Sparse Signal Recovery: Theory, Applications and Algorithms

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Outline

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Importance of Problem

Organized with Prof. Bresler a Special Session at the 1998 IEEE International Conference on Acoustics, Speech and Signal Processing

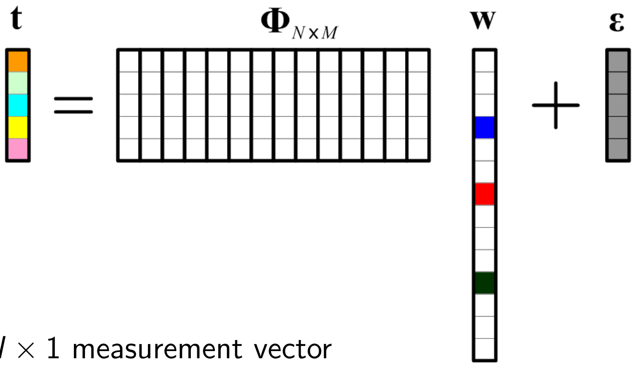
SPEC-DSP: SIGNAL PROCESSING WITH SPARSENESS CONSTRAINT

Talk Goals

- Sparse Signal Recovery is an interesting area with many potential applications
- Tools developed for solving the Sparse Signal Recovery problem are useful for signal processing practitioners to know

Outline

Problem Description



- t is $N \times 1$ measurement vector
- Φ is $N \times M$ Dictionary matrix. $M >> N$.
- w is $M \times 1$ desired vector which is sparse with K non-zero entries
- ϵ is the additive noise modeled as additive white Gaussian

Problem Statement

Noise Free Case: Given a target signal t and a dictionary Φ, find the weights w that solve:

$$\min_{w} \sum_{i=1}^{M} I(w_i \neq 0) \text{ such that } t = \Phi w$$

where $I(.)$ is the indicator function

Noisy Case: Given a target signal t and a dictionary Φ, find the weights w that solve:

$$\min_{w} \sum_{i=1}^{M} I(w_i \neq 0) \text{ such that } \|t - \Phi w\|_2^2 \leq \beta$$

Complexity

- Search over all possible subsets, which would mean a search over a total of $({}^{M}C_{K})$ subsets. Problem NP hard. With $M = 30, N = 20$, and $K = 10$ there are 3×10^{7} subsets (Very Complex)
- A branch and bound algorithm can be used to find the optimal solution. The space of subsets searched is pruned but the search may still be very complex.
- Indicator function not continuous and so not amenable to standard optimization tools.

Challenge: Find low complexity methods with acceptable performance

Outline

Applications

- Signal Representation (Mallat, Coifman, Wickerhauser, Donoho, ...)
- EEG/MEG (Leahy, Gorodnitsky, Ioannides, ...)
- Functional Approximation and Neural Networks (Chen, Natarajan, Cun, Hassibi, ...)
- Bandlimited extrapolations and spectral estimation (Papoulis, Lee, Cabrera, Parks, ...)
- Speech Coding (Ozawa, Ono, Kroon, Atal, ...)
- Sparse channel equalization (Fevrier, Greenstein, Proakis,)
- Compressive Sampling (Donoho, Candes, Tao..)

DFT Example

N chosen to be 64 in this example.

Measurement t:

$$\begin{aligned} t[n] &= \cos \omega_0 n, n = 0, 1, 2, .., N-1 \\ \omega_0 &= \frac{2\pi}{64}\frac{33}{2} \end{aligned}$$

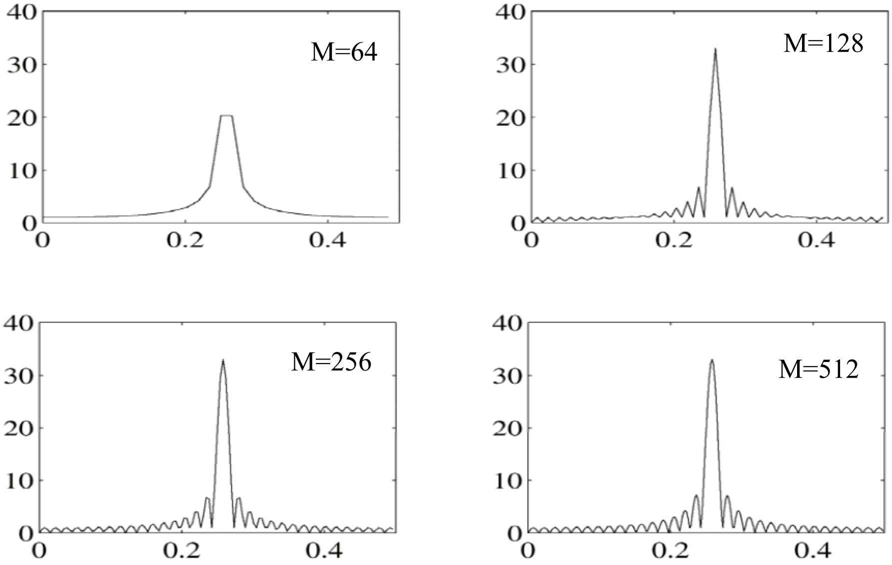
Dictionary Elements:

$$\phi_k = [1, e^{-j\omega_k}, e^{-j2\omega_k}, .., e^{-j(N-1)\omega_k}]^T, \omega_k = \frac{2\pi}{M}$$

Consider $M = 64, 128, 256$ and 512

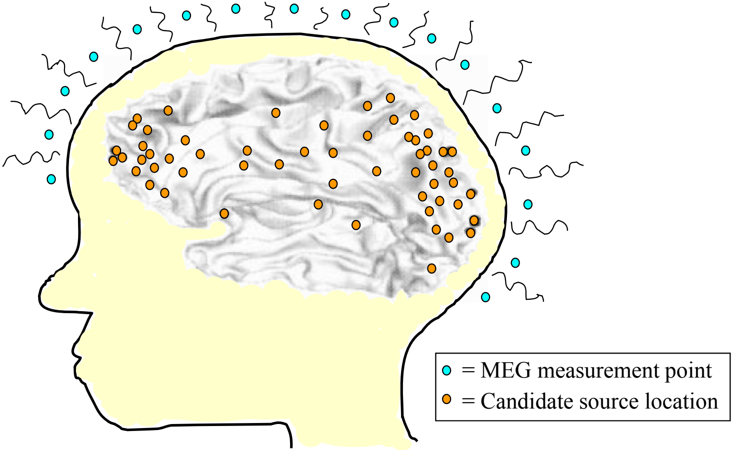
NOTE: The frequency components are included in the dictionary Φ for $M = 128, 256,$ and 512.

FFT Results with Different M



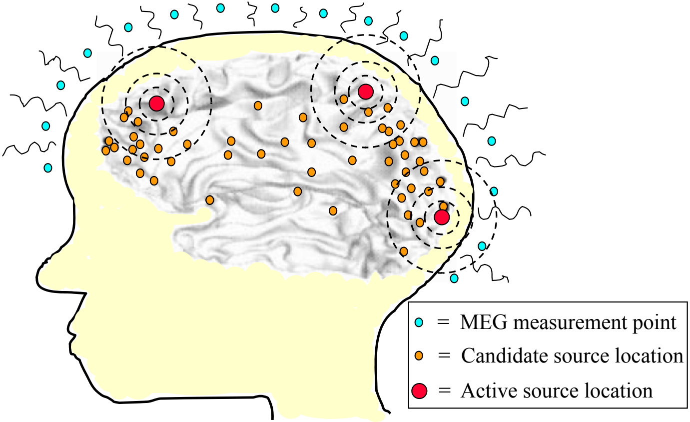
Magnetoencephalography (MEG)

Given measured magnetic fields outside of the head, the goal is to locate the responsible current sources inside the head.



MEG Example

At any given time, typically only a few sources are active (SPARSE).



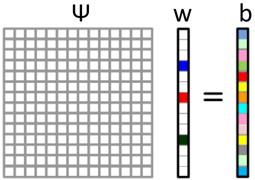
MEG Formulation

- Forming the overcomplete dictionary Φ
  - The number of rows equals the number of sensors.
  - The number of columns equals the number of possible source locations.
  - Φ_{ij} = the magnetic field measured at sensor i produced by a unit current at location j.
  - We can compute Φ using a boundary element brain model and Maxwells equations.
- Many different combinations of current sources can produce the same observed magnetic field t.
- By finding the sparsest signal representation/basis, we find the smallest number of sources capable of producing the observed field.
- Such a representation is of neurophysiological significance

Compressive Sampling

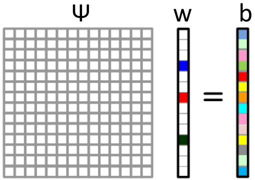
Compressive Sampling

Transform Coding

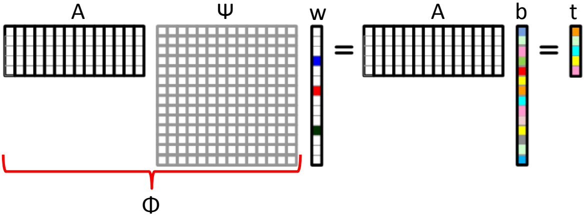


Compressive Sampling

Transform Coding

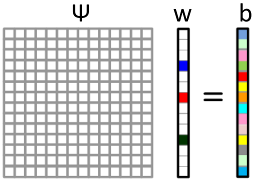


Compressive Sensing

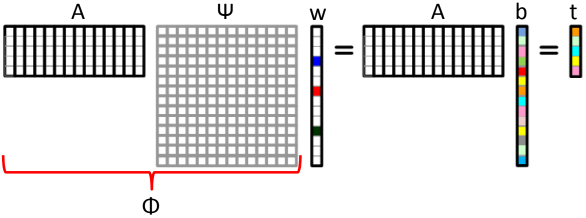


Compressive Sampling

Transform Coding



Compressive Sensing



Computation : $t \longrightarrow w \longrightarrow b$

Outline

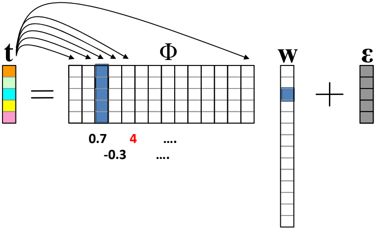
Potential Approaches

Problem NP hard and so need alternate strategies

- Greedy Search Techniques: Matching Pursuit, Orthogonal Matching Pursuit
- Minimizing Diversity Measures: Indicator function not continuous. Define Surrogate Cost functions that are more tractable and whose minimization leads to sparse solutions, e.g. ℓ_1 minimization
- Bayesian Methods: Make appropriate Statistical assumptions on the solution and apply estimation techniques to identify the desired sparse solution

Greedy Search Methods: Matching Pursuit

- Select a column that is most aligned with the current residual



- Remove its contribution
- Stop when residual becomes small enough or if we have exceeded some sparsity threshold.
- Some Variations
  - Matching Pursuit [Mallat & Zhang]
  - Orthogonal Matching Pursuit [Pati et al.]
  - Order Recursive Matching Pursuit (ORMP)

Inverse techniques

For the systems of equations $\Phi w = t$, the solution set is characterized by $\{w_s : w_s = \Phi^+ t + v, v \in \mathcal{N}(\Phi)\}$, where $\mathcal{N}(\Phi)$ denotes the null space of Φ and $\Phi^+ = \Phi^T(\Phi\Phi^T)^{-1}$.

Minimum Norm solution : The minimum ℓ_2 norm solution $w_{mn} = \Phi^+ t$ is a popular solution

Noisy Case: regularized ℓ_2 norm solution often employed and is given by

$$w_{reg} = \Phi^T(\Phi\Phi^T + \lambda I)^{-1} t$$

Problem: Solution is not Sparse

Diversity Measures

- Functionals whose minimization leads to sparse solutions
- Many examples are found in the fields of economics, social science and information theory
- These functionals are concave which leads to difficult optimization problems

Examples of Diversity Measures

$\ell_{(p\leq 1)}$ Diversity Measure

$$E^{(p)}(w) = \sum_{l=1}^{M} |w_l|^p, p \leq 1$$

Gaussian Entropy

$$H_G(w) = \sum_{l=1}^{M} \ln |w_l|^2$$

Shannon Entropy

$$H_S(w) = -\sum_{l=1}^{M} \tilde{w}_l \ln \tilde{w}_l. \text{ where } \tilde{w}_l = \frac{w_l^2}{\|w\|^2}$$

Diversity Minimization

Noiseless Case

$$\min_{w} E^{(p)}(w) = \sum_{l=1}^{M} |w_l|^p \text{ subject to } \Phi w = t$$

Noisy Case

$$\min_{w} \left(\|t - \Phi w\|^2 + \lambda \sum_{l=1}^{M} |w_l|^p \right)$$

$p = 1$ is a very popular choice because of the convex nature of the optimization problem (Basis Pursuit and Lasso).

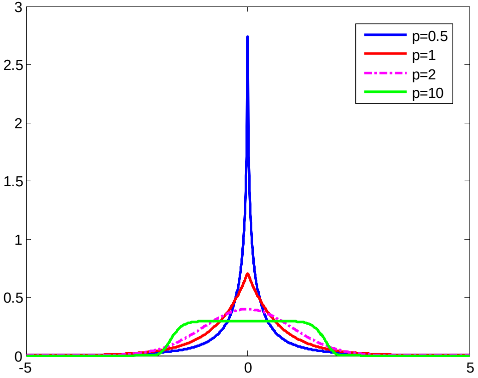
Bayesian Methods

- Maximum Aposteriori Approach (MAP)
  - Assume a sparsity inducing prior on the latent variable w
  - Develop an appropriate MAP estimation algorithm
- Empirical Bayes
  - Assume a parameterized prior on the latent variable w (hyperparameters)
  - Marginalize over the latent variable w and estimate the hyperparameters
  - Determine the posterior distribution of w and obtain a point as the mean, mode or median of this density

Generalized Gaussian Distribution

Density function: Subgaussian: $p > 2$ and Supergaussian : $p < 2$

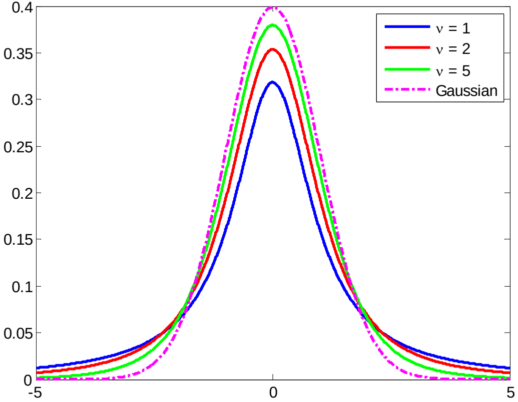
$$f(x) = \frac{p}{2\sigma\Gamma(\frac{1}{p})} \exp\left\{-\left(\frac{|x|}{\sigma}\right)^p\right\}$$



Student t Distribution

Density function:

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{\left(\frac{\nu+1}{2}\right)}$$



MAP using a Supergaussian prior

Assuming a Gaussian likelihood model for $f(t|w)$, we can find MAP weight estimates

$$
\begin{aligned}
w_{MAP} &= \arg\max_{w} \log f(w|t) \\
&= \arg\max_{w} \left(\log f(t|w) + \log f(w)\right) \\
&= \arg\min_{w} \left(\|\Phi w - t\|^2 + \lambda \sum_{l=1}^{M} |w_l|^p \right)
\end{aligned}
$$

This is essentially a regularized LS framework. Interesting range for p is $p \leq 1$.

MAP Estimate: FOCal Underdetermined System Solver (FOCUSS)

Approach involves solving a sequence of Regularized Weighted Minimum Norm problems

$$q_{k+1} = \arg\min_q \left(\|\Phi_{k+1} q - t\|^2 + \lambda \|q\|^2 \right)$$

where $\Phi_{k+1} = \Phi M_{k+1}$, and $M_{k+1} = \text{diag}(|w_{k.l}|^{1-\frac{p}{2}}$.

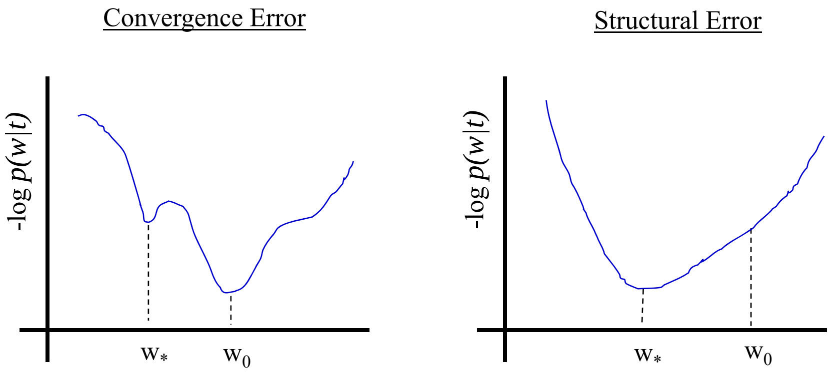
$$w_{k+1} = M_{k+1} q_{k+1}.$$

$p = 0$ is the ℓ_0 minimization and $p = 1$ is ℓ_1 minimization

FOCUSS summary

- For $p < 1$, the solution is initial condition dependent
  - Prior knowledge can be incorporated
  - Minimum norm is a suitable choice
  - Can retry with several initial conditions
- Computationally more complex than Matching Pursuit algorithms
- Sparsity versus tolerance tradeoff more involved
- Factor p allows a trade-off between the speed of convergence and the sparsity obtained

Convergence Errors vs. Structural Errors



$$\begin{aligned} w_* &= \text{solution we have converged to} \\ w_0 &= \text{maximally sparse solution} \end{aligned}$$

Shortcomings of these Methods

$p = 1$

- Basis Pursuit/Lasso often suffer from structural errors.
- Therefore, regardless of initialization, we may never find the best solution.

$p < 1$

- The FOCUSS class of algorithms suffers from numerous suboptimal local minima and therefore convergence errors.
- In the low noise limit, the number of local minima K satisfies

$$K \in \left[\binom{M-1}{N} + 1, \binom{M}{N} \right]$$

- At most local minima, the number of nonzero coefficients is equal to N, the number of rows in the dictionary.

Empirical Bayesian Method

Main Steps

- Parameterized prior $f(w|\gamma)$
- Marginalize

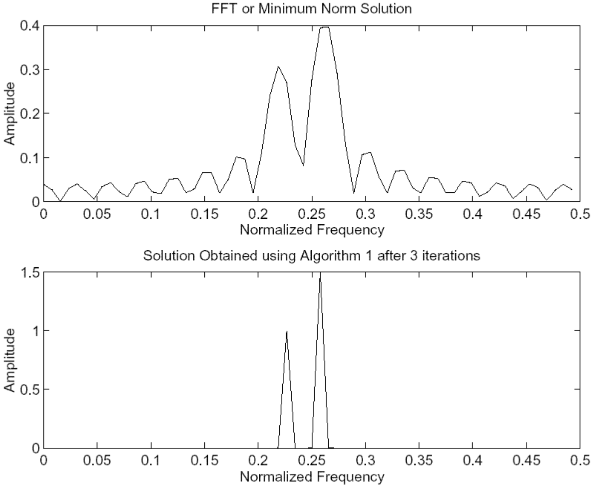
$$f(t|\gamma) = \int f(t, w|\gamma)dw = \int f(t|w)f(w|\gamma)dw$$

- Estimate the hyperparameter $\hat{\gamma}$
- Determine the posterior density of the latent variable $f(w|t, \hat{\gamma})$
- Obtain point estimate of w

Example: Sparse Bayesian Learning (SBL by Tipping)

Outline

DFT Example


FFT or Minimum Norm Solution
Amplitude
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Normalized Frequency
Solution Obtained using Algorithm 1 after 3 iterations
Amplitude
1.5
1
0.5
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Normalized Frequency

Empirical Tests

- Random overcomplete bases Φ and sparse weight vectors w_0 were generated and used to create target signals t, i.e., $t = \Phi w_0 + \epsilon$
- SBL (Empirical Bayes) was compared with Basis Pursuit and FOCUSS (with various p values) in the task of recovering w_0.

Experiment 1: Comparison with Noiseless Data

- Randomized Φ (20 rows by 40 columns).
- Diversity of the true w_0 is 7.
- Results are from 1000 independent trials.

NOTE: An error occurs whenever an algorithm converges to a solution w not equal to w_0.

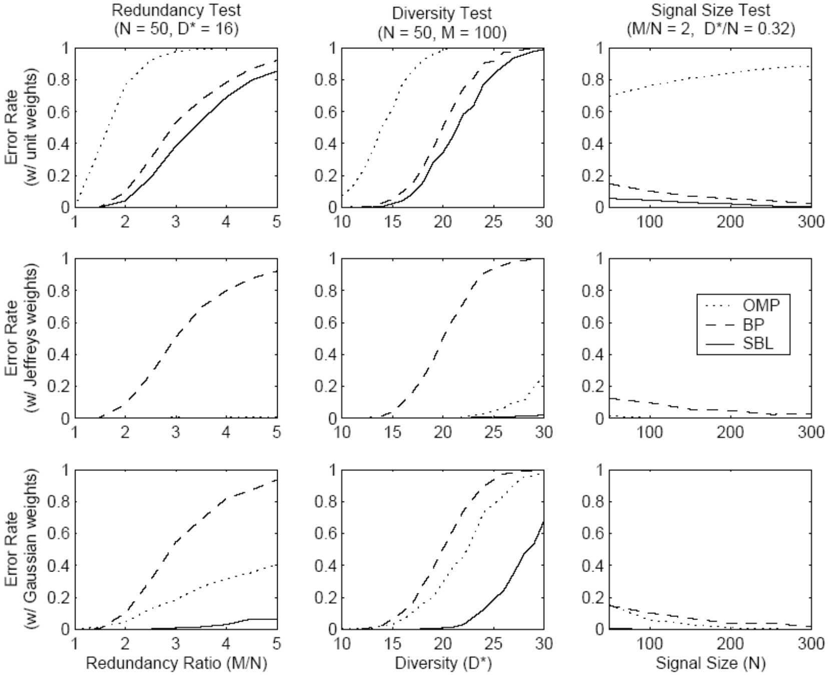
	FOCUSS ($p = 0.001$)	**FOCUSS** ($p = 0.9$)	**Basis Pursuit** ($p = 1.0$)	**SBL**
Convergence Errors	34.1%	18.1%	0.0%	7.4%
Structural Errors	0.0%	5.7%	22.3%	0.0%
Total Errors	34.1%	23.8%	22.3%	**7.4%**

Experiment II: Comparison with Noisy Data

- Randomized Φ (20 rows by 40 columns).
- Diversity of the true w_0 is 7.
- 20 db AWGN
- Results are from 1000 independent trials.

NOTE: We no longer distinguish convergence errors from structural errors.

	FOCUSS ($p = 0.001$)	**FOCUSS** ($p = 0.9$)	**Basis Pursuit** ($p = 1.0$)	**SBL**
Total Errors	52.2%	43.1%	45.5%	**21.1%**


Redundancy Test
(N = 50, D* = 16)
Diversity Test
(N = 50, M = 100)
Signal Size Test
(M/N = 2, D*/N = 0.32)
Error Rate
(w/ unit weights)
Error Rate
(w/ Jeffreys weights)
Error Rate
(w/ Gaussian weights)
OMP
BP
SBL
Redundancy Ratio (M/N)
Diversity (D*)
Signal Size (N)

MEG Example

- Data based on CTF MEG system at UCSF with 275 scalp sensors.
- Forward model based on $40,000$ points (vertices of a triangular grid) and 3 different scale factors.
- Dimensions of lead field matrix (dictionary): 275 by $120,000$.
- Overcompleteness ratio approximately 436.
- Up to 40 unknown dipole components were randomly placed throughout the sample space.
- SBL was able to resolve roughly twice as many dipoles as the next best method (Ramirez 2005).

Outline

Summary

- Discussed role of sparseness in linear inverse problems
- Discussed Applications of Sparsity
- Discussed methods for computing sparse solutions
  - Matching Pursuit Algorithms
  - MAP methods (FOCUSS Algorithm and ℓ_1 minimization)
  - Empirical Bayes (Sparse Bayesian Learning (SBL))

Expectation is that there will be continued growth in the application domain as well as in the algorithm development.