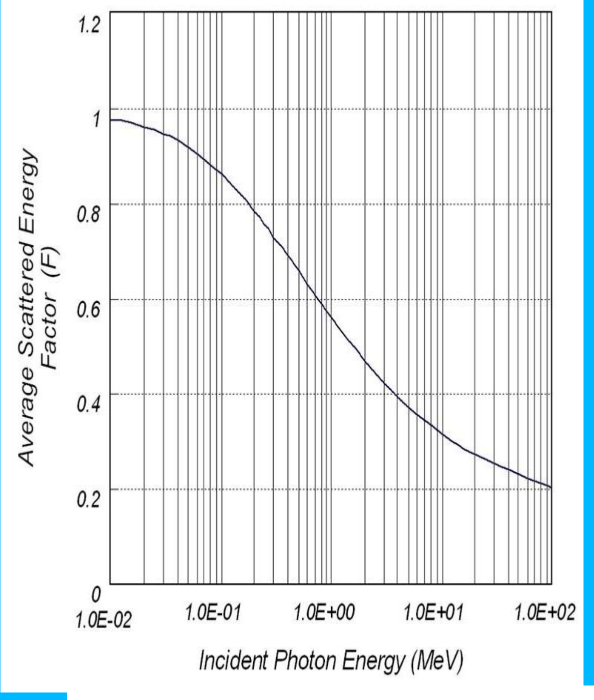


*Figure (2): The variation of reduction factor **F** with the photon*

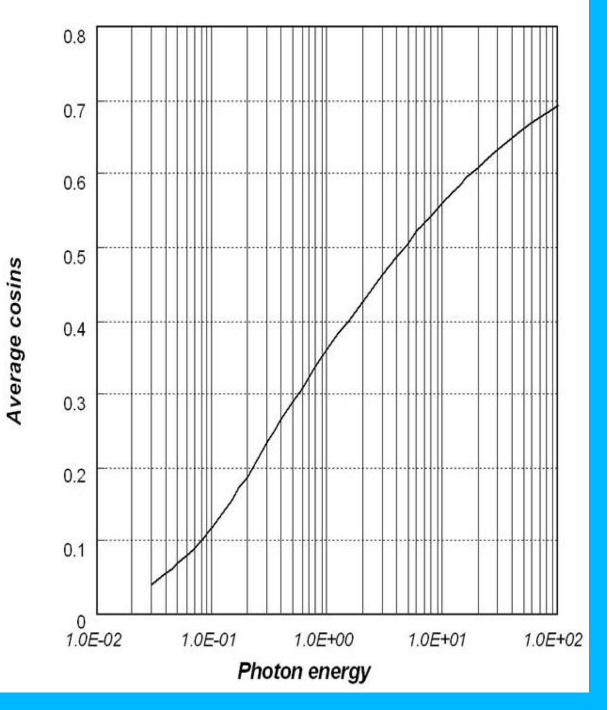


*Figure (3): The variation of average cosine **<cos θ**> with (hν)*

(7) Calculation of The Average Cosine Angle Of Scattering

The average cosine angle of the scattered photon is given by

$$\langle\cos\theta\rangle = \frac{\int_0^{\pi} \cos\theta \,\frac{d\sigma_o}{d\Omega}\, d\Omega}{\int_0^{\pi} \frac{d\sigma_o}{d\Omega}\, d\Omega} \qquad (20)$$

Where; $\frac{d\sigma_o}{d\Omega}$ as is defined in equation (15) therefore

$$\langle\cos\theta\rangle = \frac{\int_0^{\pi} \frac{\cos\theta}{[1+\alpha(1-\cos\theta)]^3}\times\left[1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)}\right]\sin\theta\, d\theta}{\int_0^{\pi} \frac{1}{[1+\alpha(1-\cos\theta)]^2}\times\left[1+\cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)}\right]\sin\theta\, d\theta} \qquad (21)$$

By solving these two integrations, Carrying out the integration term by term, we finally get;

$$\langle \cos\,\theta \rangle = \frac{\left[(1+2\alpha)^2(\alpha^3-\alpha^2-6\alpha-3)\right]\times \ln(1+2\alpha)-6\alpha^5+16\alpha^4+46\alpha^3+30\alpha^2+6\alpha}{\alpha\left[(1+2\alpha)^2(\alpha^2-2\alpha-2)\times \ln(1+2\alpha)+2\alpha^4+18\alpha^3+16\alpha^2+4\alpha\right]} \quad (22)$$

Where (α) is as defined before in equation (12). One can notice that, the average cosine depends also on the incident photon energy. The variations of <cos θ> with the photon energy (hν) is presented in figures (3) where the energetic gamma rays (MeVs') are scattered forwards, while softer rays (KeVs') are scattered sideward. The correlated understanding between Figures (2) and (3) can be explained as when an incident photon with relatively high energy is scattered then much of its energy is reduced i.e. (F) becomes small and penetrates more forward i.e. (θ) is small then *<cos θ>* becomes big till the photon energy is much reduced and the situation is interchanged. For smaller energies, the energy reduced more slowly i.e. (F) becomes big and tend to be lose to 1 and spreads more side–wards i.e. (θ) increases.

By using equation (13) for every incident gamma energy ($h\nu_o$) , the average energies of the successive Compton-scattered rays $\langle h\nu_1\rangle$, $\langle h\nu_2\rangle$,…. could be deduced. From table (1) and equations (9,13 and 18) and data from tables [24] the components of the partial attenuation coefficient could be computed

$$\frac{\tau_1}{\mu_1},\frac{\sigma_1}{\mu_1}.\frac{\tau_2}{\mu_2},\ldots,\frac{\sigma_1}{\mu_1}.\frac{\sigma_2}{\mu_2}.\frac{\sigma_3}{\mu_3}\ldots.\frac{\sigma_{m-1}}{\mu_{m-1}}\frac{\tau_m}{\mu_m} \quad (23)$$

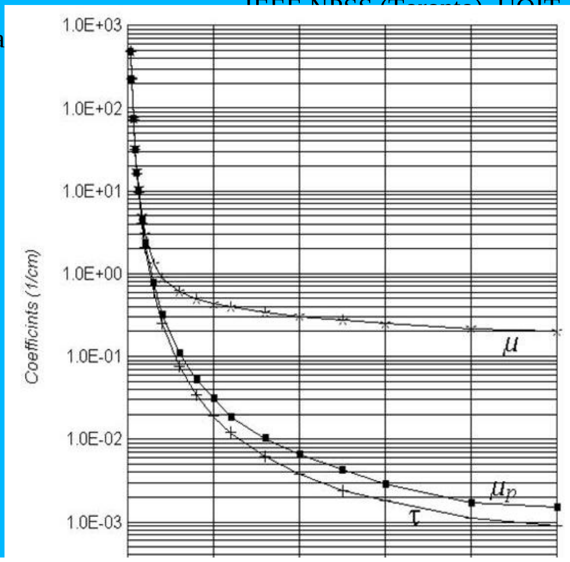


Figure (4): Variation of total attenuation, peak, and photoelectric Coefficients with the photon energy for an infinite Germanium detector

Figure (4) shows the variation of coefficients (μ, μ_p and τ_o) with the photon energy for infinite size germanium detector. One can notice that the difference between (μ_p) and (τ_o) at high energy explain the feasibility of this work.

The partial attenuation coefficient in expression (23) consider the infinite number of scattering or successive interactions that could be happened only in the infinite detector medium but actually the detection medium is finite so the longitudinal and lateral limitation should be considered.

(8) Finite Size Ge Detector

In this case, we consider the assumption that all electrons from the photoelectric effect or/and Compton scattering are completely absorbed within the detector volume. Since range of the electrons is few microns only up to about a millimeter. Therefore, the photoelectric coefficient (τ_o) would not be affected. Only Compton-components affected because of the scattered gamma ray photons that may be escape before absorption in the detector medium and contributing to previous mentioned coefficient (f_m). So the longitudinal and lateral limitation should be considered to determine the limitation number of (m) successive scattering.

(8-a) Longitudinal Limitation

Let a gamma ray of energy ($h\nu_o$) to be incident axially and normally to the detector's surface. Its path length is (L). The absorption probability in the detector is:-

$$e^{-\mu_o L} = e^{-\tau_o L} e^{-\sigma_o L} \qquad (24)$$

The mean free path of the photoelectric effect is ($1/\tau_0$); and that of Compton scattering for the first elastic collusion is ($1/\sigma_0$). The average energy of the scattered photon becomes ($h\nu_1$) with a scattering angle (ω_1).

Figure (5) illustrates the mean free paths and the scattering angles for successive scattered photon inside the cylindrical detector of dimension (2R×L). The component in the direction of incidence is clearly

$$\bar{x}_1 = \frac{1}{\sigma_1}\cos\omega_1 \tag{25}$$

Further, let this ray be scattered further to let its energy be $h\nu_2$ with a space scattering θ_2 making an angle (β) with the original direction of incidence and a rotation angle φ_2, where φ_2 is the angle of rotation of the third mean free path with respect to the plane of the first two paths.

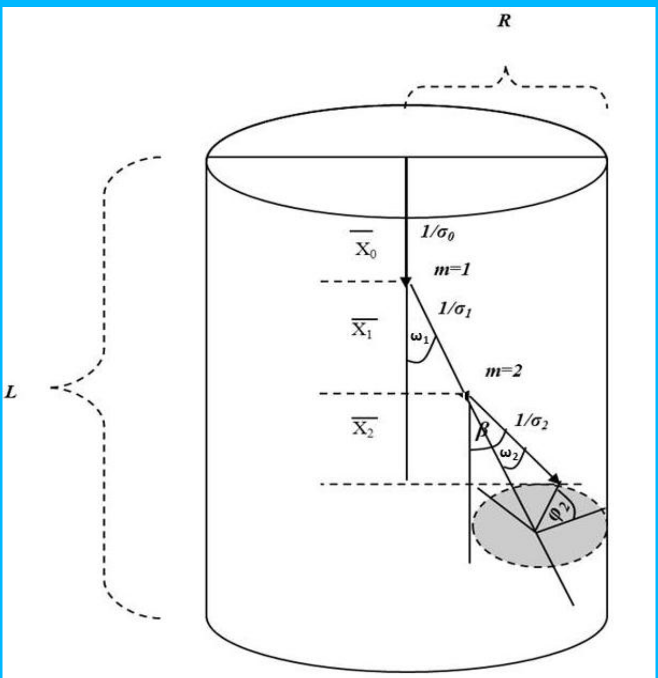


Figure (5): The mean free paths and the scattering angles for successive scattered photon inside the cylindrical detector of dimension (2R×L).

Following the same steps as followed by [25] for the scattering of neutrons

$$\cos\beta = \left[\cos\omega_1 . \cos\omega_2 - \sin\omega_1 . \sin\omega_2 . \cos\varphi_2\right] \tag{26}$$

Since all values of (φ_2) are equally probable for un-polarized rays, then the average value of the second term in the last equation is zero. Accordingly,

$$\bar{x}_2 = \frac{1}{\sigma_2} . \cos\omega_2 . \cos\omega_1 \tag{27}$$

And so on, for further scatterings. One condition to determine the suitable value of index number (m) in the fraction (f_m) is

$$L \geq \frac{1}{\sigma_o} + \frac{1}{\sigma_1}\langle \cos\omega_1 \rangle + \frac{1}{\sigma_2}\langle \cos\omega_1 \rangle\langle \cos\omega_2 \rangle + \cdots\cdots + \frac{1}{\sigma_m}\langle \cos\omega_1 \rangle \cdots \langle \cos\omega_m \rangle \qquad (28)$$

In effect, for an axial isotropic point source being situated at a distance (h) from the detector's surface the average photons ray may not fall normally on the detector surface.

Therefore, the gamma ray photons may be inter to the detector medium from any point on the upper surface with equally probability, so the average covered path of a ray is not (L) but ($\bar{d}$) as shown in the sketch figure (6).

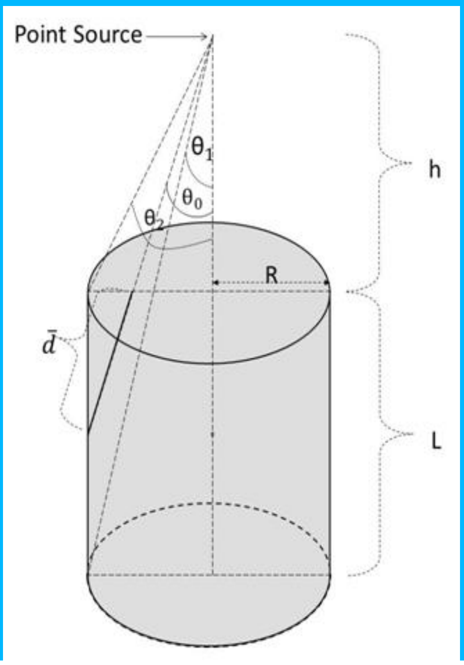


Figure (6): The average covered distance $\bar{d}$ of a ray from an axial point source at distance (h) from the upper detector face

The average covered distance of a ray from an axial point source at height (h) from the detector surface could be calculated. The allowed photon distance ($\bar{d}$) is the average distance which represents a straight line connecting the photon incident point on the detector and the photon outgoing point from the detector. Therefore ($\bar{d}$) is considered as a function in solid angle and due to the statistical nature of the radiation, the allowed photon distance ($\bar{d}$) can be expressed, form of isotropic emission by formula

$$\bar{d} = \frac{\int_\Omega d\,.\,d\Omega}{\int_\Omega d\Omega} \tag{29}$$

This integration is divided into two parts each of them is depending on the variation of ($\bar{d}$) with (θ_1) and (θ_2)

$$\bar{d} = \frac{\int_0^{\theta_1} d_1 \sin\theta\, d\theta + \int_{\theta_1}^{\theta_2} d_2 \sin\theta\, d\theta}{\int_0^{\theta_2} \sin\theta\, d\theta} \tag{30}$$

Where,

$$\theta_1 = \tan^{-1}\frac{R}{h+L} \;\; and \;\; \theta_2 = \tan^{-1}\frac{R}{h} \tag{31}$$

So from figure (6) and equations (29, 30 and 31) the average covered path could be given by equation (32).

$$\bar{d} = \frac{R\left[\tan^{-1}\left(\frac{R}{h}\right) - \tan^{-1}\left(\frac{R}{(h+L)}\right)\right] + h.\ln\left(\frac{h}{\sqrt{h^2+R^2}}\right) - (h+L).\ln\left(\frac{L+h}{\sqrt{(h+L)^2+R^2}}\right)}{\left(1 - \frac{h}{\sqrt{h^2+R^2}}\right)} \tag{32}$$

By plotting this average path distance ($\bar{d}$) with the source height distance (h) for different detector dimensions (2RxL) or volumes sizes as shown in figure (7) one can conclude that

a) For axial point source, just on the detector face, the average path distance covered in the detector is increase by increase the detector volume.

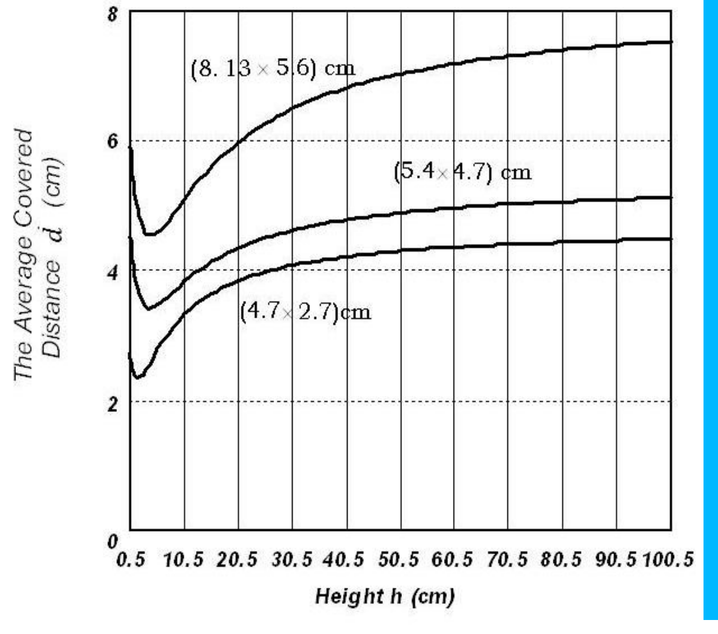


Figure (7): The variation of the average covered path with the source height for different detector volumes with dimensions (2RXL)

b) As the height (h) increase the average distance start to decrees until minimum value. This minimum values of ($\bar{d}$) is depend on the detector volume and height of the source so the significant conclusion of the minimum average distance of the gamma ray photon in the detector is correlated with the distance of the source from the detector face and the volume of the detector. Then, by replacing (L) in equation (28) with ($\bar{d}$) the longitudinal limitation becomes:-

$$\bar{d} \geq \frac{1}{\sigma_o} + \frac{1}{\sigma_1}\langle\cos\theta_1\rangle + \frac{1}{\sigma_2}\langle\cos\theta_1\rangle\langle\cos\theta_2\rangle + \cdots + \frac{1}{\sigma_m}\langle\cos\theta_1\rangle\cdots\langle\cos\theta_m\rangle \quad (33)$$

(8-b) -Lateral Limitation

The other condition for the maximum value of (m) to be considered for contributing to eligible successive Compton scatterings is the so-called lateral condition. Namely, for an axial ray falling along the detector's axis i.e. the ray enter from the central point of the upper detector face, it is straight forward to get:-

$$R \geq \frac{1}{\sigma_1}\sin\omega_1 + \frac{1}{\sigma_2}\sin(\omega_1 + \omega_2) + \cdots + \frac{1}{\sigma_m}\sin(\omega_1 + \omega_2 + \ldots\ldots\omega_m) \quad (34)$$

In fact, considering the effective averaged ray entering the detector from any point of the upper detector circular face, then the initial entering lateral distance or entrance angle (θ_0) should be considered as shown in figure (8-a) and the lateral condition becomes:

$$R \geq (h.\tan\theta_0 + \frac{1}{\sigma_0}.\sin\theta_0) + \frac{1}{\sigma_1}.\sin(\theta_0 + \omega_1) + \ldots\ldots + \frac{1}{\sigma_m}.\sin(\theta_0 + \omega_2 + \ldots\ldots\omega_m) \quad (35)$$

Where, (θ_o) is the angle of incidence of the averaged ray in the detector. It is computed from the relation:

$$\bar{d} + \frac{h}{\cos\theta_0} = \frac{R}{\sin\theta_0} \quad (36)$$

The initial angle of scattering (θ_o) can be calculated graphically as shown in fig. (8)

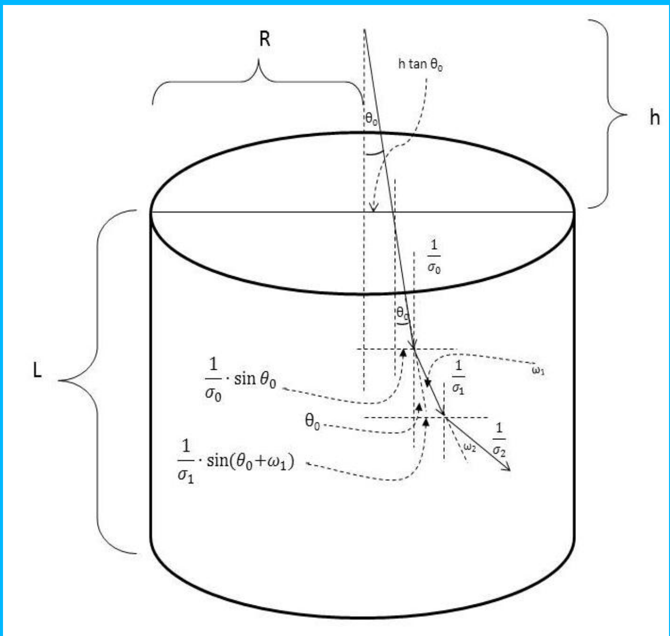


Figure (8-a): calculating lateral limitation condition

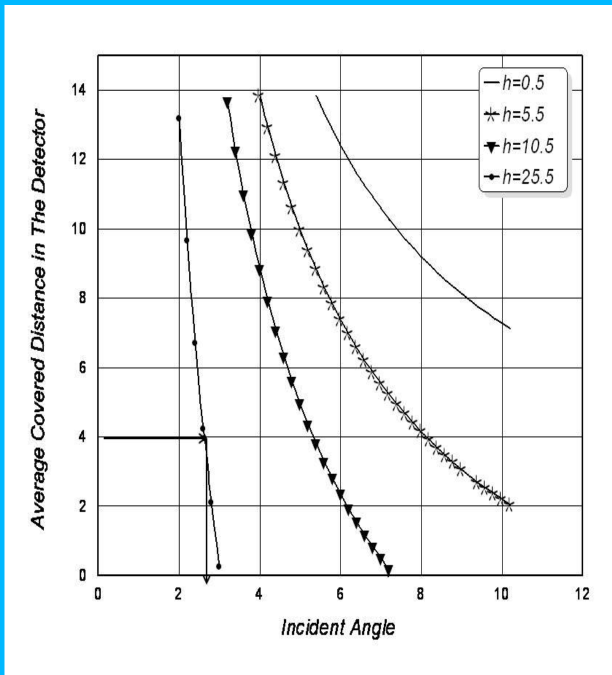


Figure (8-b): calculating the initial angle of scattering θ_o graphically

After (f_m) is calculated the photopeak coefficient can be calculated from

$$\mu_p = \tau_o + f_m \sigma_o \tag{37}$$

(9) Results

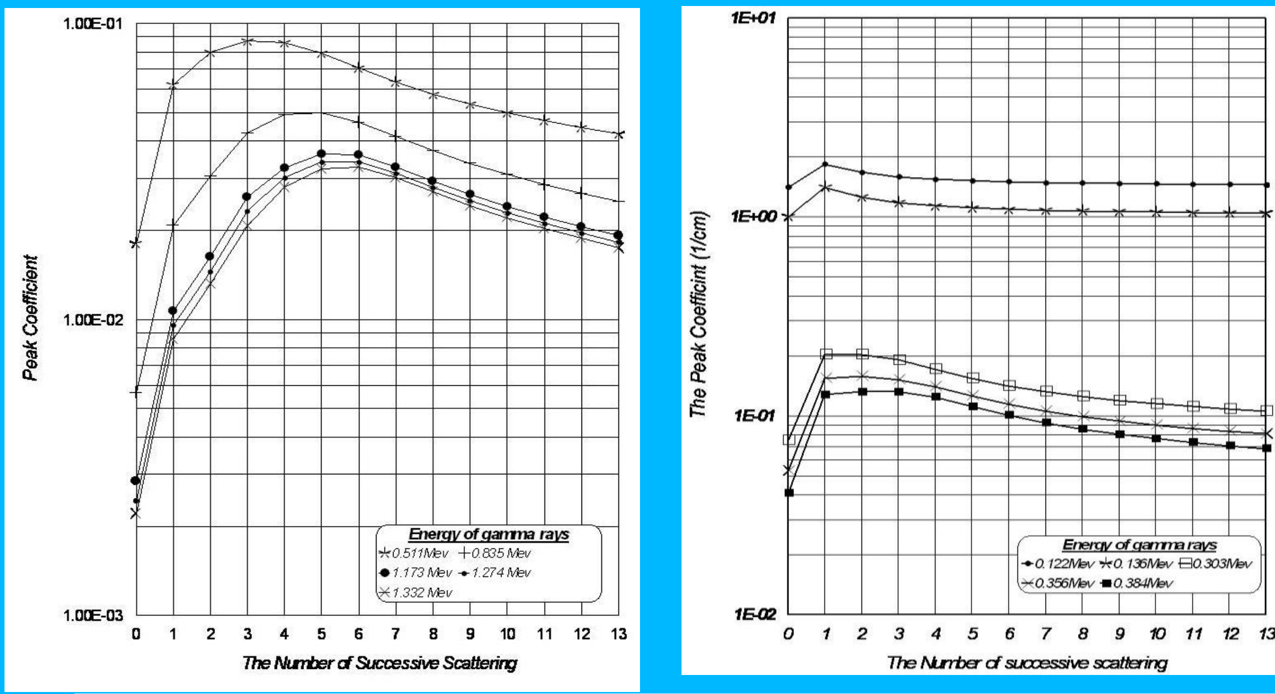
(9-1)Calculating Photo Peak Efficiency at Discreet Gamma Energy

Table(1) reproduces the peak coefficient (μ_p) for a finite size Ge detector, allowing for (m) successive Compton collision. The first column is the energy of gamma rays in MeV. The second represents the photoelectric coefficient alone (τ_o). The third column gives the values of (μ_p) for the case m=1 to m=6. All units are in cm^{-1}. In practice for customary detectors, the column m=1 is to be taken for low energies till about 0.3 MeV, while values of m=2 and 3 are used for higher energies.

*Table (1): The peak coefficient μ_p for a finite size Ge detector, allowing for **(m)** successive Compton collision.*

hv (MeV)	T_o	μ_p					
		m=1	m=2	m=3	m=4	m=5	m=6
0.015	481.1992	482.467	482.2300	482.1507	482.1110	482.087	482.071
0.020	220.3722	220.911	220.6423	220.5523	220.5073	220.480	220.462
0.030	71.3282	74.5888	74.2911	74.1907	74.1404	74.1103	74.0902
0.040	31.0863	31.7043	31.3998	31.2954	31.2431	31.2118	31.1908
0.050	16.1819	16.7999	16.4995	16.3938	16.3408	16.3090	16.2878
0.060	9.4749	10.0748	9.7886	9.6844	9.6321	9.6006	9.5797
0.080	4.0029	4.5625	4.3073	4.2070	4.1560	4.1254	4.1050
0.100	2.0440	2.5401	2.3315	2.2386	2.1901	2.1609	2.1414
0.150	0.5962	0.9723	0.8423	0.7699	0.7277	0.7015	0.6840
0.200	0.2518	0.4944	0.4527	0.4036	0.3691	0.3461	0.3304
0.300	0.0767	0.2057	0.2059	0.1934	0.1735	0.1561	0.1432
0.400	0.0341	0.1124	0.1201	0.1219	0.1152	0.1033	0.0927
0.500	0.0188	0.0644	0.0824	0.0895	0.0878	0.0804	0.0717
0.600	0.0119	0.0430	0.0598	0.0675	0.0700	0.0669	0.0601
0.800	0.0061	0.0230	0.0329	0.0452	0.0517	0.0521	0.0480
1.000	0.0038	0.0144	0.0224	0.0334	0.0399	0.0417	0.0402
1.250	0.002433	0.0095	0.0145	0.0231	0.0300	0.0340	0.0340
1.500	0.001751	0.0067	0.0107	0.0166	0.0241	0.0287	0.0297
2.000	0.0011	0.0040	0.0065	0.0111	0.0177	0.0220	0.0235
3.000	0.0006	0.0020	0.0034	0.0058	0.0101	0.0140	0.0165

Figures (9a and 9b) are showing the variation of the photopeak attenuation coefficient with number of successive scatterings at different energy for infinite detector size (Ge-Detector). From these figures one can conclude that at small energies the photo peak coefficient (μ_p) is come to maximum after first or second collision where the photons interactions terminated by photoelectric interaction. But, as the gamma ray energy going up photo peak coefficient (μ_p) increases gradually and reaches to maximum after 4; 5 or 6 successive collisions. Since these graphs for infinite detector medium then one can notice the number of collisions until 13. But, in real conditions the number of successive collisions or scattering are limited by escape from the finite detection medium or, terminated by photoelectric absorption.



Figure(9a-9b):show the variation of the photopeak attenuation coefficient with number of successive scatterings at different energy for infinite detector size(Ge-Detector).

As an example, consider a photon with energy 1.332 MeV emitted from axial point source at height 25 cm taking the cap distance 0.5 Cm under consideration from Ge detector. The calculations are traced as follows where the initial angle of entrance θ_o could be calculated graphically from figure (8-b)

$$\theta_o=2.62^o$$

and f_m can be found as follows

hv(MeV)	***m***	***μ_t (cm^{-1})***	***σ(cm^{-1})***	***τ_o(cm^{-1})***	***F***	$\langle \cos\theta \rangle$
1.332	0	0.049524	***0.258651***	***0.002209***	0.520927	-------
0.693875	1	0.069712	0.354817	0.009177	0.61164	0.387788
0.424401	2	0.090567	0.433781	0.030394	0.681997	0.3236
0.289441	3	0.118718	0.498451	0.095143	0.73522	0.271173
0.212802	4	0.159489	0.55026	0.229357	0.775783	0.228331

hv(MeV)	***m***	***θ***	***f_m***	***μ_p***
0.693875	1	67.18	0.024726	0.008606616
0.424401	2	71.12	0.042503548	0.013205668
0.289441	3	74.26	0.071514422	0.020710781
0.212802	4	76.81	***0.099470924***	***0.027943128***
0.165089	5			

$$\mu_p = \tau_o + f_m \times \sigma_o$$

$$\mu_{p=}0.002209045+(0.099470924\times 0.258651)= 0.027943128\ Cm^{-1}$$

(9-2) Relative FEPE of Germanium detector to sodium iodide detector for axial point source.

The (IEEE) Standard test procedure for germanium gamma ray detector [28] defines the relative efficiency of germanium spectrometer referenced against a ($3'' \times 3''$) NaI (Tl) detector which has an absolute efficiency of 1.2 X10^{-3} at 1.332 MeV of ^{60}Co, where the source to detector distance is 25 cm when point source used. We use our direct method to calculate the FEPF of ($3'' \times 3''$) NaI (Tl) at 25 cm at photon energy 1.332 MeV. We obtained good agreement with the result of IEEE [28].

1- Table (2) compare between the relative FEPE of germanium detector to that of NaI (Tl) detector calculated by direct method and that from other references [9,18,21,28,29,30,30] for different dimensions of HPGe and at the same energy and conditions. It is easy to noticing that the excellent agreement of the results obtained by direct method and that from of other references.

Table (2): Comparison between the relative FEPE of germanium detector to that of NaI (Tl) detector calculated by direct method and that from other references [9,18,21,28,29,30,30]

	Present work	***Reference result***	***Dimension 2RXL Cm***	***Volume Cm^3***	***Reference***
Relative Efficiency Of (Ge) to Efficiency (NaI (Tl)) of 1.332 MeV. At 25 cm form the face of the detector	8.908%	8.8%	4.3X3.75	54.46	Ref.[9]
	11.0%	10.8%	4.04X4.4	57.53	Ref.[9]
	24.514%	24.5%	5.4X4.7	108	Ref.[18],[28] and [30]
	36.407%	36%	6.05X5.88	169.04	Ref.[29]
	10.012%	10.1%	5X4.4	86.39	Ref.[21]

2- Table(3) represents the relative percentage of intrinsic Photopeak efficiency for axial point source for the energy range (0.2 MeV up to 1.2 MeV) with two detector of geometrical arrangements of volume of 2 cm^3 and volume 0.39 cm^3 at height h=5 cm. The comparison between the relative intrinsic FEPE calculated by direct method and that carried out by semi empirical formula and that measured experimentally by [19]. The agreement between these results is very good through the energy range under consideration.

Table (3): The comparison between the relative intrinsic FEPE calculated by direct method and that carried out by semi empirical formula and measured experimentally by [19].

Detector Volume (Cm^3)	***Photon Energy (MeV)***	***Relative Intrinsic Efficiency [19]***		***Direct Method (Present Work)***
		Experiment-ally	***Calculated by Semi-empirical***	
2.0	0.2	20.5X0.9	21.0	20.2908
	0.3	8.0 X 0.6	8.10	8.5258
	0.5	2.55 X 0.09	2.60	2.9303
	0.8	1.1 X 0.05	1.09	1.0566
	1.00	0.76 X 0.04	0.76	0.6631
	1.2	0.56 X 0.02	0.55	0.4844
0.39	0.2	26.0 X 2.0	27.20	20.0778
	0.3	8.6 X 0.4	9.00	8.9415
	0.5	2.50 X 0.40	2.45	2.8968
	0.8	0.87 X 0.05	0.88	1.0444
	1.00	0.56 X 0.08	0.58	0.6555
	1.2	0.39 X 0.07	0.41	0.4788

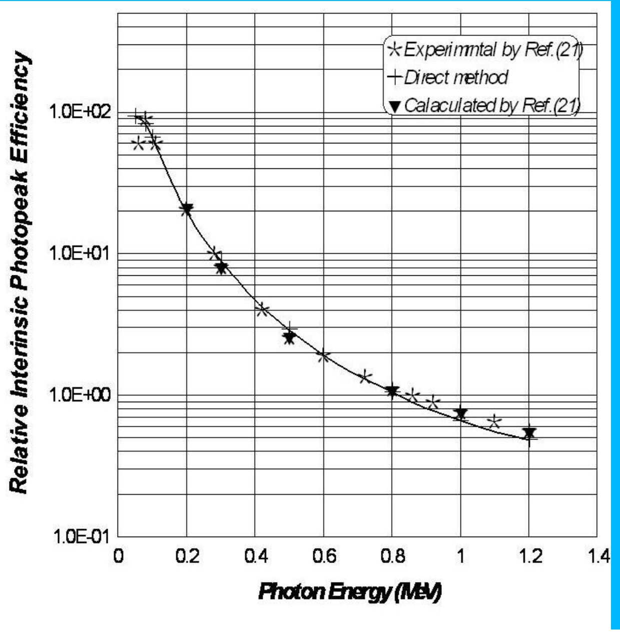
3- Table (4) gives the comparison of intrinsic photopeak efficiency for an axial point source calculated by direct method at for detector (4.4cm×5.0cm) and the distance between the source and detector is 25.7 cm for energy range from 0.2234 MeV up to 3.2536 MeV A.OWENS [19] that measure the FEPE experimentally and also compared with the result carried out by HAJNAL and KLUSEK [30] that use semi empirical method. One can see that the agreement of direct method results with the experimental and semiempirical results is very good.

Table (4:) The comparison of intrinsic photopeak efficiency for an axial point source calculated by direct method at for detector (4.4cm×5.0cm) and the distance between the source and detector is 25.7 cm for energy range from 0.2234 MeV up to 3.2536 MeV A.OWENS[19]

Photon Energy (MeV.)	***Measured Intrinsic Photopeak Efficiency ε_{ip} by Owens [21]***	***Intrinsic Photopeak Efficiency ε_{ip} using semiempirical formula [32]***	***Intrinsic Photopeak Efficiency ε_{ip} calculated by (Direct method) Present Work***
0.22343	0.575	0.56161	0.6409
0.24192	0.537	0.53623	0.6002
0.2952	0.46	0.46444	0.4778
0.30309	0.46	0.45437	0.4580
0.352	0.392	0.396758	0.4028
0.35626	0.393	0.392157	0.3974
0.38409	0.3625	0.363746	0.3628
0.6093	0.223	0.216859	0.2250
0.6616	0.1915	0.196914	0.1971
0.7687	0.158	0.165495	0.1623
0.84678	0.1445	0.148378	0.1451
1.04	0.12	0.120841	0.1122
1.17322	0.1108	0.109870	0.1148
1.23829	0.1059	0.106034	0.1079
1.27452	0.109	0.104117	0.1046
1.33224	0.0992	0.101229	0.0997
1.36021	0.098	0.099889	0.0973
1.764	0.079	0.080301	0.0745
1.77133	0.08031	0.079933	0.0742
2.204	0.061	0.063088	0.0584
2.59852	0.0533	0.053294	0.0472
3.25361	0.0425	0.04	0.05

(9-3) Graphical Comparisons of FEPE calculated by direct method and other method

1- Figures (10,11) give the comparison between the intrinsic full energy peak efficiency with the photon energy range (0.2 MeV up to 1.4 MeV) calculated by our direct method and that calculated by semi empirical formula and experimental method [12] for volume 2 cm^3 and 0.39 cm^3 the two represents the excellent agreement between direct method and semi-empirical method and experimental method.



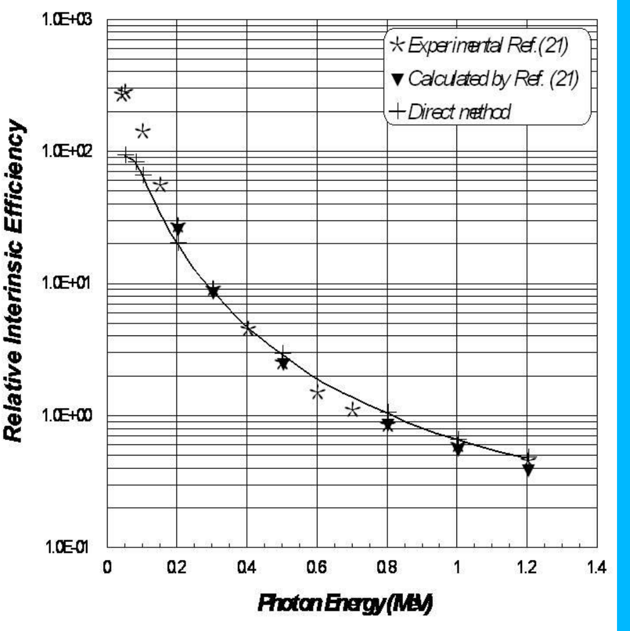


Figure (10,11): The comparison between the intrinsic full energy peak efficiency with the photon energy range (0.2 MeV up to 1.4 MeV) calculated by our direct method and that calculated by semi empirical formula and experimental method

Figure (12) present Comparison of variations of Full Energy Peak Efficiency for the energy range from 0.1 MeV and 2.5 MeV. These comparison were carried out between the present Direct Method and, that calculation by WAINIO and KNOLL, ref [7], that using Monte Carlo calculation by B.LAL et al. ref.[9] and by experimental value of CLINE ref. [31] for detector dimension of radius R=0.9 cm and depth of L=0.8 cm and the axial point source distance is 0.8 cm.

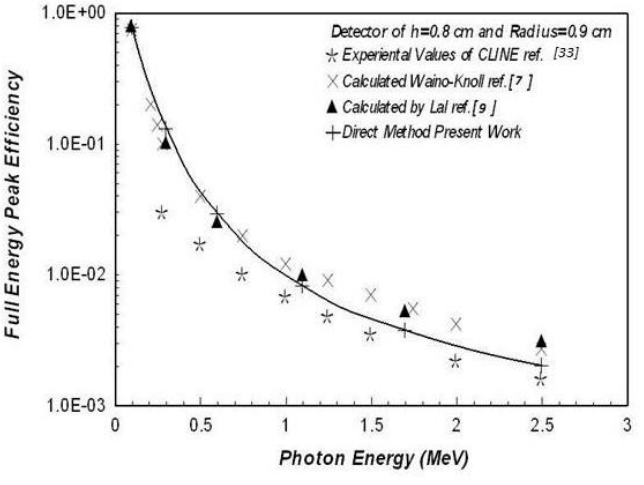


Figure (12): Comparison of the Calculated FEPE by direct method and other methods.

One can easy notice that the values of the 4-Methods are very closely to each other but the values calculated by direct method is the closest one to the experimental measurements compatibility.

(10) Conclusion

One can conclude that, an exact mathematical model to calculate directly photopeak efficiency of HPGe detector with an axial point source at different distances from the detector surface is derived successfully. The model is applicable for gamma ray energy range up to 3 MeV where, the predominant reactions considered are Compton scattering and photoelectric absorption.

The geometrical and mathematical treatment has been done to calculate the average path of the gamma ray in the detector and consequently its the lateral and longitudinal limitations in the finite detector size. Consequently,

The term photo peak coefficient was calculated accurately.

The full energy photo peak efficiencies calculated by direct mathematical model, for different detector sizes, found in an excellent agreement with other accurate published works by other methods and more closer to experimental measurements than other theoretical calculations. Finally, one can say that this work gives a good support and enhance calculations of absolute activity of γ-sources with different geometry in addition to improving the calibration of HPGe detectors.

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