IEEE WCCI 2016 Tutorial (FUZZ-4), Vancouver, July 24, 2016

A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control - Beyond Linear Matrix Inequalities -

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https://sites.google.com/site/tanaka2lab/home

Google Scholar
https://scholar.google.com/citations?user=RxHaAJwAAAAJ&hl=en
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A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control - Beyond Linear Matrix Inequalities -

The main research covered in this tutorial has been conducted in our laboratory at the University of Electro-Communications (UEC), Tokyo, Japan, in collaboration with Prof. Hua O. Wang and his laboratory at Boston University, Boston, USA. Throughout the tutorial, it will be reflected upon how to bridge enabling fuzzy model-based control frameworks with system-theoretical approaches in the development of toolkits for control of nonlinear systems.

From a nonlinear control theory point of view

Recommended prior knowledge in this tutorial
- Modern control theory
- Lyapunov stability theory

Tutorial Overview

- **Introductions**
- **Part I**
  Outline of Takagi-Sugeno (T-S) Fuzzy Model-based Control
- **Part II**
  T-S Fuzzy Model-based Control using Linear Matrix Inequalities (LMIs)
- **Part III**
  Theoretical Advances in T-S Fuzzy Model-based Control using LMIs
- **Part IV**
  Beyond LMIs: Polynomial Fuzzy Systems Control and Analysis using Sum-of- Squares (SOS)
- **Conclusions**

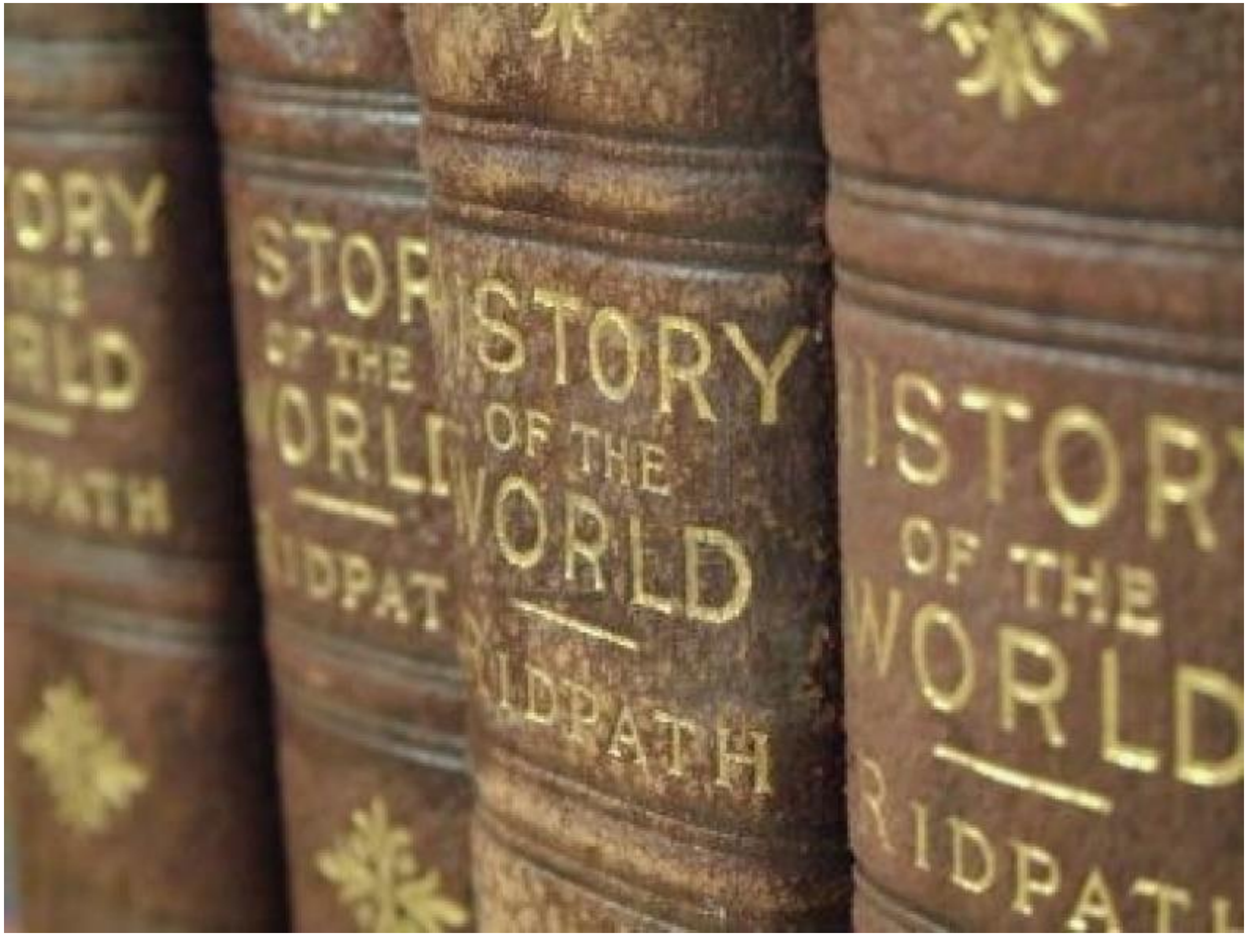
Tutorial Overview

- **Introductions**
  - **History of fuzzy control**
  - **Recent research direction in fuzzy control**

Matrices and Vectors in this tutorial

It is assumed in this tutorial that all the matrices and vectors have appropriate dimensions.
$\boldsymbol{P} \succ \boldsymbol{0}$ ($\boldsymbol{P} \succeq \boldsymbol{0}$) means that $\boldsymbol{P}$ is a positive-definite matrix (positive semidefinite matrix).

History of Fuzzy Control



K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

History of Fuzzy Control

1965 L. A. Zadeh
Fuzzy Sets

Fuzzy

1968 L. A. Zadeh
Fuzzy Algorithms

Fuzzy Control

1974 E. H. Mamdani
Application of Fuzzy Algorithms for Steam Engine Control

1982 L. P. Holmblad and J. J. Ostergaard
Control of a Cement Kiln by Fuzzy Logic

1985 T. Takagi and Sugeno
Takagi-Sugeno Fuzzy model

Fuzzy Model-based Control

1989 K. Tanaka and M. Sugeno
Stability Analysis using Lyapunov Stability Theory

1994 H. O. Wang, K. Tanaka and M. Griffin
Design and Analysis via Linear Matrix Inequality (LMI)

LMI based Design

Design and Analysis via LMIs

Recent Research Direction in Fuzzy Control

Design and Analysis via LMIs

From Simple Lyapunov Function
to Generalized Lyapunov Functions

From Takagi-Sugeno Fuzzy Model
to Polynomial Fuzzy Model

From Linear Matrix Inequality (LMI)
to Sum of Squares (SOS)

SOS based Design

2009 K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang,
Design and Analysis via SOS

Design and Analysis via SOS

Beyond LMIs

Recent Research Direction in Fuzzy Control

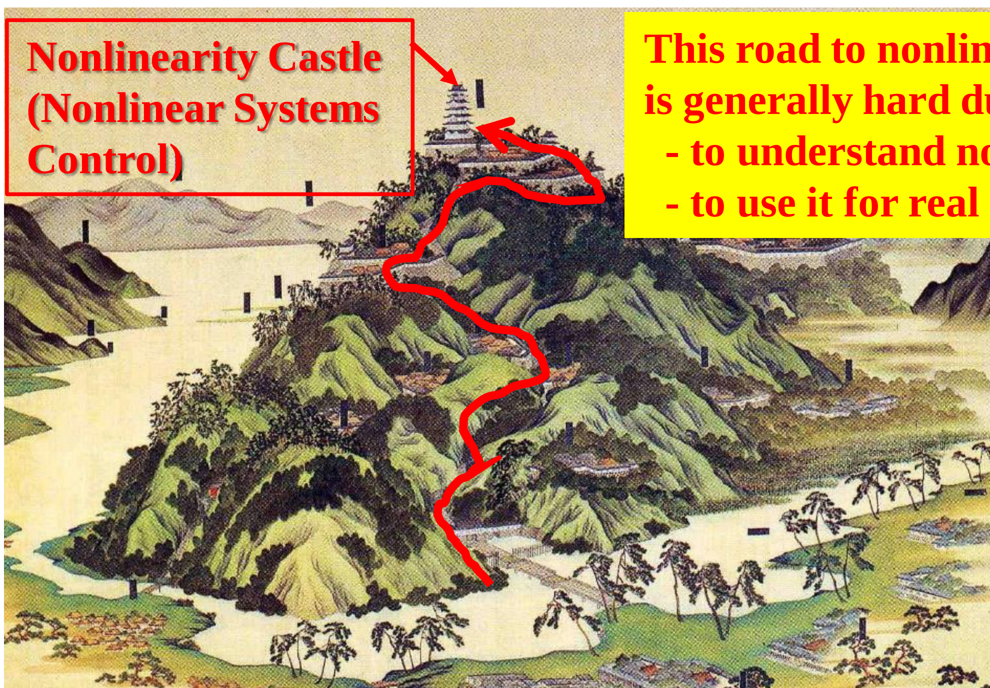
Design and Analysis via LMIs

↓

Design and Analysis via SOS

Beyond LMIs

The history of fuzzy-model based control is the history of matching to nonlinearities.



This road to nonlinearity castle (nonlinear systems control) is generally hard due to the difficulties
- **to understand nonlinear control theory**
- **to use it for real complex systems**

Fuzzy-model based Control

Fuzzy model is a good match to nonlinearities.

When nonlinearities met fuzzy model, nonlinearities became easier to contend with.

Tutorial Overview

- **Part I**

  Outline of Takagi-Sugeno (T-S) Fuzzy Model-based Control
  - **Why do we use T-S Fuzzy Model-based Control?**
  - **What is T-S Fuzzy Model-based Control?**
  - **How can we realize T-S Fuzzy Model-based Control?**

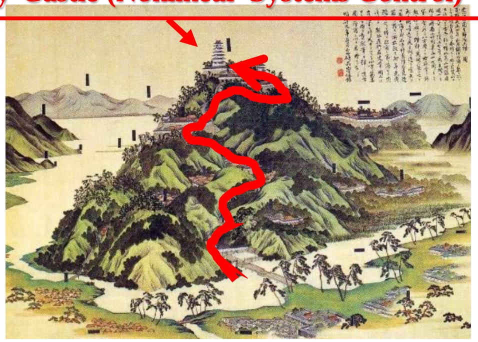
K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Why Do We Use?

Other nonlinear control techniques require special and rather involved knowledge, e.g.,

R. Sepulcher, M. Jankovic, and P. Kokotovic,
Constructive Nonlinear Control, New York: Springer-Verlag, 1997.

Nonlinearity Castle (Nonlinear Systems Control)



T-S Fuzzy Model-based Control → Simple, Natural and Effective Nonlinear Control

Fuzzy logic is a good match to nonlinearities.
When nonlinearities met fuzzy logic,
nonlinearities became easier to contend with.

K. Tanaka,
Recent Advances in Fuzzy Modeling and Control: When Nonlinearities Met Fuzzy Logic,
WCCI 2014 Invited Lecture, Beijing, July 8, 2014.

What is Fuzzy Model-based Control?

Fuzzy Model-based Control ➡
Simple, Natural and Effective Nonlinear Control

Nonlinear System

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$

Simple

Model Construction

Takagi-Sugeno Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t))\{\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)\}$$

Natural & Effective

Analysis & Design

Fuzzy Controller

$$\mathbf{u}(t) = -\sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{F}_i \mathbf{x}(t)$$

K. Tanaka and M. Sugeno,
Stability Analysis and Design of Fuzzy Control Systems,
Fuzzy Sets and Systems, Vol.45, pp.135 - 156 (1992).

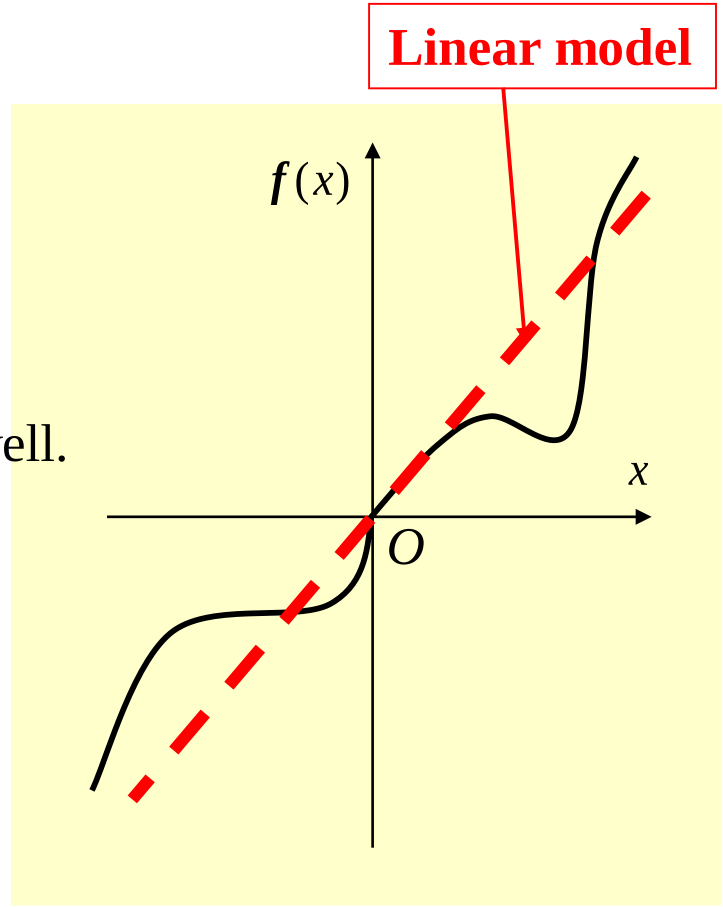
How Can We Realize?

Fuzzy Model Construction (1/4)

Nonlinear System

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$

If the nonlinearity is strong,
the linear control approach does not work well.



How Can We Realize?

Fuzzy Model Construction (2/4)

Local linearization

$$\mathbf{f}(x) = \begin{cases} a_1 x + b_1 & (d_1 < x \le d_2) \\ a_2 x + b_2 & (d_2 < x \le 0) \\ a_3 x + b_3 & (0 < x \le d_3) \\ a_4 x + b_4 & (d_3 < x \le d_4) \end{cases}$$

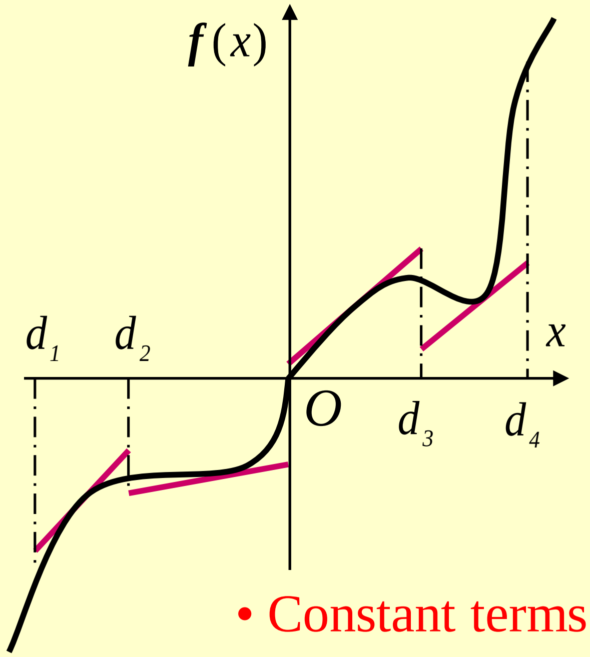
Static model

$$y = \sum_{i=1}^{4} h_i(x)\{a_i x + b_i\}$$

Dynamic model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} h_i(\mathbf{z}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) + \underline{\mathbf{D}_i})$$

T. Takagi and M. Sugeno, "Fuzzy Identification of Systems and Its Applications to Modeling and Control", IEEE Trans. on SMC 15, no. 1, pp.116-132, 1985.



- Constant terms
- Approximation

Piecewise Linear Model

How Can We Realize?

Fuzzy Model Construction (3/4)

K. Tanaka and H. O. Wang,
Fuzzy Control System Design and Analysis:
A Linear Matrix Inequality Approach,
John Wiley & Sons (2001).

Sector $a_1 x, \; a_2 x$

Fuzzy model

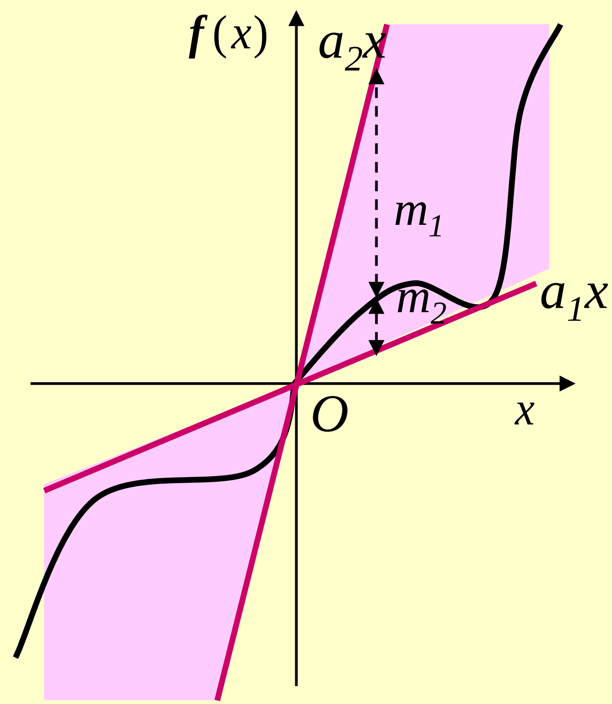
$$\dot{x} = \mathbf{f}(x) = \sum_{i=1}^{2} h_i(x) a_i x$$

$$h_1(x) = \frac{m_2(x)}{m_1(x)+m_2(x)} \quad h_2(x) = \frac{m_1(x)}{m_1(x)+m_2(x)}$$

$$h_i(x) \geq 0, \sum_{i=2}^{2} h_i(x) = 1$$

Dynamic model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} h_i(\mathbf{z}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t))$$



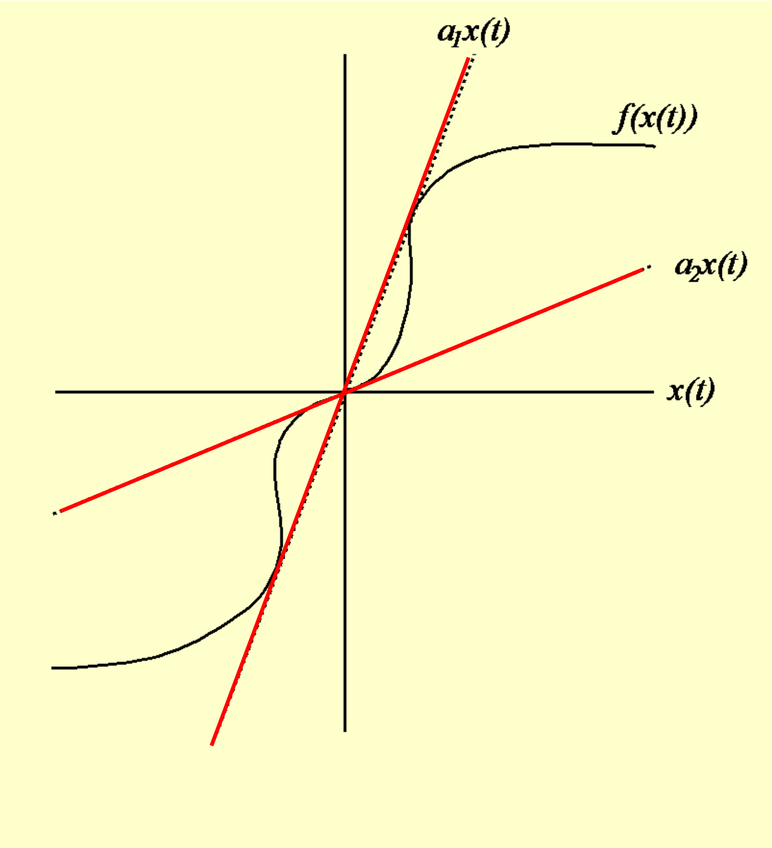
Sector nonlinearity concept

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

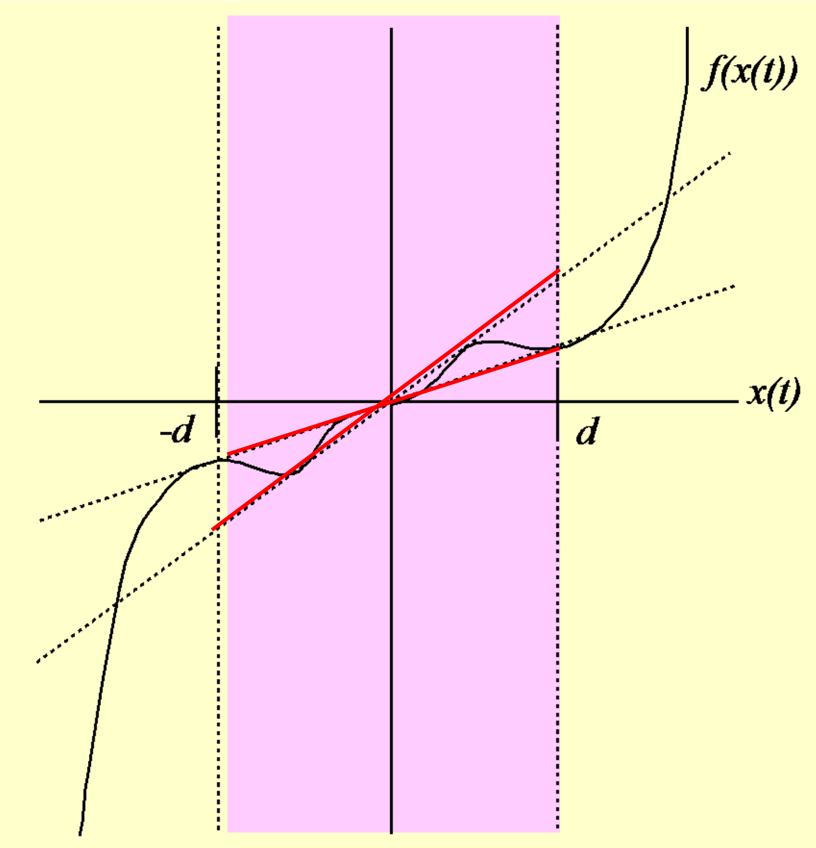
How Can We Realize?

Fuzzy Model Construction (4/4)

Consider the following nonlinear model



Global Sector



Semi-Global Sector

$$-d < x(t) < d$$

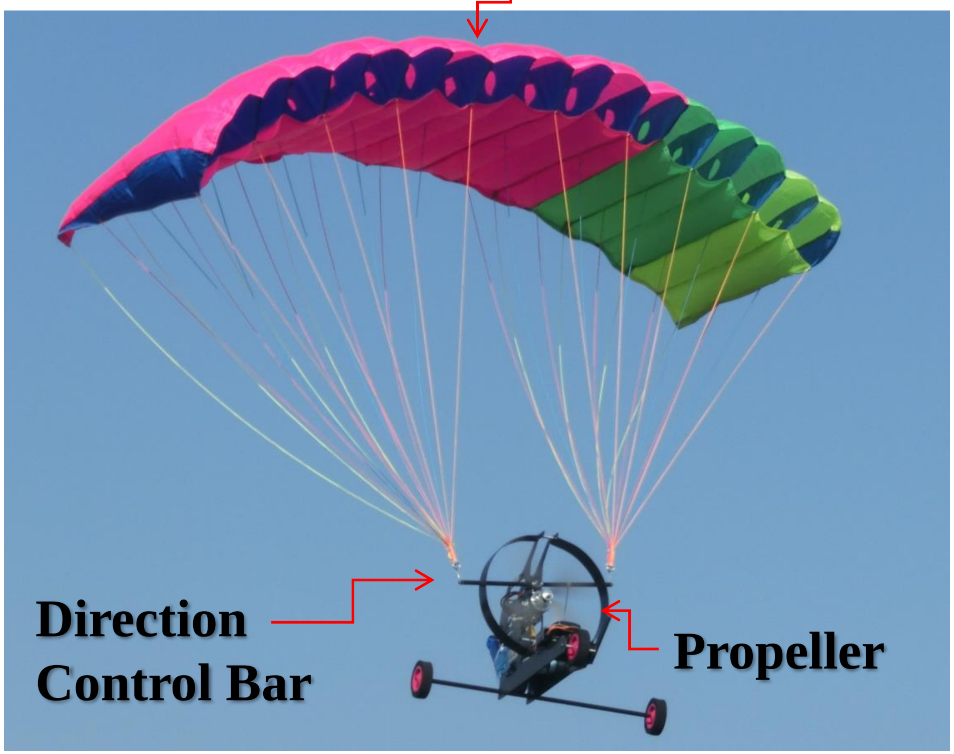
How Can We Realize?

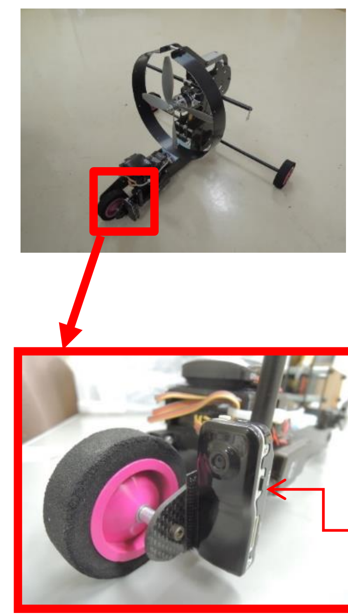
Fuzzy Model Construction Example

Powered Paraglider

- GPS
- Gyro sensor
- Acceleration sensor
- Magnetic sensor

Canopy
(Parachute)





Camera

How Can We Realize?

Fuzzy Model Construction Example

Kinematic Model of Powered Paraglider

Consider a kinematic model at the steady-state flight.

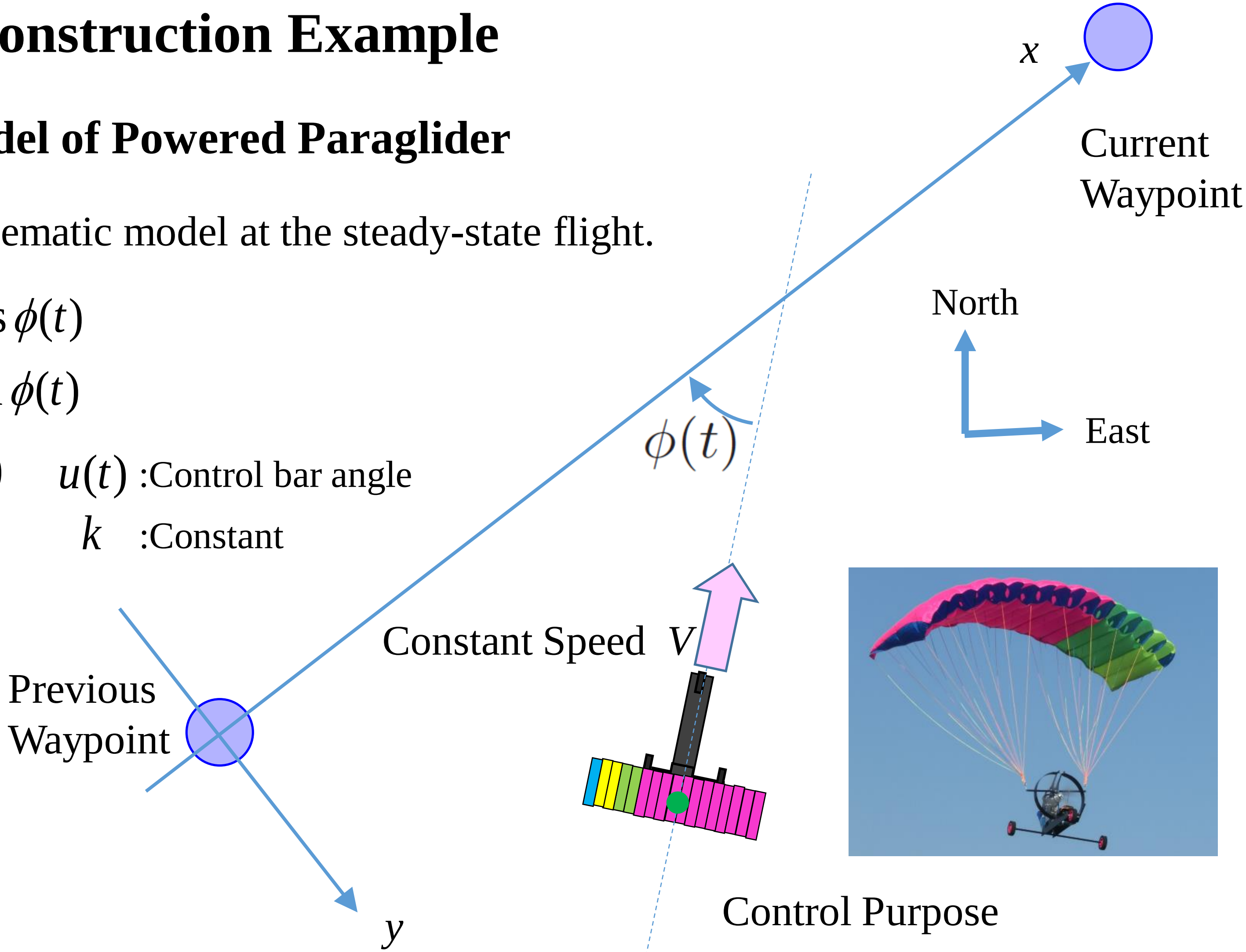
$$\dot{x}(t) = V\cos\phi(t)$$

$$\dot{y}(t) = V\sin\phi(t)$$

$$\dot{\phi}(t) = ku(t)$$

$u(t)$:Control bar angle

k :Constant



Control Purpose

$$\lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} \phi(t) = 0$$

How Can We Realize?

Fuzzy Model Construction Example

Kinematic Model of Powered Paraglider

Consider a kinematic model at the steady-state flight.

$$\dot{x}(t) = V\cos\phi(t)$$

$$\dot{y}(t) = V\sin\phi(t)$$

$$\dot{\phi}(t) = ku(t)$$

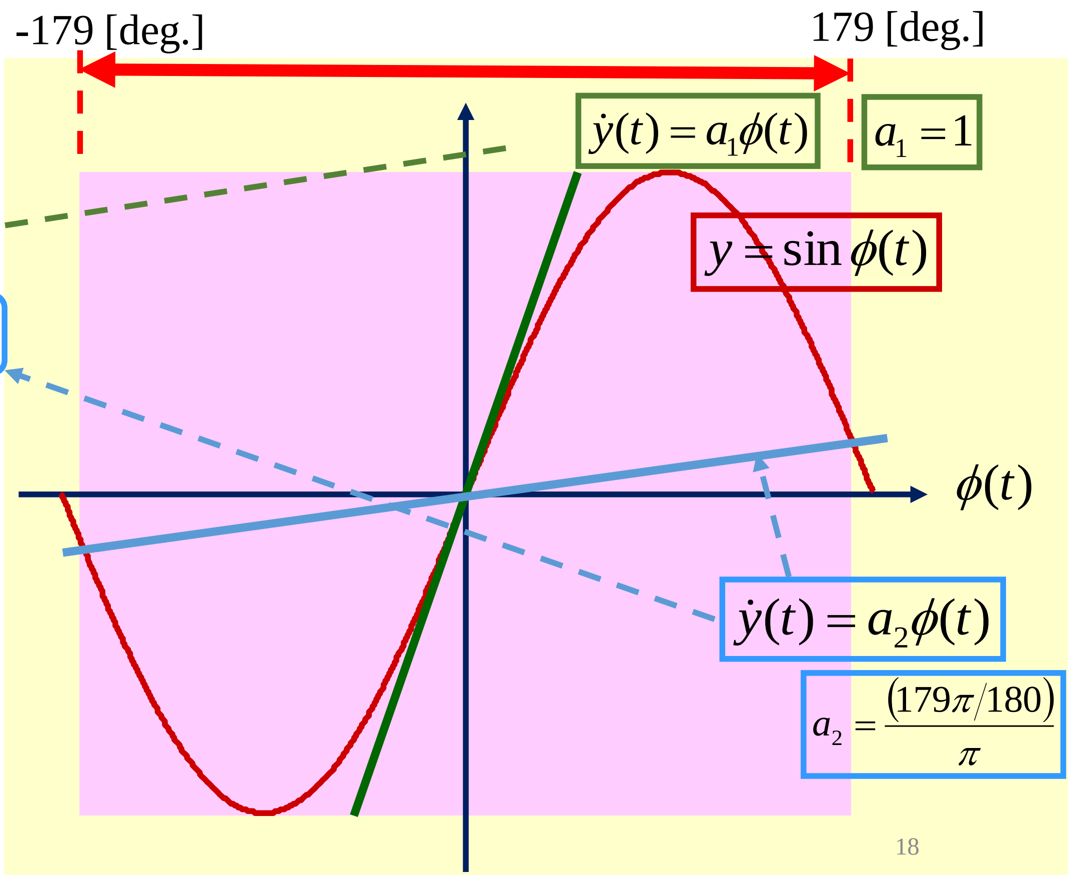
$$\sin\phi(t) = h_1(\phi(t))a_1\phi(t) + h_2(\phi(t))a_2\phi(t)$$

$$h_1(\phi(t)) + h_2(\phi(t)) = 1$$

$$0 \le h_i(\phi(t)) \le 1 \qquad i = 1,\ 2$$

$$h_1(\phi(t)) = \frac{\sin\phi(t) - a_2\phi(t)}{(a_1 - a_2)\phi(t)}$$

$$h_2(\phi(t)) = \frac{\sin\phi(t) - a_1\phi(t)}{(a_2 - a_1)\phi(t)}$$



How Can We Realize?

Fuzzy Model Construction Example

Kinematic Model of Powered Paraglider

Consider a kinematic model at the steady-state flight.

$$\dot{x}(t) = V\cos\phi(t)$$

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$$\sin\phi(t) = h_1(\phi(t))a_1\phi(t) + h_2(\phi(t))a_2\phi(t)$$

$$h_1(\phi(t)) + h_2(\phi(t)) = 1$$

$$0 \le h_i(\phi(t)) \le 1 \quad i = 1,\ 2$$

$$h_1(\phi(t)) = \frac{\sin\phi(t) - a_2\phi(t)}{(a_1 - a_2)\phi(t)}$$

$$h_2(\phi(t)) = \frac{\sin\phi(t) - a_1\phi(t)}{(a_2 - a_1)\phi(t)}$$

$$\begin{bmatrix}\dot{y}(t)\\ \dot{\phi}(t)\end{bmatrix} = \begin{bmatrix}V\sin\phi(t)\\ ku(t)\end{bmatrix} = \begin{bmatrix}h_1(\phi(t))Va_1\phi(t) + h_2(\phi(t))Va_2\phi(t)\\ ku(t)\end{bmatrix}$$

$$= \begin{bmatrix}0 & h_1(\phi(t))Va_1 + h_2(\phi(t))Va_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}y(t)\\ \phi(t)\end{bmatrix} + \begin{bmatrix}0\\ k\end{bmatrix}u(t)$$

$$= \sum_{i=1}^{2} h_i(\phi(t))\left\{\begin{bmatrix}0 & Va_i\\ 0 & 0\end{bmatrix}\begin{bmatrix}y(t)\\ \phi(t)\end{bmatrix} + \begin{bmatrix}0\\ k\end{bmatrix}u(t)\right\}$$

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} h_i(\mathbf{z}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t))$$

$$\boldsymbol{A}_i = \begin{bmatrix}0 & Va_i\\ 0 & 0\end{bmatrix},\ \boldsymbol{B}_i = \begin{bmatrix}0\\ k\end{bmatrix},\ i = 1,2$$

How Can We Realize?

Nonlinear System

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$

Takagi-Sugeno Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t))\{\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\}$$

Fuzzy Controller

$$\mathbf{u}(t) = -\sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{F}_i\mathbf{x}(t)$$

How Can We Realize?

Takagi-Sugeno Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t))\{\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\}$$

Parallel Distributed Compensation (PDC)

Fuzzy Controller

$$\mathbf{u}(t) = -\sum_{i=1}^{r} h_i(\mathbf{z}(t))\mathbf{F}_i\mathbf{x}(t)$$

[PDC]
K. Tanaka and M. Sugeno,
Stability Analysis and Design of Fuzzy Control Systems,
Fuzzy Sets and Systems, Vol.45, pp.135 - 156 (1992).

How Can We Realize?

Parallel Distributed Compensation (PDC)

Fuzzy model: $$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}$$

Rule i: IF $z_1(t)$ is M_{i1} … and $z_n(t)$ is M_{in}

Then $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)$

PDC controller: $$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i\boldsymbol{x}(t)$$

shares the same membership function

Rule i: IF $z_1(t)$ is M_{i1} … and $z_n(t)$ is M_{in}

Then $\boldsymbol{u}(t) = -\boldsymbol{F}_i\boldsymbol{x}(t)$

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

How Can We Realize?

Controller Design using LMI

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \{ A_i - B_i F_j \} x(t)$$

Design the local feedback gains such that the closed-loop system is globally asymptotically stable

Takagi-Sugeno Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t)) \{ \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) \}$$

Analysis & Design

Linear Matrix Inequality (LMI)

Fuzzy Controller

$$\mathbf{u}(t) = -\sum_{i=1}^{r} h_i(\mathbf{z}(t)) \mathbf{F}_i \mathbf{x}(t)$$

[LMI]

H. O. Wang, K. Tanaka and M. F. Griffin,
An Approach to Fuzzy Control of Nonlinear Systems,
IEEE Transactions on Fuzzy Systems, Vol.4, No.1, pp.14-23 (1996).

Tutorial Overview

- **Part II**

  T-S Fuzzy Model-based Control using Linear Matrix Inequalities (LMIs)

  - **Local linear system stability does not imply the global stability of T-S fuzzy systems**
  - **Lyapunov stability theory**
  - **What are LMIs?**
  - **Three important theorems**
  - **Basic stabilization condition**

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

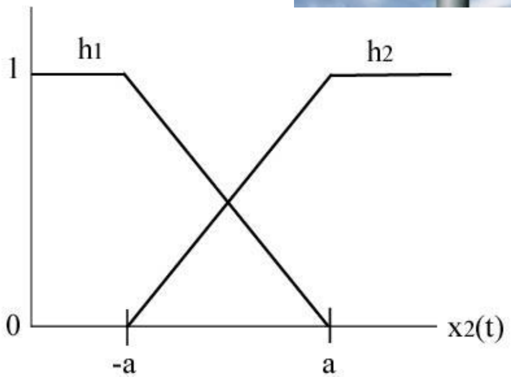
Local stability does not imply global stability

Consider the following (discrete) fuzzy system

$$\mathbf{x}(t+1)=\sum_{i=1}^{2} h_i(\mathbf{z}(t))\mathbf{A}_i\mathbf{x}(t) \qquad \boldsymbol{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$h_i(\mathbf{z}(t))\geq 0 \qquad \sum_{i=1}^{r} h_i(\mathbf{z}(t))=1 \qquad \mathbf{z}(t)=x_2(t)$$

$$\mathbf{A}_1=\begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}_2=\begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}$$



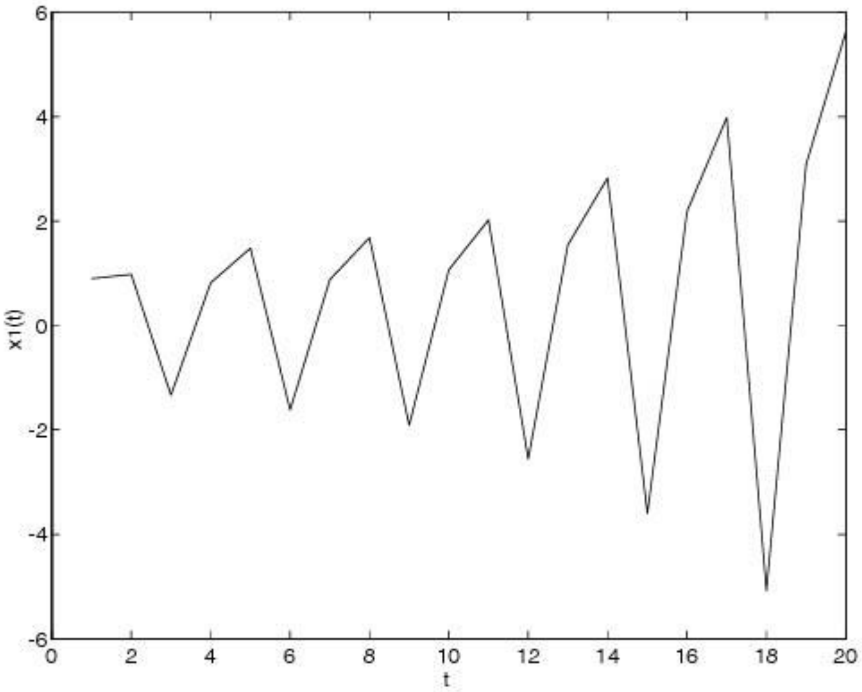
All the A_i are stable matrices

A natural question is that this fuzzy system is stable?

K. Tanaka and M. Sugeno,
Stability Analysis and Design of Fuzzy Control Systems,
Fuzzy Sets and Systems, Vol.45, pp.135 - 156 (1992).

Local stability does not imply global stability

Response for $\mathbf{x}(0) = \begin{bmatrix} 0.90 & -0.70 \end{bmatrix}^T$



This fuzzy system is not stable

K. Tanaka and M. Sugeno,
Stability Analysis and Design of Fuzzy Control Systems,
Fuzzy Sets and Systems, Vol.45, pp.135 - 156 (1992).

Lyapunov Stability Theory

Aleksandr Mikhailovich Lyapunov
1857 - 1918

$\dot{x} = f(x), \ x \in R^n$ $\quad f$ is a nonlinear function

Check the stability of dynamic system without solving the given differential equations

Globally Asymptotically Stable

(i) $V(0) = 0$ **and** $V(x) > 0, \quad \forall x \neq 0$

(ii) $||x|| \to \infty \Rightarrow V(x) \to \infty$

(iii) $\dot{V}(x) < 0, \quad \forall x \neq 0$

H. K. Khalil, Nonlinear Systems, 2nd ed., Englewood Cliffs, NJ: Prentice Hall, 1996.

What are LMIs?

A linear matrix inequality (LMI) is any constraint of the form

$$\boldsymbol{A}(\boldsymbol{x}) := A_0 + x_1 A_1 + \ldots + x_N A_N \prec \boldsymbol{0}$$

where

- $x = (x_1, \ldots, x_N)$ is a vector of unknown scalar (the decision or optimization variable)
- $A_0, \ldots, A_N$ are given matrices
- " $\succ \mathbf{0}$" stands for "positive definite"
- " $\prec \mathbf{0}$" stands for "negative definite"

$$\boldsymbol{A}(\boldsymbol{x}) \succ \boldsymbol{0} \Rightarrow -\boldsymbol{A}(\boldsymbol{x}) \prec \boldsymbol{0}$$

$$\boldsymbol{A}(\boldsymbol{x}) \prec \boldsymbol{B}(\boldsymbol{x}) \Rightarrow \boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x}) \prec \boldsymbol{0}$$

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

What are LMIs?

- If a matrix inequality is an LMI, then every term of the matrix inequality has only one LMI variable (the decision variable which should be obtained by an LMI solver).
- For example, $\boldsymbol{A}$ and $\boldsymbol{B}$ are known matrices, and $\boldsymbol{X}$ and $\boldsymbol{M}$ are LMI variables

$$\underline{\boldsymbol{X}}\boldsymbol{A}^T + \boldsymbol{A}\underline{\boldsymbol{X}} - \underline{\boldsymbol{M}}^T\boldsymbol{B}^T - \boldsymbol{B}\underline{\boldsymbol{M}} \prec 0 \quad \Rightarrow \quad \text{(LMI)}$$

$$\begin{bmatrix} \underline{\boldsymbol{X}} & \underline{\boldsymbol{X}}\boldsymbol{A}^T - \underline{\boldsymbol{M}}^T\boldsymbol{B}^T \\ \boldsymbol{A}\underline{\boldsymbol{X}} - \boldsymbol{B}\underline{\boldsymbol{M}} & \underline{\boldsymbol{X}} \end{bmatrix} \succ 0 \quad \Rightarrow \quad \text{(LMI)}$$

However, the following matrix inequality is NOT an LMI (with respect to $\boldsymbol{P}$ and $\boldsymbol{F}$).

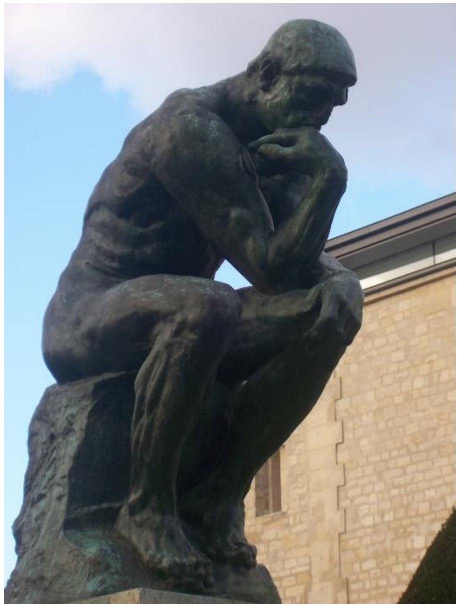
$$\boldsymbol{A}^T\underline{\boldsymbol{P}} + \underline{\boldsymbol{P}}\boldsymbol{A} - \boxed{\underline{\boldsymbol{P}}\boldsymbol{B}\underline{\boldsymbol{F}} - \underline{\boldsymbol{F}}^T\boldsymbol{B}^T\underline{\boldsymbol{P}}} \prec 0 \quad \Rightarrow \quad \text{(BMI)}$$

Not LMI

BMI: Bilinear Matrix Inequality

Three Important Theorems

- Schur Complement
- *S*-procedure
- Finsler's Lemma



Three Important Theorems

Schur Complement

The following conditions are equivalent.

(1) $$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{12}^T & \boldsymbol{\Theta}_{22} \end{bmatrix} \succ \boldsymbol{0}$$

(2) $$\boldsymbol{\Theta}_{11} \succ \boldsymbol{0} \;\; and \;\; \boldsymbol{\Theta}_{22} - \boldsymbol{\Theta}_{12}^T \boldsymbol{\Theta}_{11}^{-1} \boldsymbol{\Theta}_{12} \succ \boldsymbol{0}$$

(3) $$\boldsymbol{\Theta}_{22} \succ \boldsymbol{0} \;\; and \;\; \boldsymbol{\Theta}_{11} - \boldsymbol{\Theta}_{12} \boldsymbol{\Theta}_{22}^{-1} \boldsymbol{\Theta}_{12}^T \succ \boldsymbol{0}$$

Three Important Theorems

S-procedure

$$\zeta^T T_0 \zeta > 0 \text{ for all } \zeta \neq 0 \text{ such that } \zeta^T T_i \zeta \geq 0, \quad i = 1, \dots, p. \quad (1)$$

It is obvious that if

there exists $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that $T_0 - \sum_{i=1}^{p} \tau_i T_i \succ 0$

then (1) holds.

Three Important Theorems

Finsler's Lemma

Lemma [Finsler]: Let $x \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that $\mathrm{rank}\,(\mathcal{B}) < n$. The following statements are equivalent:

i) $x^T \mathcal{Q} x < 0, \quad \text{for all } \mathcal{B}x = 0, \quad x \neq 0.$

ii) $\exists\, \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$

iii) $\exists\, \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

UEC TOKYO

Basic Stabilization Condition

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(t)) h_j(\boldsymbol{z}(t)) \{\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{F}_j\} \boldsymbol{x}(t)$$

Design the local feedback gains such that the closed-loop system is globally asymptotically stable

Takagi-Sugeno Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\mathbf{z}(t)) \{\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)\}$$

Analysis & Design

Linear Matrix Inequality (LMI)

Fuzzy Controller

$$\mathbf{u}(t) = -\sum_{i=1}^{r} h_i(\mathbf{z}(t)) \mathbf{F}_i \mathbf{x}(t)$$

[LMI]
H. O. Wang, K. Tanaka and M. F. Griffin,
An Approach to Fuzzy Control of Nonlinear Systems,
IEEE Transactions on Fuzzy Systems, Vol.4, No.1, pp.14-23 (1996).

Basic Stabilization Condition

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\{\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j\}\boldsymbol{x}(t)$$

Quadratic Lyapunov Function $V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$

$$\boldsymbol{P} = \boldsymbol{X}^{-1} \succ \boldsymbol{0}$$

$$\dot{V}(\boldsymbol{x}(t)) = \dot{\boldsymbol{x}}^T(t)\boldsymbol{X}^{-1}\boldsymbol{x}(t) + \boldsymbol{x}^T(t)\boldsymbol{X}^{-1}\dot{\boldsymbol{x}}(t)$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\boldsymbol{x}^T(t)\boldsymbol{G}_{ij}\boldsymbol{x}(t)$$

BMI

$$\boldsymbol{G}_{ij} = (\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j)^T\boldsymbol{X}^{-1} + \boldsymbol{X}^{-1}(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j)$$

Non-convex $\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\boldsymbol{G}_{ij} \prec \boldsymbol{0}$

Basic Stabilization Condition

Multiplying the inequality on the left and right by $\boldsymbol{X}$

Convex $$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\boldsymbol{Q}_{ij} \prec \boldsymbol{0}$$

$$\boldsymbol{Q}_{ij} = \boldsymbol{X}\boldsymbol{G}_{ij}\boldsymbol{X}$$

$$\boldsymbol{Q}_{ij} = \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{B}_i\boldsymbol{M}_i - \boldsymbol{M}_i^T\boldsymbol{B}_i^T$$

$$\boldsymbol{M}_i = \boldsymbol{F}_i\boldsymbol{X}$$

LMI Stabilization Condition

$$\boldsymbol{Q}_{ii} \prec 0$$

$$\boldsymbol{Q}_{ij} + \boldsymbol{Q}_{ji} \preceq 0$$

$$\boldsymbol{F}_i = \boldsymbol{M}_i\boldsymbol{X}^{-1}$$

Basic Stabilization Condition

Stable Controller Design Example

Kinematic Model of Powered Paraglider

$$\dot{y}(t) = V \sin\phi(t)$$

$$\dot{\phi}(t) = ku(t) \qquad u(t) \text{ :Control bar angle}$$

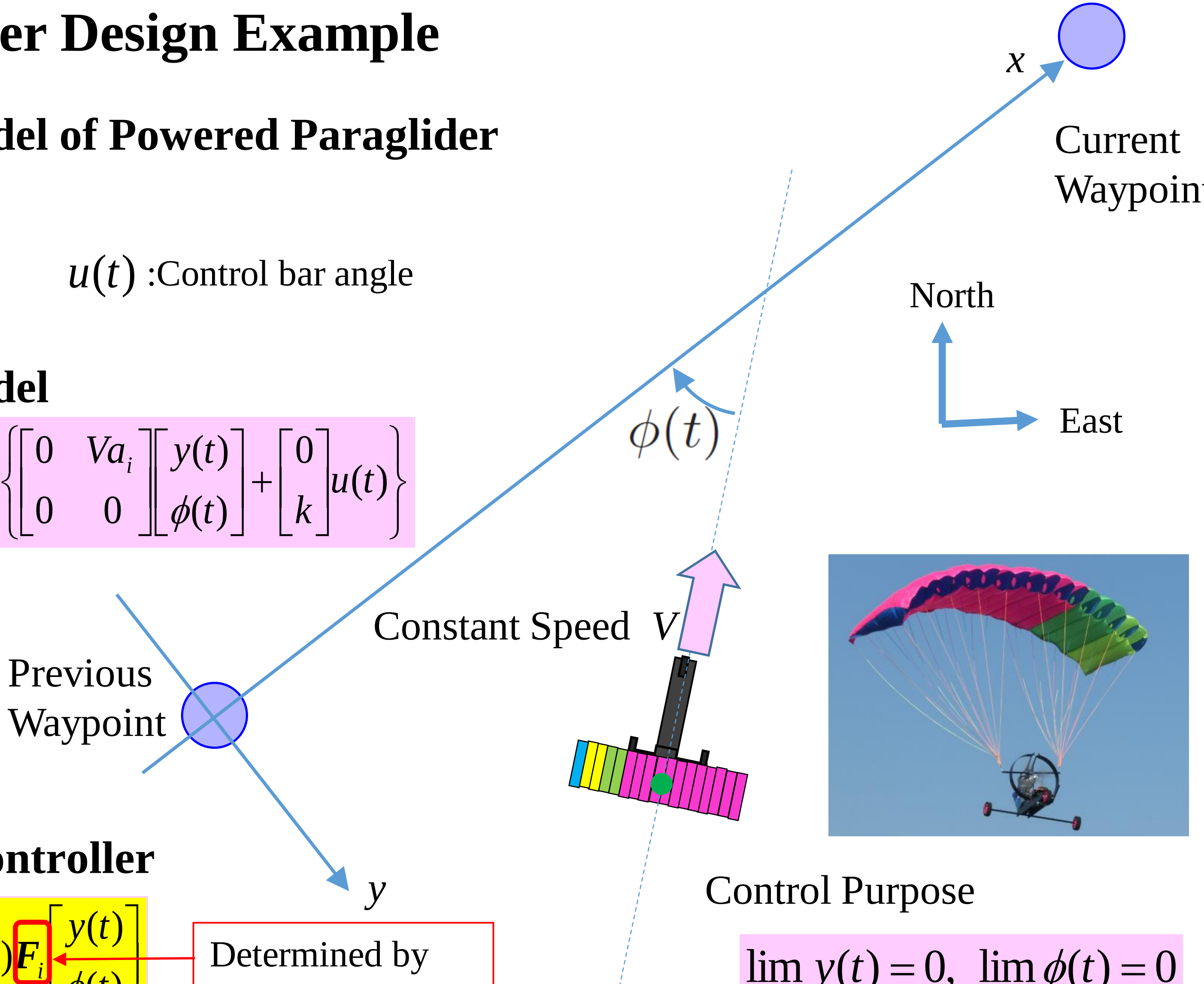
T-S Fuzzy Model

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\phi}(t) \end{bmatrix} = \sum_{i=1}^{2} h_i(\phi(t)) \left\{ \begin{bmatrix} 0 & Va_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \phi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} u(t) \right\}$$

T-S Fuzzy Controller

$$u(t) = -\sum_{i=1}^{2} h_i(\phi(t)) F_i \begin{bmatrix} y(t) \\ \phi(t) \end{bmatrix}$$

F_i: Determined by solving the LMIs



Control Purpose

$$\lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} \phi(t) = 0$$

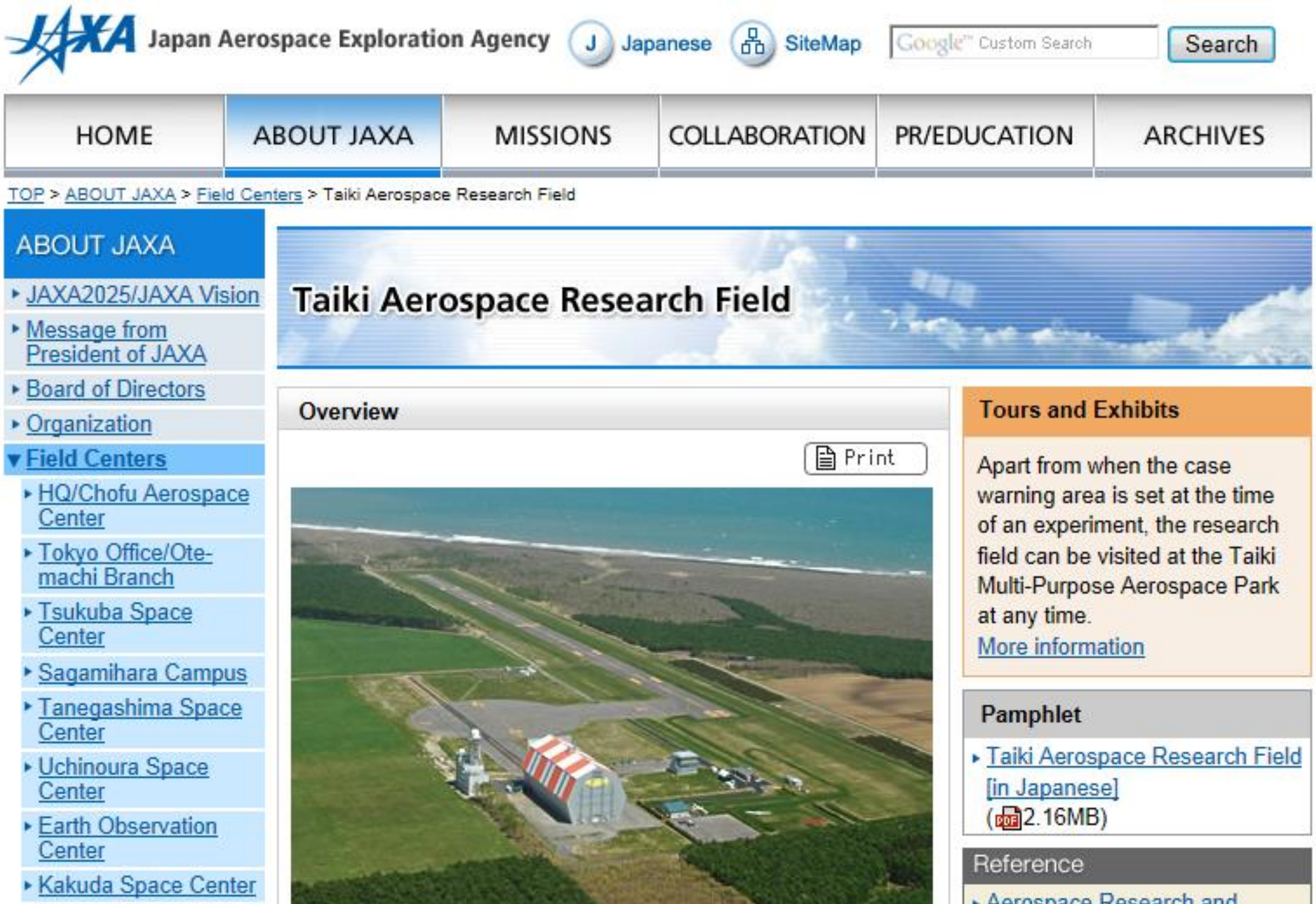
Powered Paraglider (PPG)

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Basic Stabilization Condition

Flight Experiments

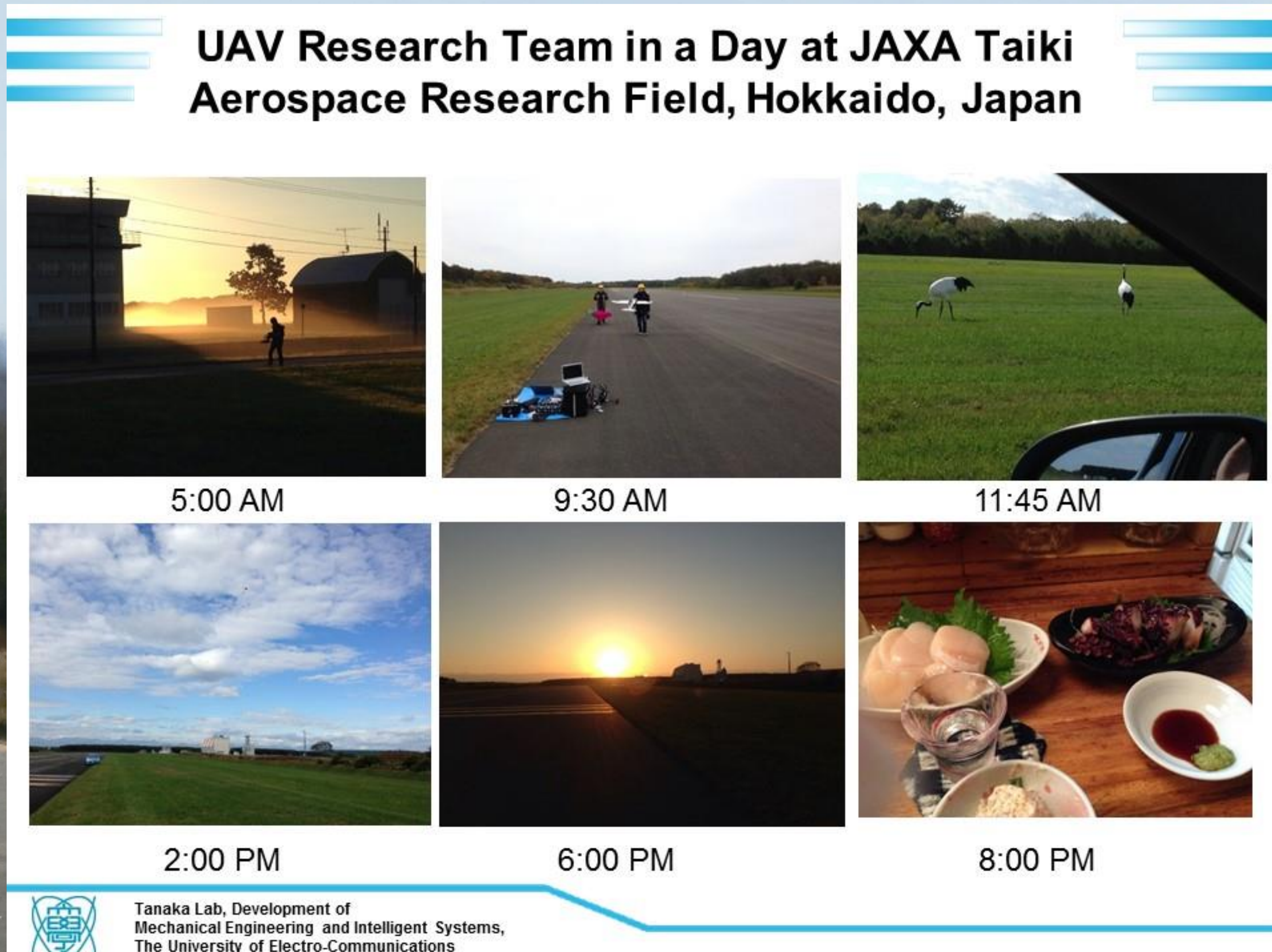
JAXA(Japan Aerospace Exploration Agency)



JAXA Taiki Aerospace Research Field

JAXA Taiki Aerospace Research Field

JAXA Taiki Aerospace Research Field

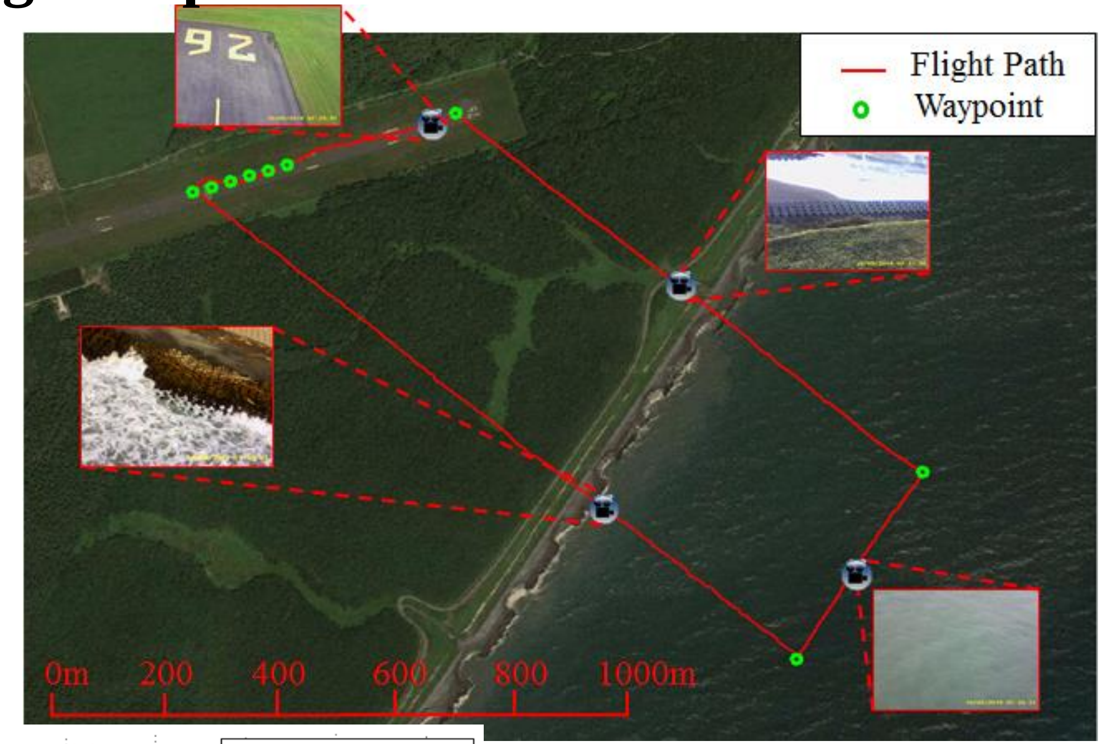


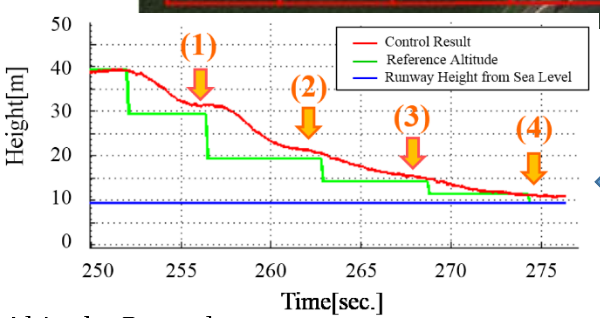
JAXA Taiki Aerospace Research Field

PPG Flight via
Fuzzy Model-based Control

Basic Stabilization Condition

Flight Experiments





Altitude Control

M Tanaka, H Kawai, K Tanaka, HO Wang, Development of an autonomous flying robot and its verification via flight control experiment, IEEE Int. Conf. on Robotics and Automation (ICRA), pp.4439-4444 2013

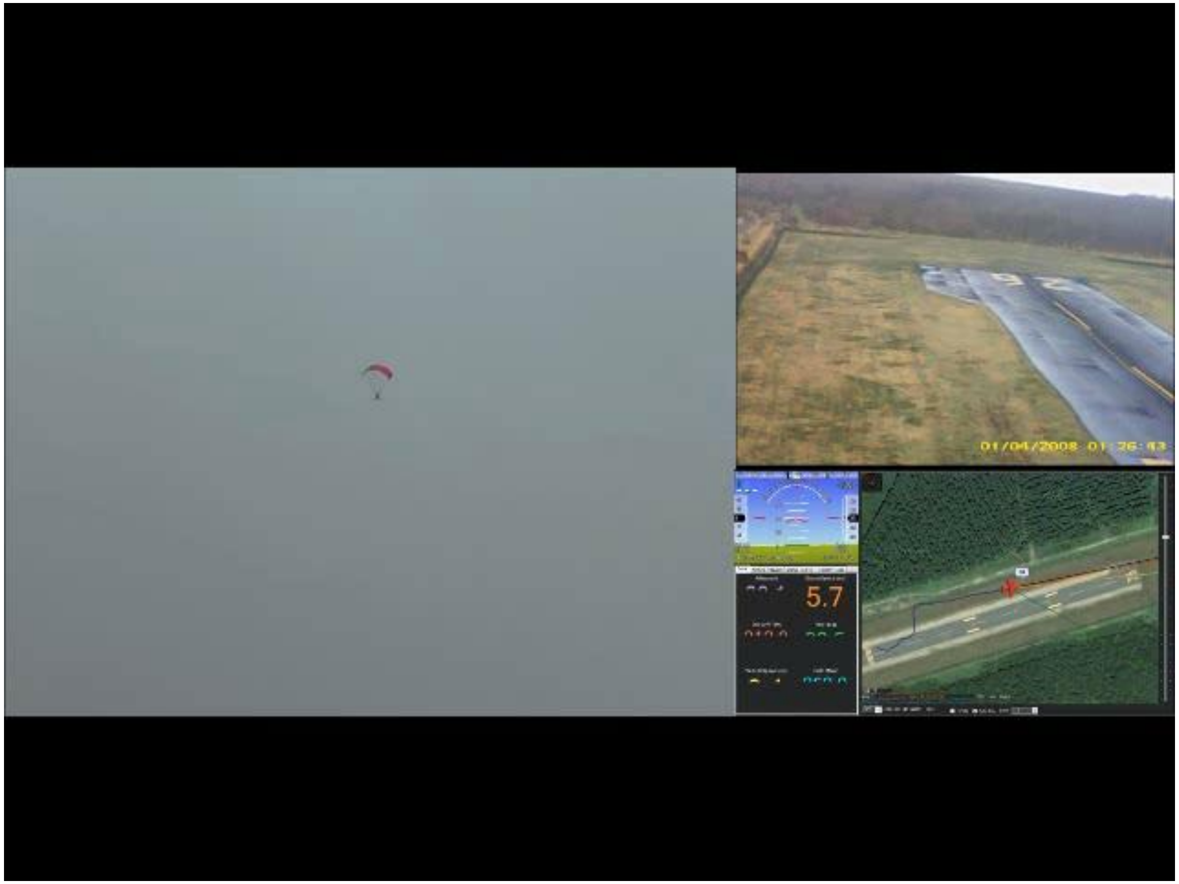
Stable control including automatic landing is realized via the LMI-based fuzzy controller

Automatic Landing

Basic Stabilization Condition

Flight Experiments (Video)

Stable waypoint following control and altitude control + automatic landing!



https://www.youtube.com/watch?v=t5agDDQ7lQM

Tutorial Overview

- **Part III**

  Theoretical Advances in T-S Fuzzy Model-based Control using LMIs

  - ❖**Double Summation Relaxation**
  - ❖**Generalized Lyapunov Function Approaches**
  - ❖**Other Relaxations**

Double Fuzzy Summation Relaxation

Basic Stabilization Condition

Multiplying the inequality on the left and right by X

Convex $$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))Q_{ij} \prec 0 \qquad Q_{ij} = XG_{ij}X$$

$$Q_{ij} = A_iX + XA_i^T - B_iM_i - M_i^TB_i^T \qquad M_i = F_iX$$

LMI Stabilization Condition

$$Q_{ii} \prec 0$$

$$Q_{ij} + Q_{ji} \preceq 0$$

$$F_i = M_iX^{-1}$$

Double Fuzzy Summation

Double is a great value?

A number of studies on double fuzzy summation relaxation have been reported in stability and stabilization of fuzzy control systems. Those will be introduced in the tutorial.

Generalized Lyapunov Function

Simple Quadratic Lyapunov Function

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

LF:Lyapunov Function

Fuzzy LF , Multiple LF, Weighting dependent LF

$$V(\boldsymbol{x}(t)) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t)$$

Piecewise LF, Switched LF

$$V(\mathbf{x}(t)) = \max_{1 \le i \le N} \{V_i(\mathbf{x}(t))\}, \qquad V_i(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Polynomial LF

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{x}(t))\boldsymbol{x}(t)$$

Generalized Lyapunov Function

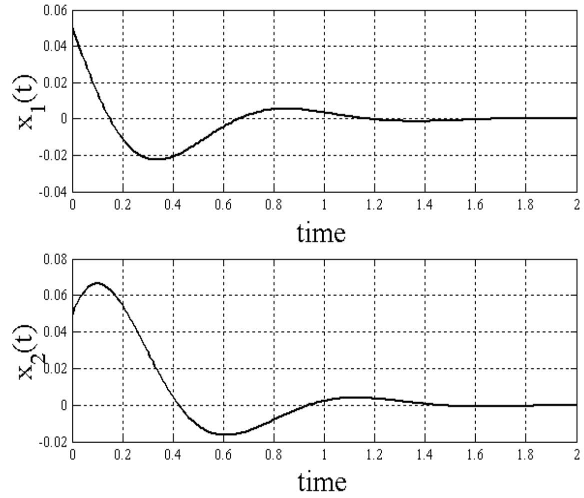
Fuzzy Lyapunov Function

Fuzzy System

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} h_i(\mathbf{z}(t))\mathbf{A}_i\mathbf{x}(t)$$

$$\mathbf{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

$$h_1(\mathbf{z}(t)) = \frac{1+\sin x_1(t)}{2} \quad h_2(\mathbf{z}(t)) = \frac{1-\sin x_1(t)}{2}$$



Time responses.

Quadratic Lyapunov Function

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Quadratic Lyapunov functions satisfying the LMI stabilization condition do not exist

K. Tanaka, T. Hori and H. O. Wang, " A Multiple Lyapunov Function Approach to Stabilization of Fuzzy Control Systems", IEEE Transactions on Fuzzy Systems, Vol.11, No.4, pp.582-589, August 2003.

Generalized Lyapunov Function

Fuzzy Lyapunov Function

Fuzzy System

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} h_i(\mathbf{z}(t))\mathbf{A}_i\mathbf{x}(t) \quad \mathbf{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

$$h_1(\mathbf{z}(t)) = \frac{1+\sin x_1(t)}{2} \quad h_2(\mathbf{z}(t)) = \frac{1-\sin x_1(t)}{2}$$

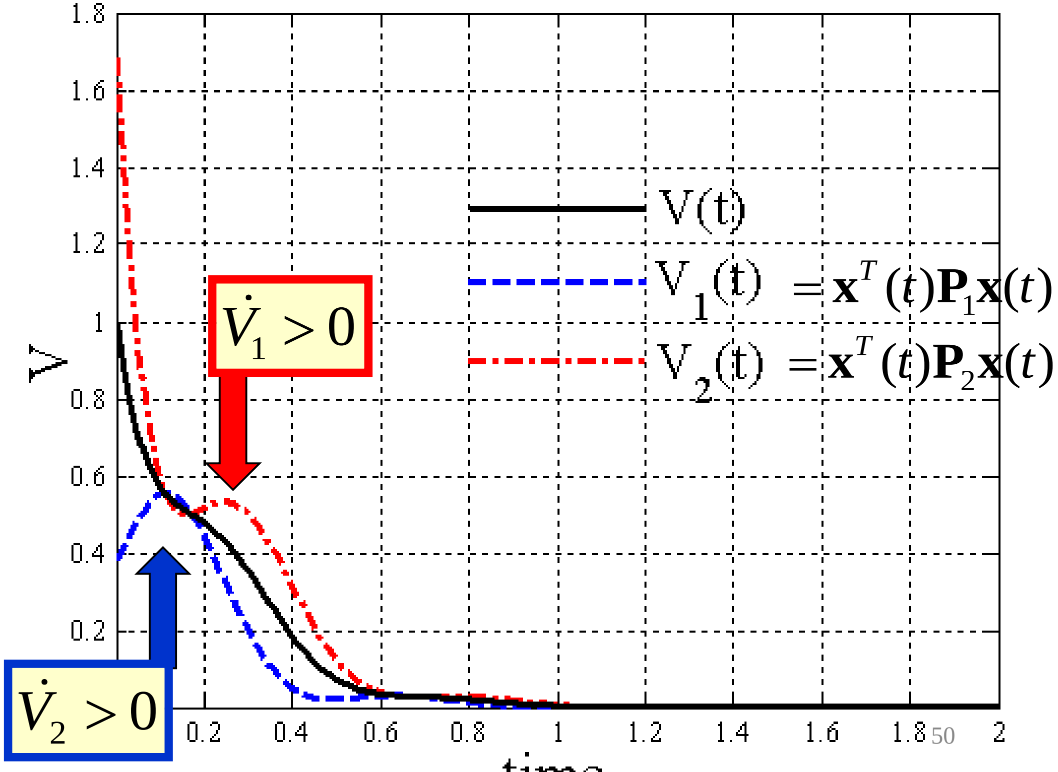
Fuzzy Lyapunov Function

$$\mathrm{V}(\boldsymbol{x}(t)) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(t))V_i(\boldsymbol{x}(t))$$

$$= \sum_{i=1}^{2} h_i(\boldsymbol{z}(t))\boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t)$$

$$\boldsymbol{P}_1 = \begin{bmatrix} 86.95 & -29.80 \\ -29.80 & 125.61 \end{bmatrix} > \boldsymbol{0},$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 647.98 & -46.83 \\ -46.83 & 118.24 \end{bmatrix} > \boldsymbol{0}$$



Generalized Lyapunov Function

Simple Quadratic Lyapunov Function

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

LF:Lyapunov Function

Fuzzy LF , Multiple LF, Weighting dependent LF

$$V(\boldsymbol{x}(t)) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t)$$

Piecewise LF, Switched LF

$$V(\mathbf{x}(t)) = \max_{1\le i\le N}\{V_i(\mathbf{x}(t))\}, \qquad V_i(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Polynomial LF

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{x}(t))\boldsymbol{x}(t)$$

Generalized Lyapunov Function

Piecewise Lyapunov Function

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - x_2 - g\, x_1$$

$$g \in [0, k]$$

Compare the maximum *k* guaranteeing stability conditions

Fuzzy model

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(z(t))\{\boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{B}_i u(t)\}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}, \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$h_1(z(t)) = \frac{k - g(t)}{k}, \ \ h_2(z(t)) = \frac{g(t)}{k} \qquad z(t) = g(t)$$

Generalized Lyapunov Function

Piecewise Lyapunov Function

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - x_2 - g\,x_1$$

$$g \in [0, k]$$

Compare the maximum k guaranteeing stability conditions

Quadratic Lyapunov Function $k_{\max} = 3.82$

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Piecewise Lyapunov Function [1] $k_{\max} = 4.7$

$$V(\mathbf{x}(t)) = \max\{\mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t), \mathbf{x}^T(t)\mathbf{P}_2\mathbf{x}(t)\}$$

[1]L. Xie, S. Shishkin and M. Fu, "Piecewise Lyapunov Functions for Robust Stability of Linear Time-Varying Systems", Systems & Control Letters 31 pp.165-171, 1997.

Generalized Lyapunov Function

Simple Quadratic Lyapunov Function

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

LF:Lyapunov Function

Fuzzy LF , Multiple LF, Weighting dependent LF

$$V(\boldsymbol{x}(t)) = \sum_{i=1}^{r} h_i(\mathbf{z}(t))\boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t)$$

Piecewise LF, Switched LF

$$V(\mathbf{x}(t)) = \max_{1\le i\le N}\{V_i(\mathbf{x}(t))\}, \qquad V_i(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Polynomial LF

$$V(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{x}(t))\boldsymbol{x}(t)$$

Generalized Lyapunov Function

Polynomial Lyapunov Function

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - x_2 - g\,x_1$$

$$g \in [0, k]$$

Compare the maximum *k* guaranteeing stability conditions

Quadratic Lyapunov Function $k_{\max} = 3.82$

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Piecewise Lyapunov Function [1] $k_{\max} = 4.7$

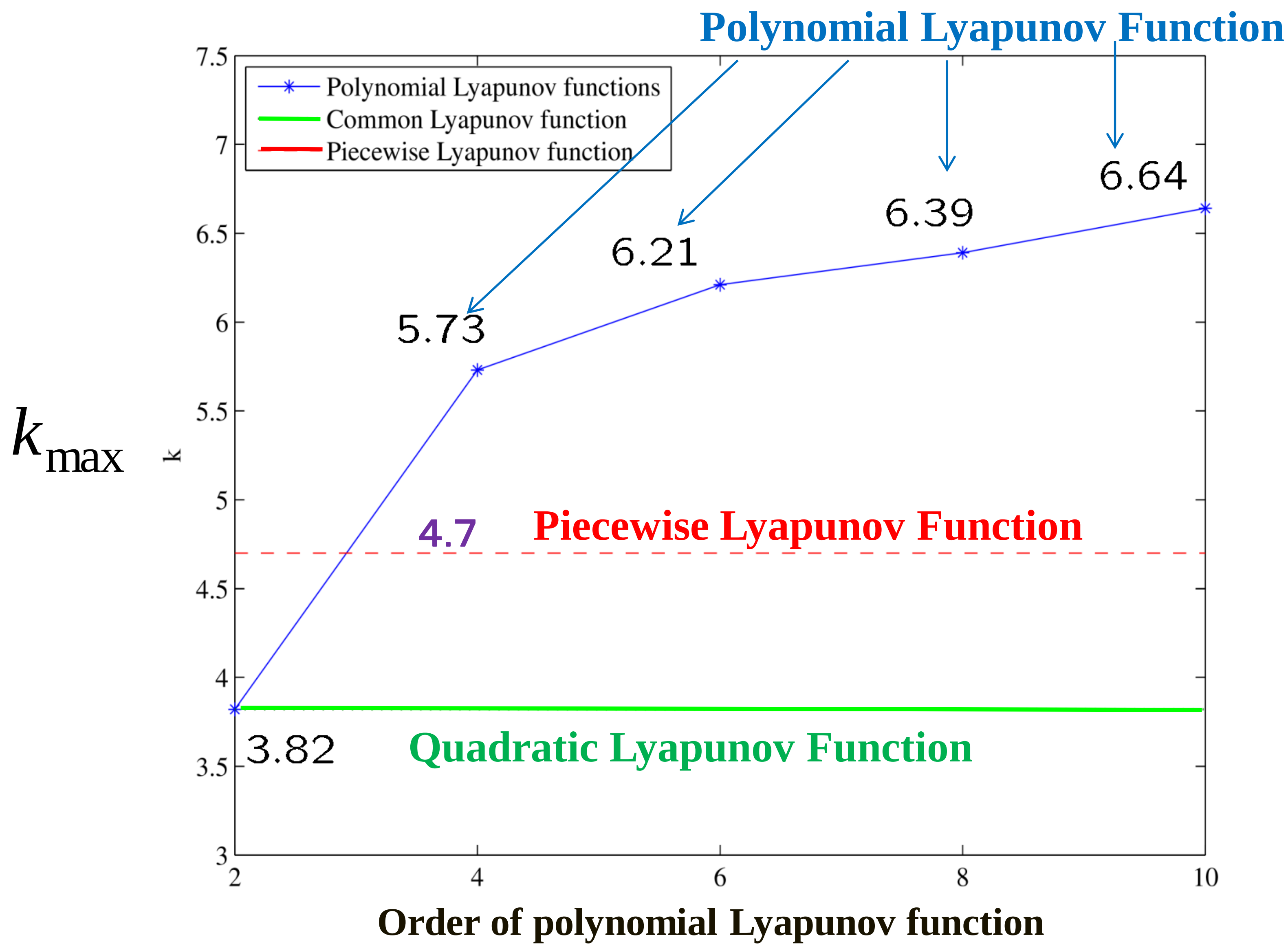
$$V(\mathbf{x}(t)) = \max\{\mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t), \mathbf{x}^T(t)\mathbf{P}_2\mathbf{x}(t)\}$$

[1]L. Xie, S. Shishkin and M. Fu, “Piecewise Lyapunov Functions for Robust Stability of Linear Time-Varying Systems”, Systems & Control Letters 31 pp.165-171, 1997.

Polynomial Lyapunov Function [2] $k_{\max} = ??$

[2] K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang, ``A Sum of Squares Approach to Modeling and Control of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems", IEEE Transactions on Fuzzy Systems, Vol.17, No.4, pp.911-922, August 2009.

Generalized Lyapunov Function



Generalized Lyapunov Function

Polynomial Lyapunov Function

Second order polynomial (quartaic) Lyapunov function

$$V(\mathbf{x}) = 27.4x_1^2 + 6.97x_1x_2 + 7.02x_2^2$$

Fourth order polynomial Lyapunov function

$$\begin{aligned} V(\mathbf{x}) &= 271.0x_1^4 + 83.5x_1^3x_2 + 157.0x_1^2x_2^2 \\ &\quad +38.7x_1x_2^3 + 12.6x_2^4 \end{aligned}$$

Sixth order polynomial Lyapunov function

$$\begin{aligned} V(\mathbf{x}) &= 2330.0x_1^6 + 713.0x_1^5x_2 + 1920.0x_1^4x_2^2 \\ &\quad +889.0x_1^3x_2^3 + 553.0x_1^2x_2^4 + 108.0x_1x_2^5 \\ &\quad +23.1x_2^6 \end{aligned}$$

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Generalized Lyapunov Function

Polynomial Lyapunov Function

Eighth order polynomial Lyapunov function

$$\begin{aligned} V(\mathbf{x}) &= 3990.0x_1^8 + 1580.0x_1^7x_2 + 4680.0x_1^6x_2^2 \\ &+2560.0x_1^5x_2^3 + 1850.0x_1^4x_2^4 + 675.0x_1^3x_2^5 \\ &+284.0x_1^2x_2^6 + 49.2x_1x_2^7 + 7.44x_2^8 \end{aligned}$$

Tenth order polynomial Lyapunov function

$$\begin{aligned} V(\mathbf{x}) &= 28100.0x_1^{10} + 10200.0x_1^9x_2 \\ &+40100.0x_1^8x_2^2 + 29100.0x_1^7x_2^3 \\ &+24900.0x_1^6x_2^4 + 10400.0x_1^5x_2^5 \\ &+5630.0x_1^4x_2^6 + 1990.0x_1^3x_2^7 \\ &+609.0x_1^2x_2^8 + 89.0x_1x_2^9 + 10.4x_2^{10} \end{aligned}$$

Generalized Lyapunov Function

More relaxed stability results by other generalized Lyapunov functions will be presented in the tutorial.

Generalized Lyapunov Function

Polynomial LF is most powerful!

SOS Approach ☆ Beyond LMIs

From LMI

To SOS

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Other Relaxations

Other relaxations will be introduced in the tutorial.

Tutorial Overview

- **Part IV**

  Beyond LMIs: Polynomial Fuzzy Systems Control and Analysis using Sum-of- Squares (SOS)

  - **What is Sum of Squares (SOS)**
  - **What is polynomial fuzzy systems (PFS) control**
  - **T-S fuzzy model VS Polynomial fuzzy model**
  - **SOS-based Design**
  - **Design Example**
  - **Micro Helicopter Control Example**
  - **Recent Topics**

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

What is SOS?

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

What is SOS?

A multivariate polynomial $p(x_1, ..., x_n)$ is a sum of squares (SOS) if there exist polynomials $f_1(\boldsymbol{x}), ..., f_m(\boldsymbol{x})$ such that

$$p(\boldsymbol{x}) = \sum_{i=1}^{m} f_i^2(\boldsymbol{x})$$

e.g. $p(\boldsymbol{x}) = (x_1 + x_2)^2 + (x_1 x_2 - x_2)^2$

Clearly, $p(\boldsymbol{x})$ is an SOS $\Rightarrow$ $p(\boldsymbol{x}) \geq 0$

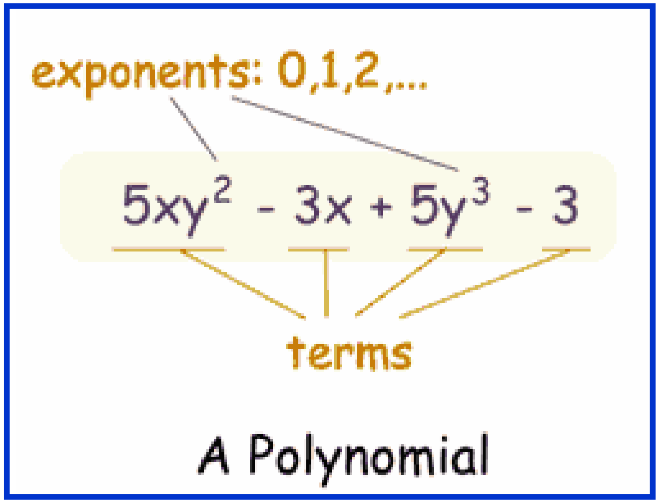
$p(\boldsymbol{x}) - \epsilon(\boldsymbol{x})$ is an SOS, where $\epsilon(\boldsymbol{x}) > 0 \ at \ \boldsymbol{x} \neq 0$

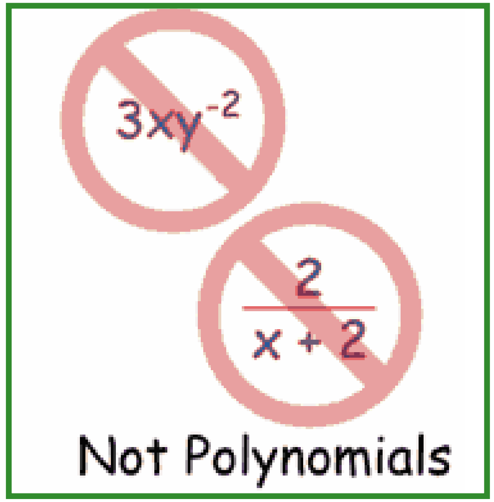
$\Rightarrow p(\boldsymbol{x}) > 0 \ at \ \boldsymbol{x} \neq 0$

Positive definite polynomial

What is SOS?

SOS ⊂ Polynomial ⊂ Polynomial-related

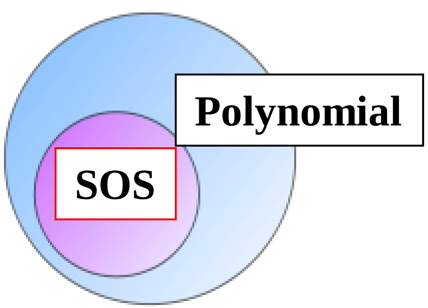




New Definition

Fraction $\frac{x^3 + x^2 + x + 1}{x^2 + x + 1}$

- Polynomial denominator (PD)
- Polynomial numerator (PN)
- But not polynomial in general
- Polynomial only if the PD is zero order



Keep in mind that inverse of polynomials are NOT polynomials.

Polynomial-related

What is SOS?

LMI ⊂ Convex SOS ⊂ SOS

Convex SOS with zero order = LMI

Convex SOS

Find $\boldsymbol{P}(\boldsymbol{x})$ such that $\boldsymbol{x}^T \boldsymbol{P}(\boldsymbol{x}) \boldsymbol{x}$ is SOS

$\boldsymbol{P}(\boldsymbol{x})$:polynomial matrix in $\boldsymbol{x}$

For example,

$$\boldsymbol{P}(\boldsymbol{x}) = \begin{bmatrix} a_0 + a_1 x_1 + a_2 x_2 & b_0 + b_1 x_1 + b_2 x_2 \\ b_0 + b_1 x_1 + b_2 x_2 & c_0 + c_1 x_1 + c_2 x_2 \end{bmatrix}$$

Reduction of order

$$\boldsymbol{P}(\boldsymbol{x}) = \begin{bmatrix} a_0 & b_0 \\ b_0 & c_0 \end{bmatrix} \Rightarrow \boldsymbol{P}$$

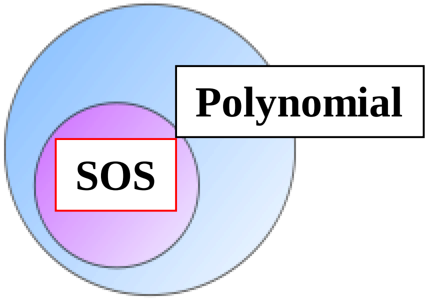
Find $\boldsymbol{P}$ such that $\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} > 0$

What is SOS?

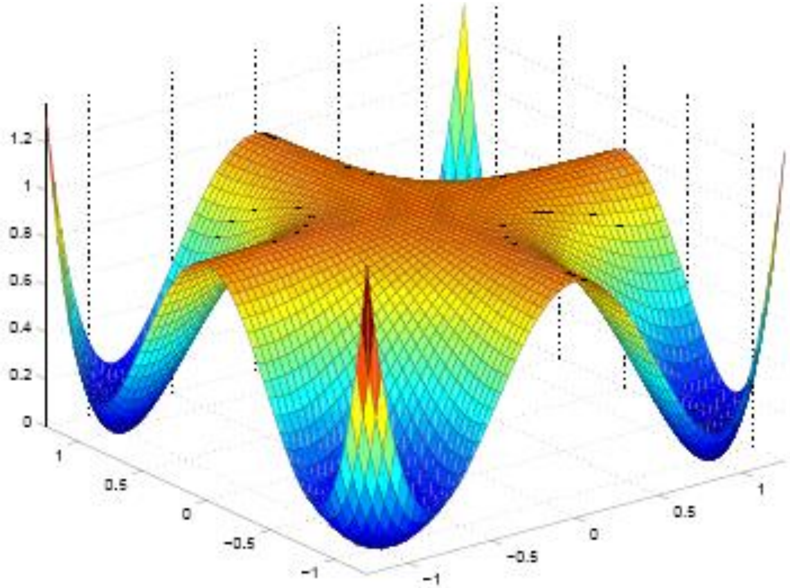
Properties of SOS

A polynomial $M(x)$ is an SOS ⇨ $M(\boldsymbol{x}) \geq 0$

How about the converse? **NO!**



$$M(x_1, x_2) = x_1^2 x_2^4 + x_2^2 x_1^4 - 3x_1^2 x_2^2 + 1$$



$$M(x_1, x_2) \geq 0$$

However, $M(x_1, x_2)$ is not an SOS

UEC TOKYO

What is SOS?

Properties of SOS

$\boldsymbol{x}^T(t)\boldsymbol{F}(\boldsymbol{x}(t))\boldsymbol{x}(t)$ is an SOS, where $\boldsymbol{x}(t) \in R^N$

➡ Is it satisfied that $F(\boldsymbol{x}(t)) \succeq 0$ for all $\boldsymbol{x}(t)$?

NO!

$$[x_1 \quad x_2]\begin{bmatrix} 1 & x_1x_2 \\ x_1x_2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1^2x_2^2 + x_2^2 \text{ is an SOS}$$

However $\begin{bmatrix} 1 & x_1x_2 \\ x_1x_2 & 1 \end{bmatrix}$ is not PSD for all $\boldsymbol{x}$

Hence, to show $F(\boldsymbol{x}(t)) \succeq 0$ for all $\boldsymbol{x}(t)$, we need

$\boldsymbol{v}^T(t)F(\boldsymbol{x}(t))\boldsymbol{v}(t)$ is an SOS, where $\boldsymbol{v}(t) \in R^N$

independent of x(t)

➡ It is clear that $F(\boldsymbol{x}(t)) \succeq 0$ for all $\boldsymbol{x}(t)$

What is PFS Control?

Polynomial +
Fuzzy approach
is most powerful.

What is PFS Control?

Polynomial + fuzzy approach is most powerful.

Nonlinear System

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$

Polynomial-related Lyapunov Function

$$V(\boldsymbol{x}(t)) = \hat{\mathbf{x}}^T(t)\boldsymbol{P}(\tilde{\mathbf{x}}(t))\hat{\mathbf{x}}(t)$$

Polynomial Fuzzy Model

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\mathbf{x}}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}$$

$$\hat{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}$$

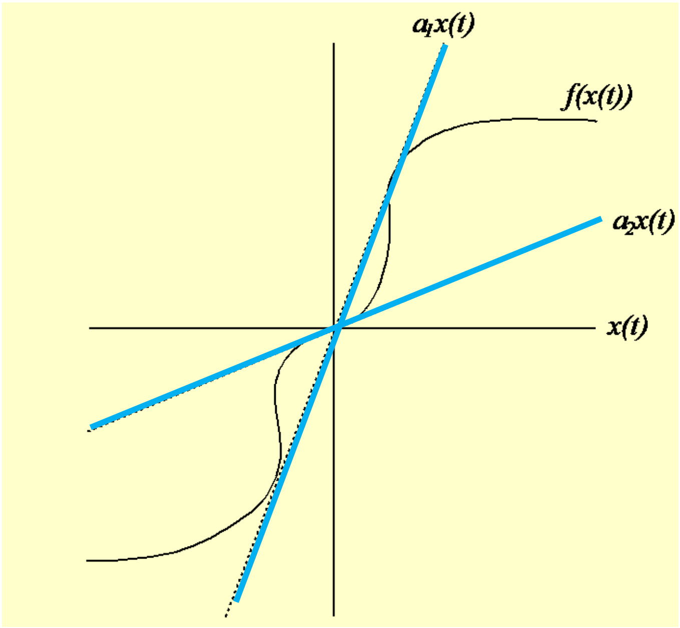
Polynomial-related Fuzzy Controller

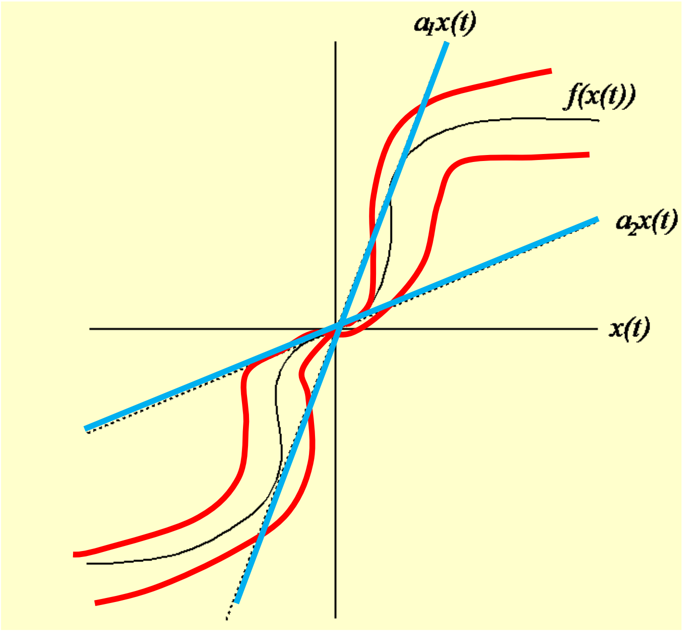
$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\mathbf{x}}(t)$$

K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang,
A Sum of Squares Approach to Modeling and Control
of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems,
IEEE Transactions on Fuzzy Systems, Vol.17, No.4, pp.911-922, August 2009.

T-S Fuzzy Model vs Polynomial Fuzzy Model

Takagi-Sugeno Fuzzy Model → **Polynomial Fuzzy Model**





Linear Sector → **Nonlinear Sector**

Linear Matrix Inequality (LMI) → **Sum of Squares (SOS)**

A. Sala and C. Arino
Polynomial Fuzzy Models for Nonlinear Control:A Taylor Series Approach
IEEE TRANSACTIONS ON FUZZY SYSTEMS, VOL. 17, NO. 6, pp.1284-1295, 2009

UEC TOKYO

SOS-based Design

$$\dot{\boldsymbol{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))-\boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(t)$$

Design the local feedback gains such that the closed-loop system is (globally asymptotically) stable

Polynomial Fuzzy Model

$$\dot{\boldsymbol{x}}(t)=\sum_{i=1}^{r}h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(t)+\boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}$$

Sum of Squares (SOS)

Polynomial-related Fuzzy Controller

$$\boldsymbol{u}(t)=-\sum_{i=1}^{r}h_i(\boldsymbol{z}(t))\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(t)$$

K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang,
A Sum of Squares Approach to Modeling and Control of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems,
IEEE Transactions on Fuzzy Systems, Vol.17, No.4, pp.911-922, August 2009.

SOS-based Design

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(t)$$

Polynomial-related Lyapunov Function

$$V(\boldsymbol{x}(t)) = \hat{\boldsymbol{x}}^T(t)\boldsymbol{P}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(t)$$

$$\boldsymbol{P}(\tilde{\boldsymbol{x}}(t)) = \boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t)) \succ \boldsymbol{0}$$

$$\dot{V}(\boldsymbol{x}(t)) = \dot{\hat{\boldsymbol{x}}}^T(t)\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{x}}^T(t)\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t))\dot{\hat{\boldsymbol{x}}}(t)$$

$$+\hat{\boldsymbol{x}}^T(t)\left(\sum_{k=1}^{n}\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k(t)}(\tilde{\boldsymbol{x}}(t))\dot{x}_k(t)\right)\hat{\boldsymbol{x}}(t)$$

$$\hat{\boldsymbol{x}}^T(t)\dot{\boldsymbol{P}}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(t)$$

$$\dot{x}_k(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{A}_i^k(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(t)$$

$$\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t))\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k(t)}(\tilde{\boldsymbol{x}}(t))\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t)) = -\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k(t)}(\tilde{\boldsymbol{x}}(t))$$

$$\boldsymbol{T}(\boldsymbol{x}(t)) = \left[T^{ij}(\boldsymbol{x}(t))\right], \qquad T^{ij}(\boldsymbol{x}) = \frac{\partial \hat{x}_i}{\partial x_j}(\boldsymbol{x})$$

SOS-based Design

$$\dot{V}(\boldsymbol{x}(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\hat{\boldsymbol{x}}^T(t)\boldsymbol{S}_{ij}(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(t)$$

$$\begin{aligned}\boldsymbol{S}_{ij}(\boldsymbol{x}(t)) = {} & (\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t)))^T \boldsymbol{T}^T(\boldsymbol{x}(t))\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t)) \\ & + \boldsymbol{T}(\boldsymbol{x}(t))\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t))(\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))) \\ & + \sum_{k \in \boldsymbol{K}} \frac{\partial \boldsymbol{X}^{-1}}{\partial x_k(t)}(\tilde{\boldsymbol{x}}(t))\boldsymbol{A}_i^k(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(t)\end{aligned}$$

Non-convex

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t))\boldsymbol{S}_{ij}(\boldsymbol{x}(t)) \prec \boldsymbol{0}$$

SOS-based Design

Multiplying the inequality on the left and right by $X(\tilde{x}(t))$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))L_{ij}(x(t)) \prec 0$$

Convex

$$L_{ij}(x(t)) = X(\tilde{x}(t))S_{ij}(x(t))X(\tilde{x}(t))$$

$$L_{ii}(x(t)) \prec 0$$

$$L_{ij}(x(t)) + L_{ji}(x(t)) \preceq 0$$

or

$$-L_{ii}(x(t)) \succ 0$$

$$-\left(L_{ij}(x(t)) + L_{ji}(x(t))\right) \succeq 0$$

SOS-based Design

$$-\boldsymbol{L}_{ii}(\boldsymbol{x}(t)) \succ 0$$

$$-\left(\boldsymbol{L}_{ij}(\boldsymbol{x}(t)) + \boldsymbol{L}_{ji}(\boldsymbol{x}(t))\right) \succeq 0$$

Convex SOS Condition

$$\boldsymbol{v}_1^T(t)\left(\boldsymbol{L}_{ii}(\boldsymbol{x}(t)) - \varepsilon(\boldsymbol{x}(t))\right)\boldsymbol{v}_1(t) \ \ is \ \ SOS$$

$$-\boldsymbol{v}_2^T(t)\left(\boldsymbol{L}_{ij}(\boldsymbol{x}(t)) + \boldsymbol{L}_{ji}(\boldsymbol{x}(t))\right)\boldsymbol{v}_2(t) \ \ is \ \ SOS$$

$\varepsilon(\boldsymbol{x}(t))$ **:positive definite polynomial**

K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang,
A Sum of Squares Approach to Modeling and Control
of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems,
IEEE Transactions on Fuzzy Systems, Vol.17, No.4, pp.911-922, August 2009.

SOS-based Design

Convex SOS Design Condition

If there exist a symmetric polynomial matrix $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and a polynomial matrix $\boldsymbol{M}_i(\boldsymbol{x})$ satisfying the following SOS conditions, the polynomial fuzzy model can be stabilized by the fuzzy controller.

$$\boldsymbol{v}^T(\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\boldsymbol{v} \ is \ SOS$$

$V(\mathbf{x}(t)) > 0$

$$-\boldsymbol{v}^T\Big(\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) + \boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x})$$
$$-\boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) + \boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x})$$
$$+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_i^T(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \sum_{k \in \boldsymbol{K}} \frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})$$
$$-\sum_{k \in \boldsymbol{K}} \frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \epsilon_{2ij}(\boldsymbol{x})\boldsymbol{I}\Big)\boldsymbol{v} \ is \ SOS \quad i \le j,$$

$\dot{V}(\mathbf{x}(t)) < 0$

Positive definite polynomial $\epsilon_1(\boldsymbol{x}) > 0 \ (\boldsymbol{x} \neq 0)$ $\epsilon_{2ij}(\boldsymbol{x}) \geq 0$

$\boldsymbol{v} \in R^N$ is a vector that is independent of $\boldsymbol{x}$

$\mathbf{T}(\mathbf{x}) \in \mathbf{R}^{N \times n}$ is a polynomial matrix whose (i, j)-th entry is given by $T^{ij}(\mathbf{x}) = \frac{\partial \hat{x}_i}{\partial x_j}(\mathbf{x})$.

UEC TOKYO

SOS-based Design

Convex SOS Design Condition

If there exist a symmetric polynomial matrix $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and a polynomial matrix $\boldsymbol{M}_i(\boldsymbol{x})$ satisfying the following SOS conditions, the polynomial fuzzy model can be stabilized by the fuzzy controller.

$$\boldsymbol{v}^T(\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\boldsymbol{v} \ is \ SOS$$

$V(\boldsymbol{x}(t)) > 0$

$$\begin{aligned}-\boldsymbol{v}^T \Big(&\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) + \boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) \\ &-\boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) + \boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x}) \\ &+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_i^T(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \sum_{k\in\boldsymbol{K}} \frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\ &- \sum_{k\in\boldsymbol{K}} \frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \epsilon_{2ij}(\boldsymbol{x})\boldsymbol{I}\Big)\boldsymbol{v} \ is \ SOS \quad i \le j,\end{aligned}$$

$\dot{V}(\boldsymbol{x}(t)) < 0$

A stabilizing feedback gain $\boldsymbol{F}_i(\boldsymbol{x})$ can be obtained from the solutions $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{M}_i(\boldsymbol{x})$ as $\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})$.

Lyapunov function $V(\boldsymbol{x}(t)) = \hat{\boldsymbol{x}}^T(t)\boldsymbol{P}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}^T(t)\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(t)$

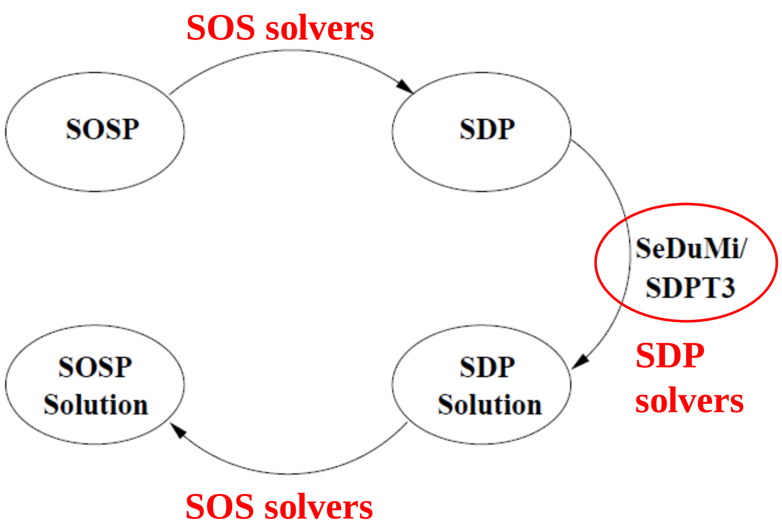
If $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is a constant matrix, then the stability holds globally.

SOS-based Design

How to Solve Convex SOS?

- A variety of control theory problems can be expressed as SOS problems (SOSPs).
- If a problem is formulated in terms of SOS, then it can be solved by efficient convex optimization algorithms

  (the "SOS solvers").
- SOS solvers: SOSOPT, SOSTOOLS, etc



Check/solve $p(x)$ is SOS

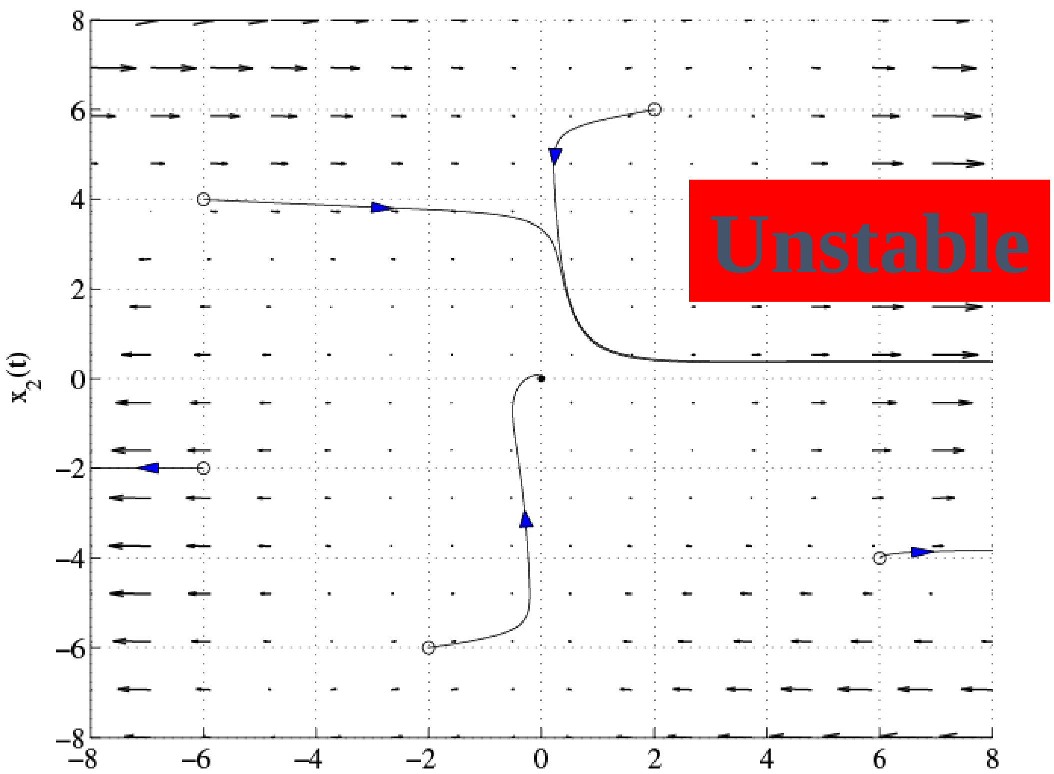
$$p(\boldsymbol{x}) = \boldsymbol{z}^T(\boldsymbol{x})\boldsymbol{Q}\boldsymbol{z}(\boldsymbol{x}), \quad \boldsymbol{Q} = \boldsymbol{Q}^T$$

Check/solve $\boldsymbol{Q} \succeq 0$

Design Example

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 + x_1 u$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

Without control



Behavior in $x_1(t)$-$x_2(t)$ plane

Design Example

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 + x_1 u$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

Takagi-Sugeno fuzzy model

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{8} h_i(\boldsymbol{z})\{\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\}$$

PDC fuzzy controller

$$\boldsymbol{u} = -\sum_{i=1}^{8} h_i(\boldsymbol{z}) \boldsymbol{F}_i \boldsymbol{x}$$

We can NOT design a PDC fuzzy controller guaranteeing global stability by solving the existing LMI conditions

Design Example

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 + x_1 u$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

Takagi-Sugeno fuzzy model

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{8} h_i(\boldsymbol{z})\{\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\}$$

PDC fuzzy controller

$$\boldsymbol{u} = -\sum_{i=1}^{8} h_i(\boldsymbol{z}) \boldsymbol{F}_i \boldsymbol{x}$$

Polynomial fuzzy model

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(\boldsymbol{z})\{\underline{\boldsymbol{A}_i(\boldsymbol{x})}\hat{\boldsymbol{x}} + \underline{\boldsymbol{B}_i(\boldsymbol{x})}\boldsymbol{u}\}$$

Polynomial-related fuzzy controller

$$\boldsymbol{u} = -\sum_{i=1}^{2} h_i(\boldsymbol{z}) \underline{\boldsymbol{F}_i(\boldsymbol{x})}\hat{\boldsymbol{x}}$$

Design Example

Polynomial Fuzzy Model Construction

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}$$

$A_i(x)$ and $B_i(x)$ are permitted to be polynomial matrices in x.

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1^1 x_2 - x_2^2 & 1 \\ -1 & -1 \end{bmatrix} \qquad h_1(\boldsymbol{z}) = \frac{\sin x_1 + 0.2172 x_1}{1.217 x_1}$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1^1 x_2 - x_2^2 & 1 \\ 0.2172 & -1 \end{bmatrix} \qquad h_2(\boldsymbol{z}) = \frac{x_1 - \sin x_1}{1.217 x_1}$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \qquad \boldsymbol{x} = [x_1 \ \ x_2]^T, \ \hat{\boldsymbol{x}} = \boldsymbol{x}, \ \boldsymbol{u} = u$$

Design Example

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 + x_1 u$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

Design
- stable controller

by solving the SOS conditions

Polynomial fuzzy model

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(\boldsymbol{z})\{\underline{\boldsymbol{A}_i(\boldsymbol{x})}\hat{\boldsymbol{x}} + \underline{\boldsymbol{B}_i(\boldsymbol{x})}\boldsymbol{u}\}$$

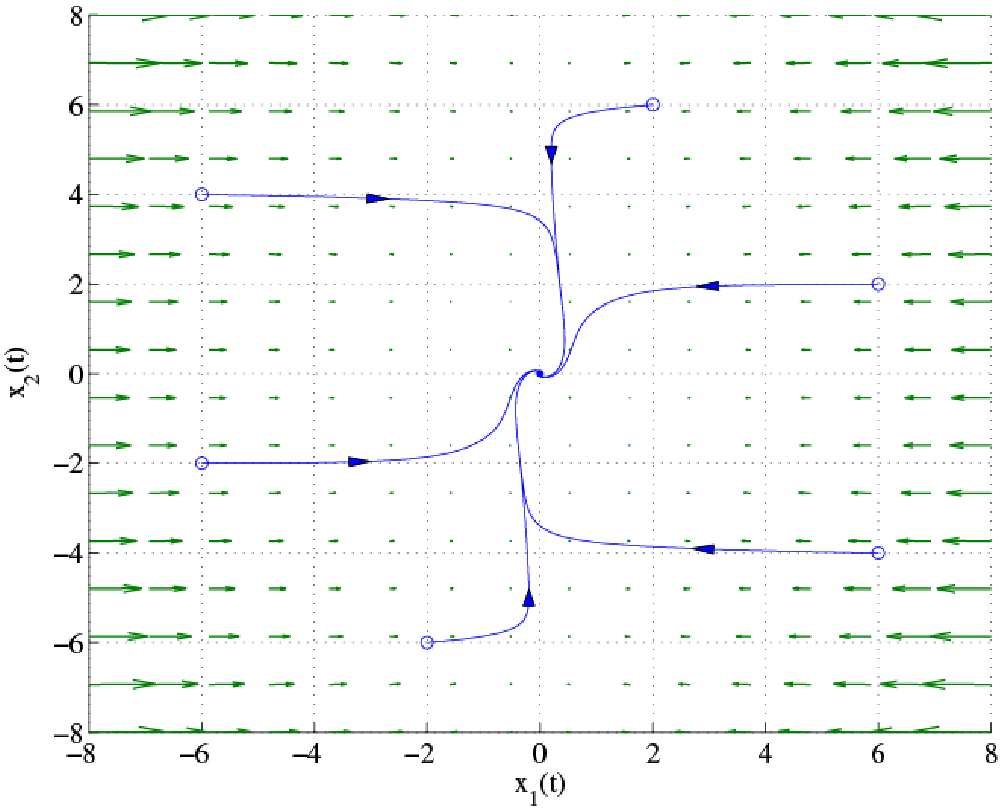
Polynomial-related fuzzy controller

$$\boldsymbol{u} = -\sum_{i=1}^{2} h_i(\boldsymbol{z})\underline{\boldsymbol{F}_i(\boldsymbol{x})}\hat{\boldsymbol{x}}$$

K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Design Example

Behavior in $x_1(t)$-$x_2(t)$ plane



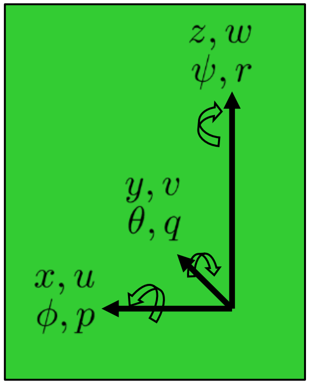
Stabilized by SOS controller

With control

K. Tanaka, H. Yoshida, H. Ohtake and H. O. Wang, A Sum of Squares Approach to Modeling and Control of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems, IEEE Transactions on Fuzzy Systems, Vol.17, No.4, pp.911-922, 2009.

Micro Helicopter Control Example

Weight	190g
Blade diameter	350mm



Co-axial counter rotating helicopter X.R.B produced by HIROBO

Micro Helicopter Control Example

Micro Helicopter Dynamics

$$\dot{u}(t) = -\frac{a}{I_z}\psi(t)v(t) + \frac{1}{m}U_X(t)$$

$$\dot{v}(t) = \frac{a}{I_z}\psi(t)u(t) + \frac{1}{m}U_Y(t)$$

$$\dot{w}(t) = \frac{1}{m}U_Z(t)$$

$e_x(t) = x(t) - x_{\mathrm{ref}}$, $e_y(t) = y(t) - y_{\mathrm{ref}}$, and $e_z(t) = z(t) - z_{\mathrm{ref}}$

$\psi(t) \in [-\pi \ \pi]$

Takagi-Sugeno Fuzzy Model

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(t))\boldsymbol{C}_i\boldsymbol{x}(t)$$

where $\boldsymbol{z}(t) = \psi(t)$ and

$$\boldsymbol{x}(t) = [u(t)\ v(t)\ w(t)\ e_x(t)\ e_y(t)\ e_z(t)]^{\mathrm{T}}$$

$$\boldsymbol{u}(t) = [U_X(t)\ U_Y(t)\ U_Z(t)]^{\mathrm{T}}.$$

Micro Helicopter Control Example

Takagi-Sugeno Fuzzy Model

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i\left(\boldsymbol{z}(t)\right)\left\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\right\}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{2} h_i\left(\boldsymbol{z}(t)\right)\boldsymbol{C}_i\boldsymbol{x}(t)$$

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & -\frac{a\pi}{I_Z} & 0 & 0 & 0 & 0 \\ \frac{a\pi}{I_Z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & \frac{a\pi}{I_Z} & 0 & 0 & 0 & 0 \\ -\frac{a\pi}{I_Z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$h_1\left(\psi(t)\right) = \frac{\psi(t) + \pi}{2\pi} \qquad h_2\left(\psi(t)\right) = \frac{\pi - \psi(t)}{2\pi}.$$

Micro Helicopter Control Example

Micro Helicopter Dynamics (Takagi-Sugeno Fuzzy Model)

Guaranteed Cost Controller Design

LMI controller

$$\boldsymbol{u}(t) = -\sum_{i=1}^{2} h_i(z)\boldsymbol{F}_i\boldsymbol{x}(t)$$

SOS controller

$$\boldsymbol{u}(t) = -\sum_{i=1}^{2} h_i(z)\underline{\boldsymbol{F}_i(\boldsymbol{x}(t))}\hat{\boldsymbol{x}}(t)$$

Comparison of performance function values

$$J = \int_0^{\infty} \{\boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\}dt$$

Kazuo Tanaka, Hiroshi Ohtake and Hua O. Wang,
Guaranteed Cost Control of Polynomial Fuzzy Systems via a Sum of Squares Approach,
IEEE Transactions on Systems, Man and Cybernetics Part B, Vol.39, No.2, pp.561-567 April, 2009.

Micro Helicopter Control Example

Case I $\boldsymbol{Q}=\boldsymbol{I}$, $\boldsymbol{R}=0.1\boldsymbol{I}$
Case II $\boldsymbol{Q}=\boldsymbol{I}$, $\boldsymbol{R}=\boldsymbol{I}$
Case III $\boldsymbol{Q}=\boldsymbol{I}$, $\boldsymbol{R}=10\boldsymbol{I}$

Comparison of performance function values J

	Case I	Case II	Case III
LMI controller	0.67286	1.5522	3.8873
SOS controller (Order of M is 2)	0.57539	1.0388	2.3350
SOS controller (Order of X is 2)	0.60621	1.0861	2.6191
Reduction rate of J [%]	14.48593	33.07563	39.9326

LMI controller

SOS controller

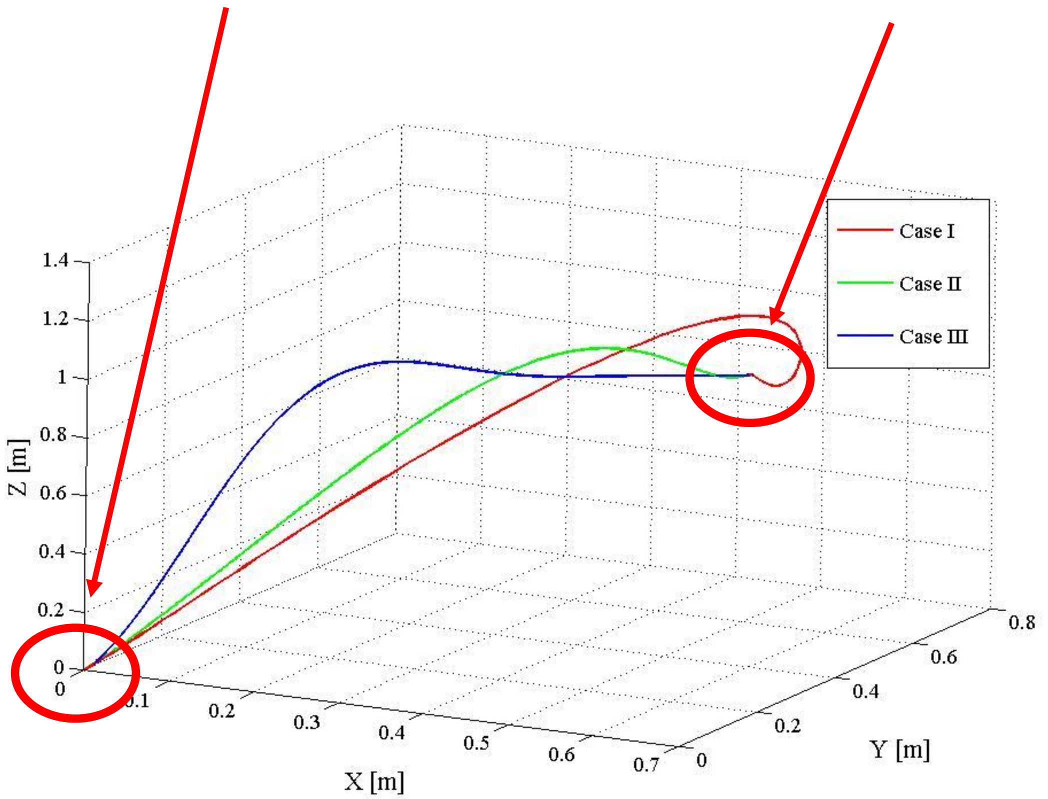
Max 39% reduction

SOS design approach provides
better control results than the LMI design approach.

Kazuo Tanaka, Hiroshi Ohtake and Hua O. Wang,
Guaranteed Cost Control of Polynomial Fuzzy Systems via a Sum of Squares Approach,
IEEE Transactions on Systems, Man and Cybernetics Part B, Vol.39, No.2, pp.561-567 April, 2009.

Micro Helicopter Control Example

$$[x_0\ y_0\ z_0\ \psi_0]^T = [0\ 0\ 0\ \tfrac{1}{2}\pi]^T \longrightarrow [x_{ref}\ y_{ref}\ z_{ref}\ \psi_{ref}]^T = [0.6\ 0.4\ 1.0\ 0]^T$$



Case I $Q=I$, $R=0.1I$
Case II $Q=I$, $R=I$
Case III $Q=I$, $R=10I$

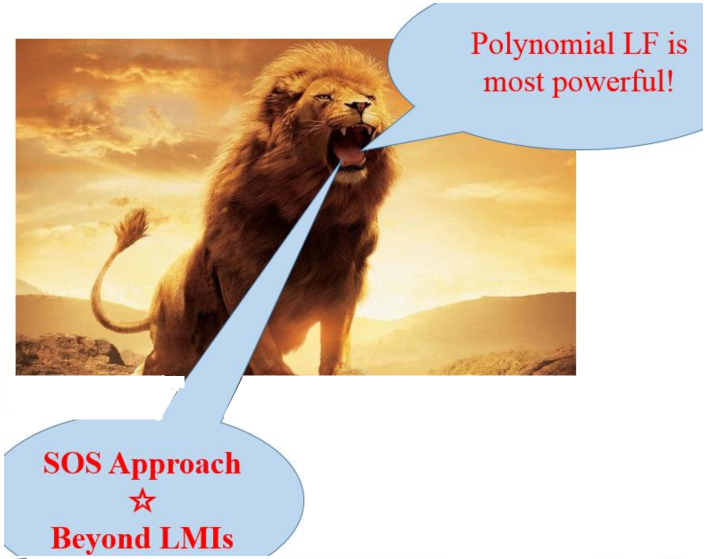
Control Results (Order of X is 0, Order of M is 2)

Kazuo Tanaka, Hiroshi Ohtake and Hua O. Wang,
Guaranteed Cost Control of Polynomial Fuzzy Systems via a Sum of Squares Approach,
IEEE Transactions on Systems, Man and Cybernetics Part B, Vol.39, No.2, pp.561-567 April, 2009.

Recent Topics

Some recent topics in SOS-based design will be presented in the tutorial.

Conclusions



K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

Tutorial Overview

- Introductions
- Part I
  Outline of Takagi-Sugeno (T-S) Fuzzy Model-based Control
- Part II
  T-S Fuzzy Model-based Control using Linear Matrix Inequalities (LMIs)
- Part III
  Theoretical Advances in T-S Fuzzy Model-based Control using LMIs
- Part IV
  Beyond LMIs: Polynomial Fuzzy Systems Control and Analysis using Sum-of- Squares (SOS)
- Conclusions

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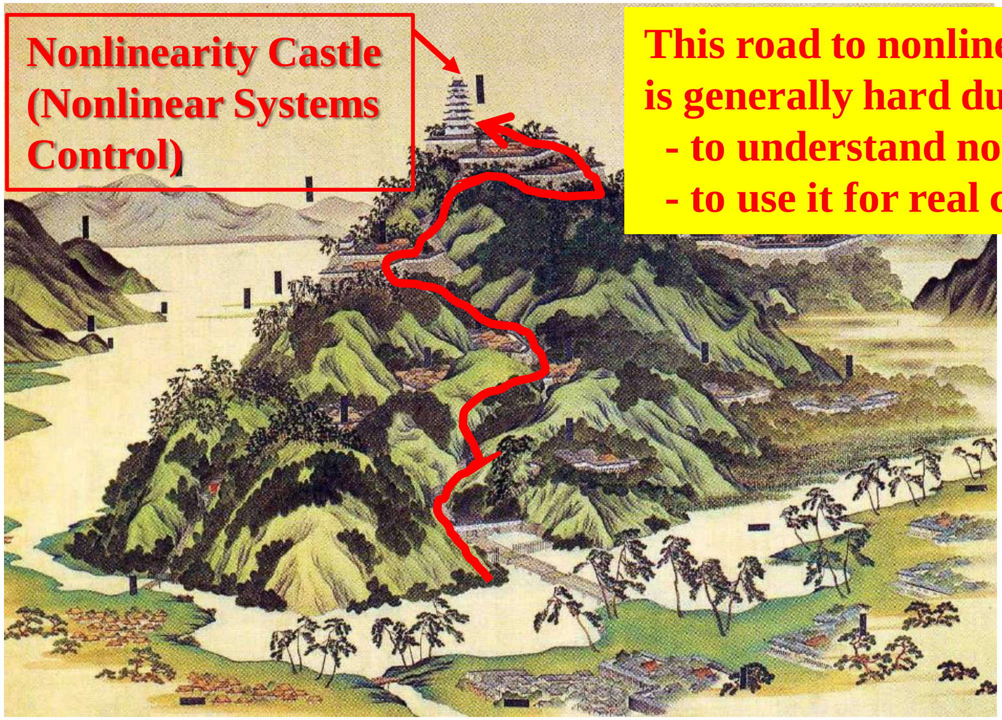
BOSTON UNIVERSITY

Conclusions

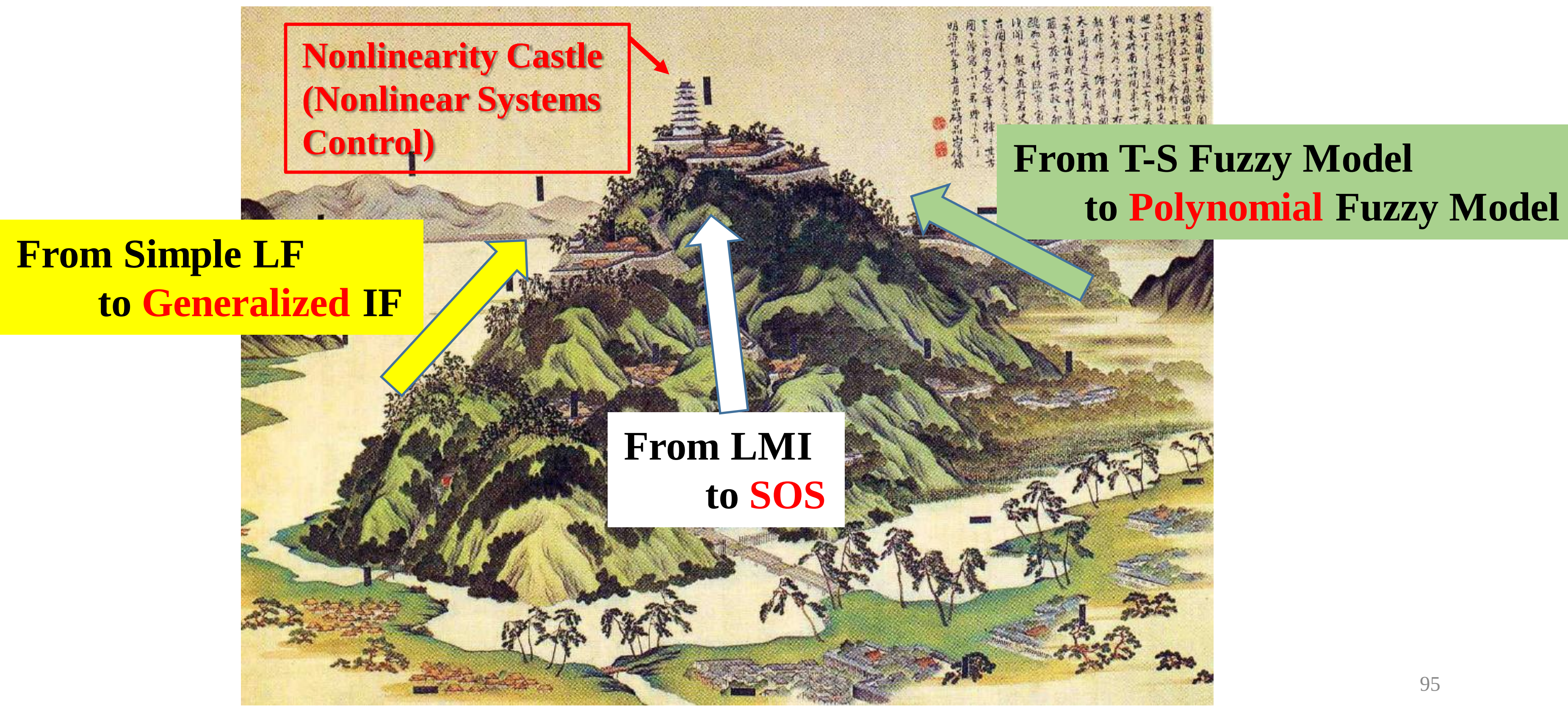


This road to nonlinearity castle (nonlinear systems control) is generally hard due to the difficulties
- **to understand nonlinear control theory**
- **to use it for real complex systems**

Conclusions

The history of fuzzy-model based control is the history of matching to nonlinearities.

When nonlinearities met fuzzy logic, nonlinearities became easier to contend with.



K. Tanaka, A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control:Beyond Linear Matrix Inequalities, The University of Electro-Communications (UEC) , Tokyo, Japan

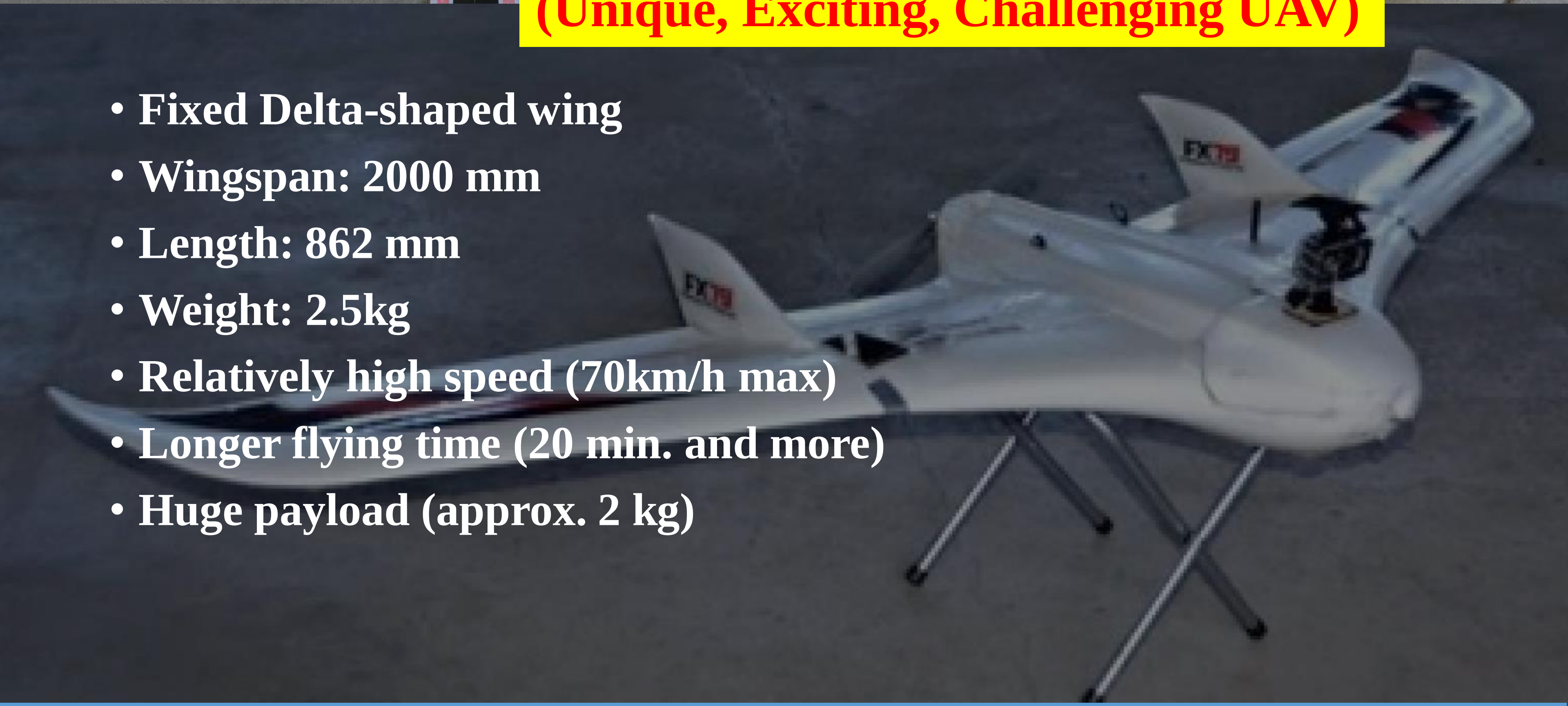
Future Research

- What's coming next?
  LMI → SOS → ?

- Great & Challenging Applications
  Unmanned aerial vehicle (UAV) control applications

More challenging

UEC-UAV
(Unique, Exciting, Challenging UAV)



- Fixed Delta-shaped wing
- Wingspan: 2000 mm
- Length: 862 mm
- Weight: 2.5kg
- Relatively high speed (70km/h max)
- Longer flying time (20 min. and more)
- Huge payload (approx. 2 kg)

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Manual control by students.
Actually very tough to realize stable control by human.


十勝毎

（第3種郵便物認可）

風を計算 無人飛行成功

電通大が大樹で実験

ラジコン応用、低価格で

【大樹】電気通信大学情報理工学部知能機械工学科（東京）の学生が4月29日～5月9日、町多目的航空公園で、カメラで地上を撮影する小型飛行機の実験を行った。上空での自動操縦時に風の影響を計算し、予定通りの経路を飛行できるかどうかを確かめた。

同学科の田中一男教授の研究室は2011年から、災害時に上空から被害の状況を確認できる無人機を開発している。市販のラジコンを応用するため、低価格での撮影が可能となる。これまでも大樹で実験を繰り返しており、自動での離着

固定翼機を持つ尾鷲さんと黒田さん（後列左から）。前列はパラグライダー型の機体を持つ岩瀬さん

陸やあらかじめ設定し
点の撮影などに成功し
る。

今回は田中教授のほ
4年生の尾鷲真士さ
(22)、岩瀬惟真（ありま
さん(21)、黒田真生さ
(22)が実験に取り組ん
パラグライダーとプロ
で飛ぶ機体（全長40ｾﾝ
さ30ｾﾝ、幅45ｾﾝ、重さ
1㌔）と、より広範囲を
できる固定翼の機体（
ﾒｰﾄﾙ、重さ2・2㌔）を
コン、自動操縦で飛ば
飛行回数はそれぞれ、
で1日6、7回に及ん

パラグライダーの機
は上空20～40ﾒｰﾄﾙでの飛
に、風の影響を計算し
ら指定した経路を飛ぶ
ができるかどうか、固
機では高度、速度を保
飛ぶことができるかど
が今回の実験ポイント
方とも想定通り飛行さ
ことができた。

尾鷲さんは「機体の
に時間がかかり難し
た。実験で得られたデ
を解析して、精度を高
いきたい」と話してい
（伊藤亮

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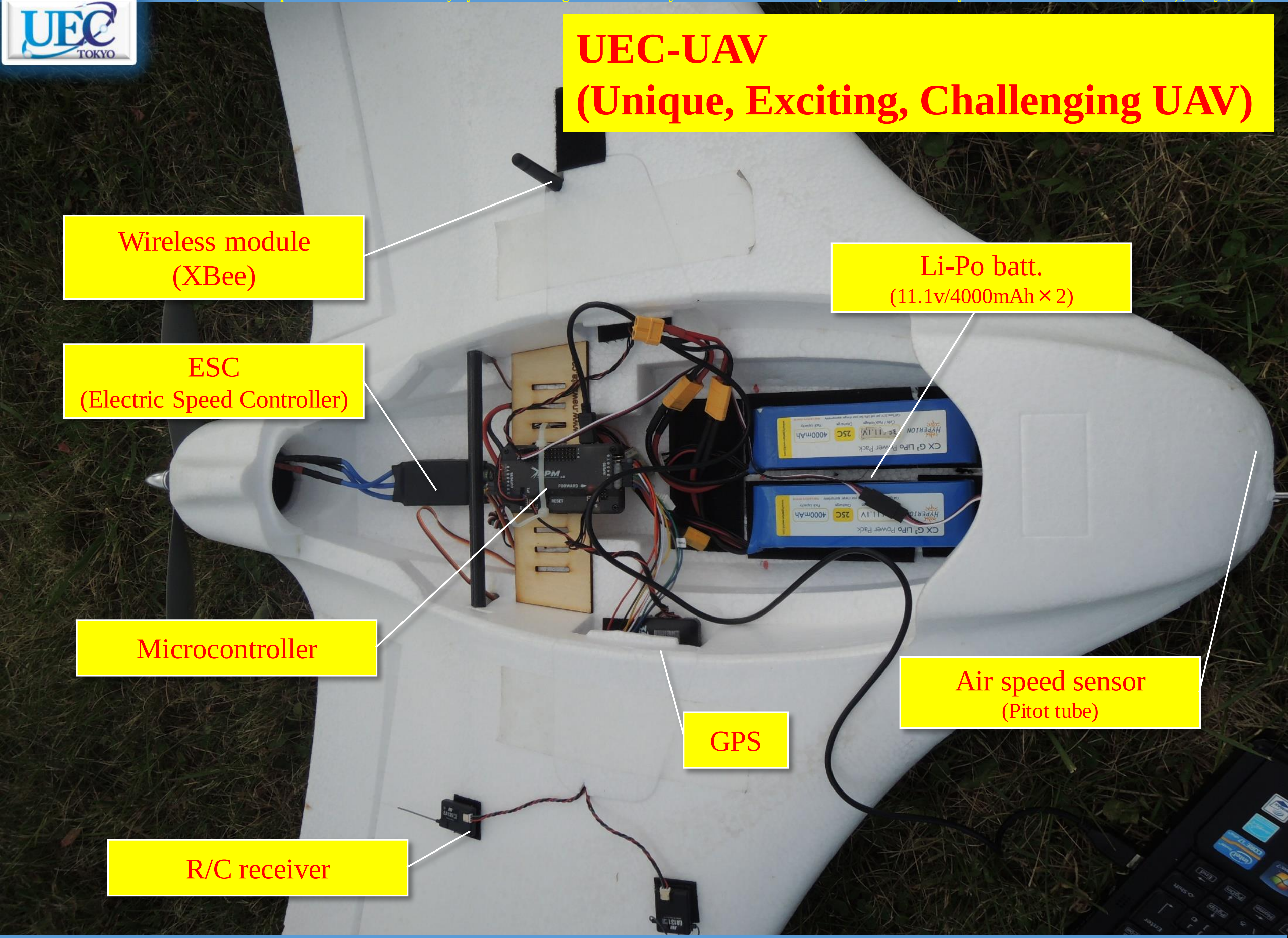


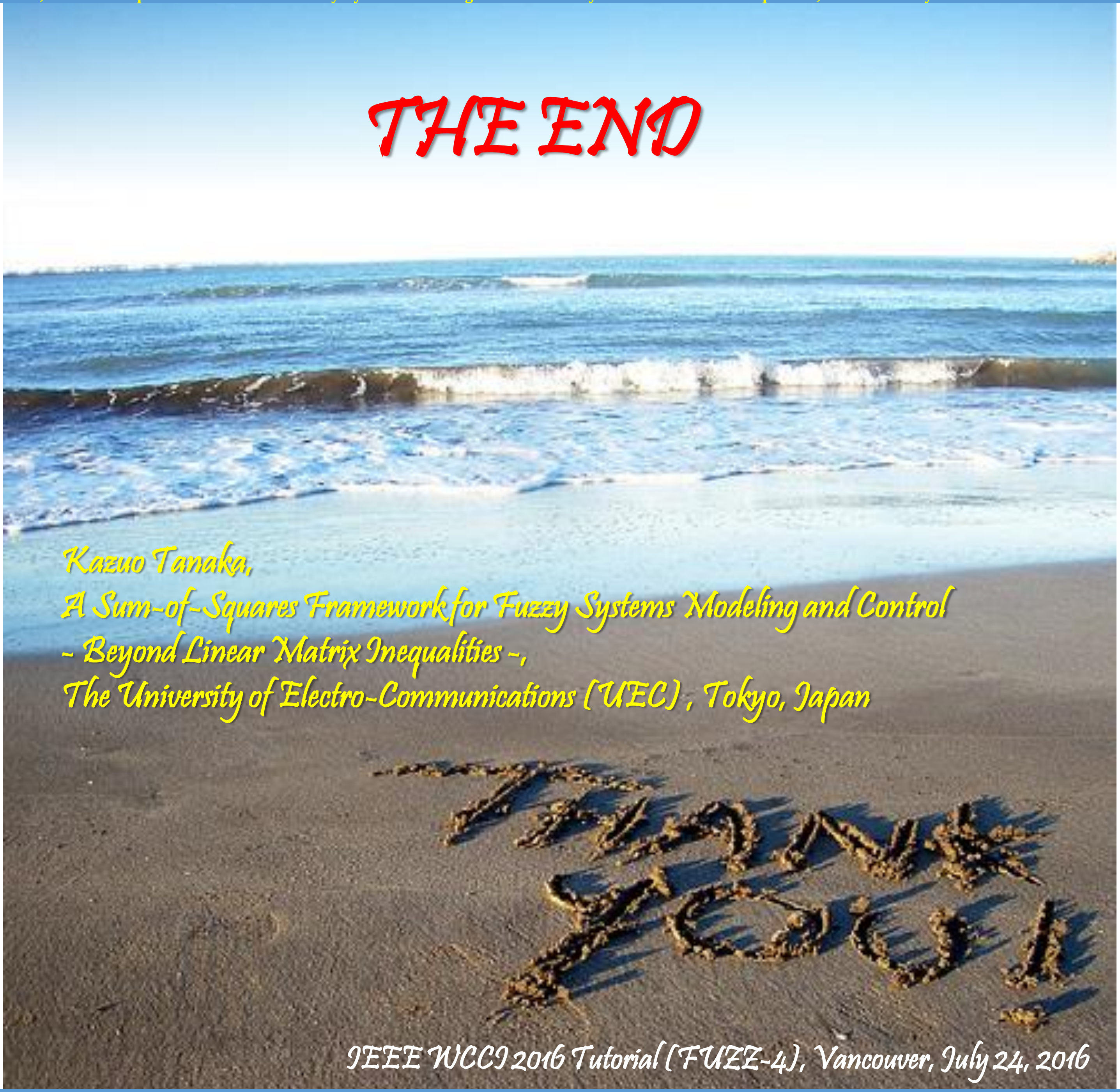
JAXA Taiki Aerospace Research Field

Future Research

Our UAV (UEC-UAV) Control (Video)

https://www.youtube.com/watch?v=z3m5ChL4Sx0

THE END



Kazuo Tanaka,
A Sum-of-Squares Framework for Fuzzy Systems Modeling and Control
- Beyond Linear Matrix Inequalities -,
The University of Electro-Communications (UEC) , Tokyo, Japan

IEEE WCCI 2016 Tutorial (FUZZ-4), Vancouver, July 24, 2016