Learning Sparsifying Transforms for Signal, Image, and Video Processing

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Work with

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- Luke Pfister

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Andrew T. Yang Fellowship

Overview

Today we will see that

* Learned Sparsity Models are valuable and well-founded tools for modeling data
* The Transform Learning formulation has computational and performance advantages
* When used in imaging and image and video processing, Transform Learning leads to state-of-the-art results

Outline

- Sparse signal models – Why and How?
  - Synthesis Dictionaries
  - Sparsifying Transforms
- Basic Transform Learning
- Variations on Transform Learning
  - Union of Transforms for inverse problems
  - Online Transform Learning for big data and video denoising
  - A filter bank formulation of Transform Learning
- Conclusions

Why Sparse Modeling?

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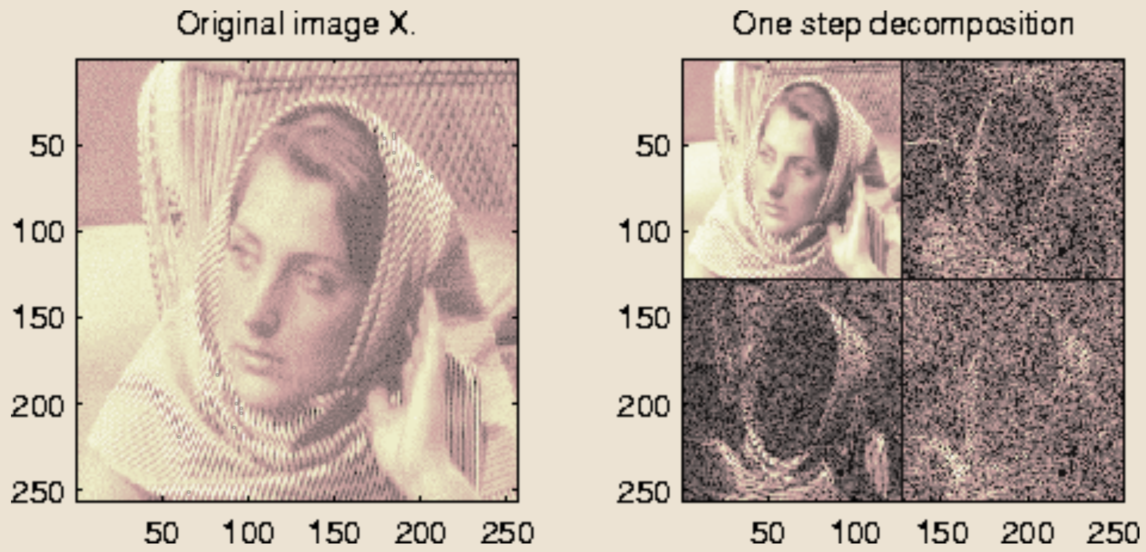

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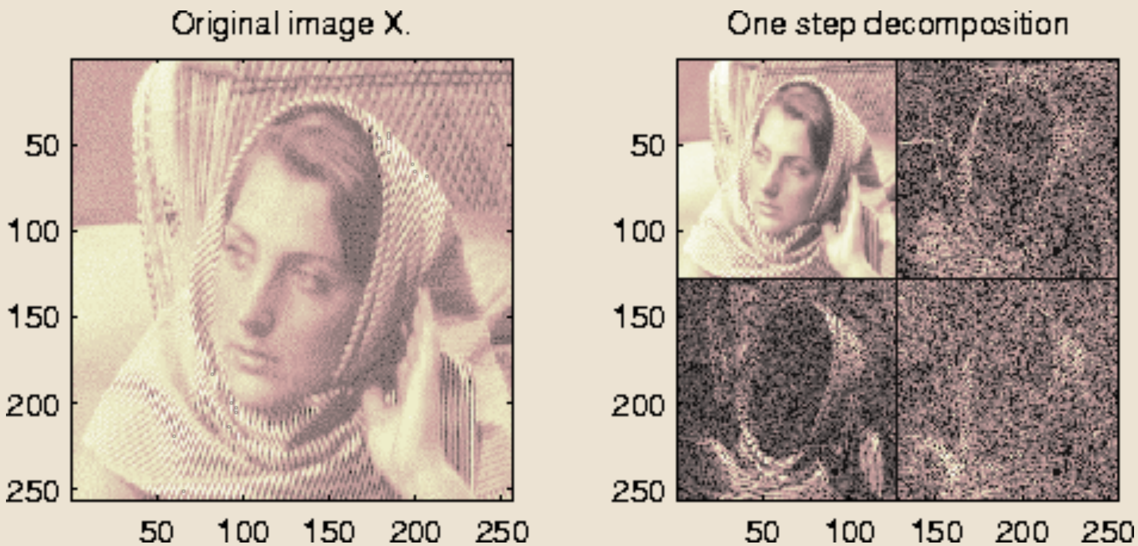
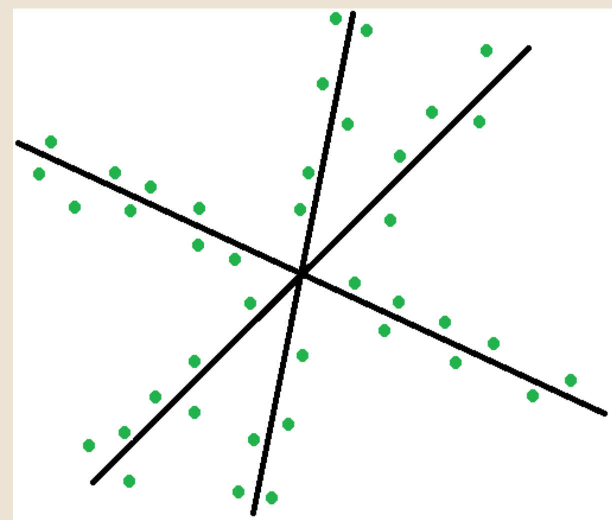


- ✓ Image data usually lives in low dimensional subspaces



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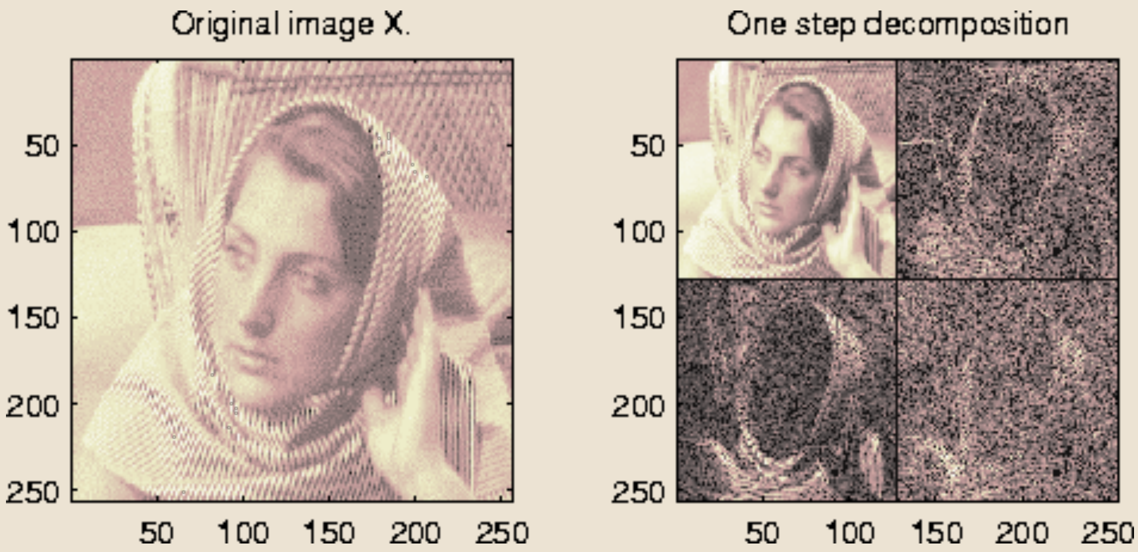

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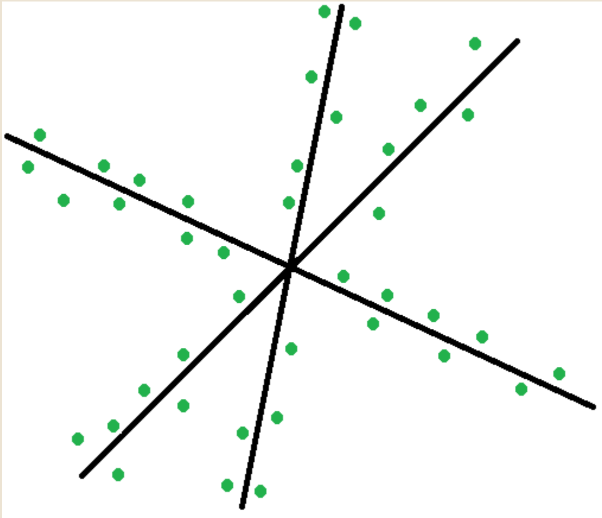
Applications:



- Compact representations (compression)
- Regularization in inverse problems
  - Denoising
  - recovery from degraded data
  - Compressed Sensing
- Classification

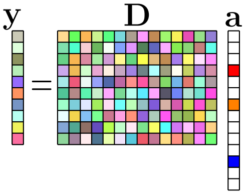
Introduction to Sparse Signal Models

* The Synthesis Dictionary Model
* Learning Synthesis Dictionaries
* The Transform Model

Sparse Representations: Model

- We model $\mathbf{y} \in \mathbb{R}^N$ as

$$\mathbf{y} = \mathbf{D}\mathbf{a}, \quad \|\mathbf{a}\|_0 \leq s$$



- $\mathbf{a}$ is a sparse coefficient vector
- $\mathbf{D} \in \mathbb{R}^{n \times K}$ is a dictionary. Can be square ($n = K$) or rectangular ($n < K$)
- Columns of $\mathbf{D}$ are called atoms
- $\mathbf{y}$ belongs to a union of subspaces spanned by s atoms of $\mathbf{D}$

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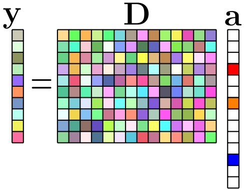

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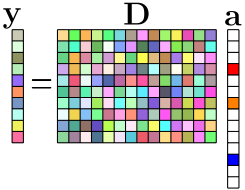

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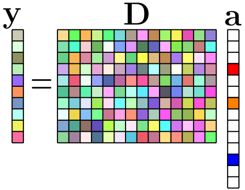

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Sparse Representations: the Synthesis Model

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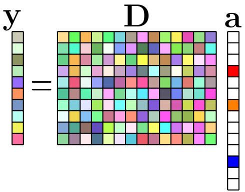

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Sparse Representations: Sparse Coding

- Given an overcomplete $\mathbf{D}$ and vector $\mathbf{y}$, how can we find the sparsest $\mathbf{a}$ such that $\mathbf{y} = \mathbf{Da}$?
- Solve

$$\min_{a} \quad \|\mathbf{a}\|_0 \quad \text{subject to} \quad \mathbf{y} = \mathbf{Da}$$

- NP-Hard![1] Look for approximate solutions
  - Convex Relaxation
    - Basis pursuit
  - Greedy Algorithms
    - Orthogonal Matching Pursuit (OMP)

[1] Natarajan, 1995

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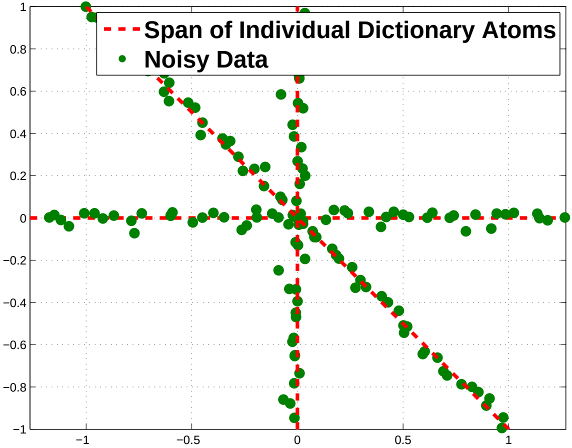
Sparse Representations: Denoising

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$s = 1$$



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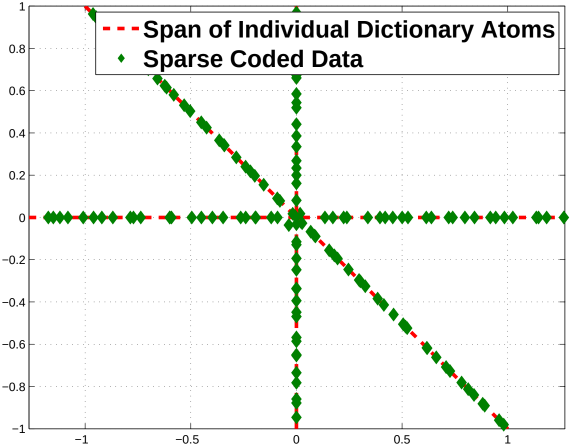

What are good dictionaries for sparse representation of signals and images?

The more sparse the representation, the better!

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Analytic Dictionaries

- Design dictionary around a predefined set of functions
  - Fourier
  - Wavelet
  - Curvelet
  - Contourlet

    ⋮

- Fast implementations

- But, hard to design effective dictionaries in high dimensions

Adaptive Dictionaries

- Adaptively learn dictionary from data itself
- Karhunen-Loève/PCA: fit low-dim subspace to minimize ℓ_2 error

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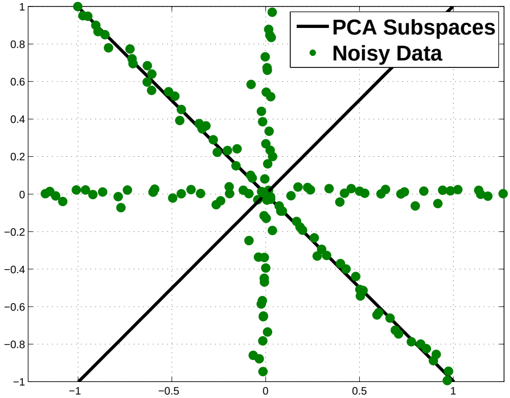


Dictionary Learning

- Training data: $\{\mathbf{y}_j\}_{j=1}^{M} \in \mathbb{R}^N$
- Want:

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{D}\mathbf{a}_1, & \|\mathbf{a}_1\|_0 \leq s \\ \mathbf{y}_2 &= \mathbf{D}\mathbf{a}_2, & \|\mathbf{a}_2\|_0 \leq s \\ &\vdots \\ \mathbf{y}_M &= \mathbf{D}\mathbf{a}_M, & \|\mathbf{a}_M\|_0 \leq s \end{aligned}$$

- **Dictionary Learning:** Given a set of training signals $\{\mathbf{y}_j\}_{j=1}^{M}$ formed into a matrix $\mathbf{Y} \in \mathbb{R}^{N \times M}$, we seek to find $\mathbf{D} \in \mathbb{R}^{N \times K}$, $\mathbf{A} \in \mathbb{R}^{K \times M}$ such that $\mathbf{Y} \approx \mathbf{D}\mathbf{A}$ with $\|\mathbf{a}_j\|_0 \leq s$

Summary: Learning Synthesis Dictionary Models

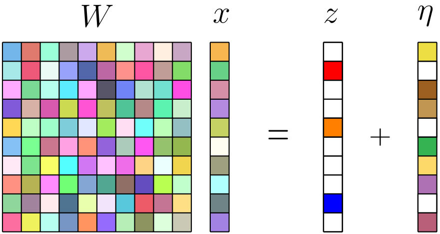
The Good

- Sparsity in an appropriate dictionary is a powerful model; learned sparsity models even better!
- Many successful applications: denoising, in-painting, image super resolution, compressed sensing(MRI, CT), classification, etc.

The Bad

- Synthesis sparse coding solved repeatedly for learning is NP-hard
- Approximate synthesis sparse coding algorithms can be computationally expensive
- The synthsis dictionary learning problem is highly non-convex, and algorithms can get stuck in bad local minima

A Classical Alternative: Transform Sparsity



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- z: Sparse Code
- $Wx \approx$ sparse
- Approximation error in the transform domain: $\|\eta\|_2 \ll \|z\|_2$

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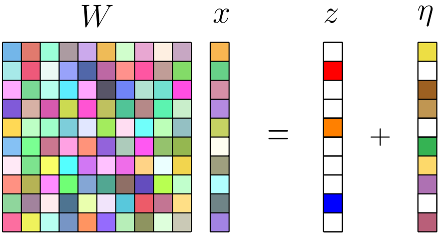

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Transform Sparse Coding

$$z^* = \arg\min_z \frac{1}{2}\|Wx - z\|_2^2$$

subject to $\|z\|_0 \leq s$ sparsity constraint

Easy Exact Solution:

$z^* \triangleq H_s(Wx)$ Thresholding to s largest elements

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Transform Sparse Coding

- Penalized Form

$$z^* = \min_z \frac{1}{2}\|Wx - z\|_2^2 + \nu\|z\|_0$$

Easy Exact Solution:

$$z_i^* = \begin{cases} [Wx]_i & |[Wx]_i| \geq \sqrt{\nu} \\ 0 & \text{else} \end{cases}$$

$$\triangleq \mathcal{T}_\nu(Wx) \quad \text{Hard Thresholding}$$

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Summary of the Models

- SM : finding x with given D is NP-hard.

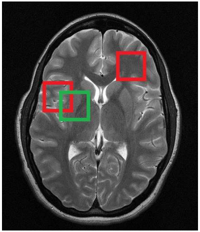
$$y = Dx + e\,, \ \|x\|_0 \leq s \tag{1}$$

- NSAM : finding q with given Ω is NP-hard.

$$y = q + e\,, \ \|\Omega q\|_0 \leq m - t \tag{2}$$

- TM : finding x with given W is easy $\Rightarrow$ efficiency in applications.

$$Wy = x + \eta\,, \ \|x\|_0 \leq s \tag{3}$$



Patches of image

- $Y_j = R_j y$, $j = 1,..N$: jth image patch, vectorized.
- $Y = [\, Y_1 \,|\, Y_2 \,|\, ... \,|\, Y_N \,] \in \mathbb{R}^{n \times N}$: matrix of vectorized patches - training signals

Basic Transform Learning Formulation[6]

$$(\mathrm{P0}) \quad \min_{W,X} \overbrace{\|WY - X\|_F^2}^{\text{Sparsification Error}} \quad \text{s.t.} \quad \|X_i\|_0 \leq s \;\; \forall\, i$$

- $Y = [\, Y_1 \,|\, Y_2 \,|\, ... \,|\, Y_N \,] \in \mathbb{R}^{n \times N}$: matrix of training signals
- $X = [\, X_1 \,|\, X_2 \,|\, ... \,|\, X_N \,] \in \mathbb{R}^{n \times N}$: matrix of sparse codes of Y_i
- $W \in \mathbb{R}^{n \times n}$: square transform
- Sparsification error - measures deviation of data in transform domain from perfect sparsity

6 [Ravishankar & Bresler ICIP 2012, TSP 2013, TSP 2015]

Basic Transform Learning Formulation[6]

$$(\text{P0}) \quad \min_{W,X} \overbrace{\|WY - X\|_F^2}^{\text{Sparsification Error}} + \overbrace{\lambda \left(\|W\|_F^2 - \log|\det W| \right)}^{\text{Regularizer } \triangleq \lambda v(W)}$$
$$\text{s.t.} \quad \|X_i\|_0 \leq s \ \forall\ i$$

- $Y = [\, Y_1 \,|\, Y_2 \,|\, ... \,|\, Y_N \,] \in \mathbb{R}^{n\times N}$: matrix of training signals
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- Sparsification error - measures deviation of data in transform domain from perfect sparsity
- $\lambda > 0$. Regularizer cost $v(W)$ prevents trivial solutions and fully controls condition number of W

[6] [Ravishankar & Bresler ICIP 2012, TSP 2013, TSP 2015]

Alternating Algorithm for Transform Learning

- (P0) solved by alternating between updating X and W.

- **Sparse Coding Step** solves for X with fixed W.

$$\min_{X} \|WY - X\|_F^2 \ \text{ s.t. } \ \|X_i\|_0 \leq s \ \forall \ i \qquad (1)$$

  - **Easy** problem: Solution $\hat{X}$ computed exactly by zeroing out all but the s largest magnitude coefficients in each column of WY.

- **Transform Update Step** solves for W with fixed X.

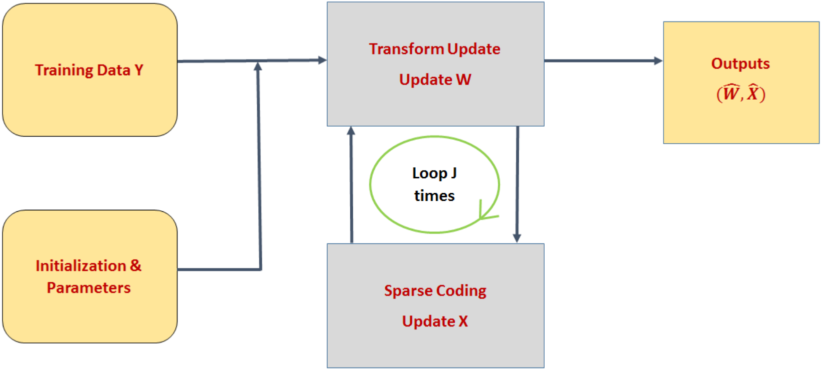
$$\min_{W} \|WY - X\|_F^2 + \lambda \left(\|W\|_F^2 - \log |\det W| \right) \qquad (2)$$

  - **Closed-form solution:**

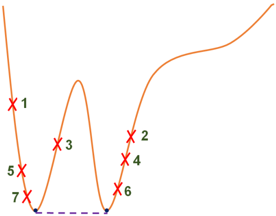
$$\hat{W} = 0.5U \left(\Sigma + \left(\Sigma^2 + 2\lambda I \right)^{\frac{1}{2}} \right) Q^T L^{-1} \qquad (3)$$

    - $YY^T + \lambda I = LL^T$, and $L^{-1}YX^T$ has a full SVD of $Q\Sigma U^T$.

Algorithm A1 for Square Transform Learning



Convergence Guarantees[7]



Theorem 1

For each initialization of Algorithm A1, the objective converges to a local minimum, and the iterates converge to an equivalence class (same function values) of local minimizers.

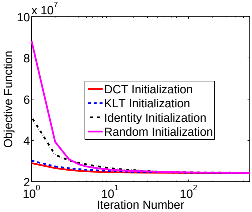
Corollary 1

Algorithm A1 is globally convergent (i.e., from any Initialization) to the set of local minimizers in the problem.

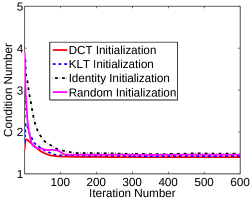
[7] [Ravishankar & Bresler, TSP 2015]

Convergence with Various Initializations

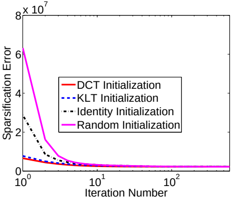
Barbara - 8×8 patches



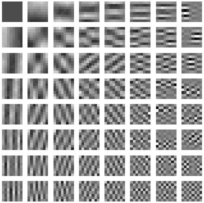
Objective Function



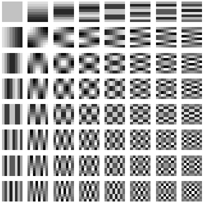
$\kappa(W)$



Sparsification Error ($s = 11$)



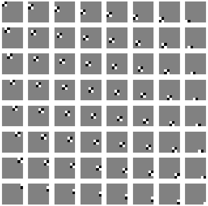
Learned W - DCT Init



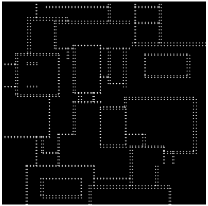
2D DCT

Piecewise-Constant Images

Image



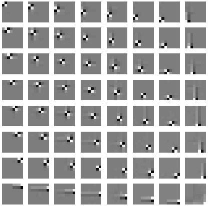
Finite difference (FD)
$\kappa(W) = 113.5$



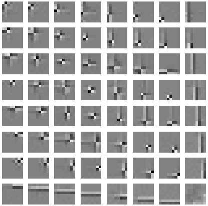
Sparse Result
$s = 5$

- 2D FD obtained as kronecker product of two square 1D-FD matrices - exact sparsifier for patches of image for $s \geq 5$.
- However, the 2D FD transform is poorly conditioned.

Well-Conditioned Adaptive Transforms Perform Well!



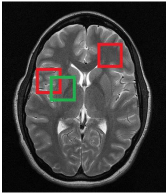
Learnt (FD Init)
$\kappa(W) = 15.35$



Learnt (FD Init)
$\kappa(W) = 5.77$

- The learnt transforms provide almost zero NSE ($\sim 10^{-4}/10^{-5}$).
- Such well-conditoned transforms perform better than poorly conditioned ones in applications such as denoising.
- For $s < 5$, the learnt well-conditioned transforms provide significantly lower NSE at the same s, than FD.

Computational Advantages



Patches of image

- Synthesis/Analysis K-SVD[8,9] for N training samples and $D \in \mathbb{R}^{n \times K}$: cost per iteration (dominated by sparse coding):
  $O(Nn^3) \propto \text{(Image Size)} \times \text{(pixels in patch)}^3$
- Transform Learning Algorithm A1 for N training samples and $W \in \mathbb{R}^{n \times n}$: $O(Nn^2) \propto \text{(Image Size)} \times \text{(pixels in patch)}^2$
- In 2D with $p \times p$ patches $\Rightarrow$reduction of computations in the order by p^2
- In 3D with $p \times p \times p$ patches $\Rightarrow$ reduction of computations in the order by p^3 (=1000X for $p = 10$)

[8] [Aharon, Elad & Bruckstein '06] [9] [Rubinstein, Peleg & Elad '13]

Noisy Image
PSNR = 24.60 dB

64×256 Synthesis D
PSNR = 31.50 dB

64×64 W $(\kappa = 1.3)$
PSNR = 31.66 dB

- Transform learning-based denoising is better and highly efficient (17X faster) compared to overcomplete K-SVD denoising.

Compressed Sensing with a Learned Transform

Review: Compressed Sensing (CS)

- CS enables accurate recovery of images from far fewer measurements than the number of unknowns
  - Sparsity of image in transform domain or dictionary
  - Measurement procedure incoherent with transform
  - **Reconstruction non-linear**
- Conventional CS Reconstruction problem -

$$\min_{x} \overbrace{\|Ax - y\|_2^2}^{\text{Data Fidelity}} + \lambda \overbrace{\|\Psi x\|_0}^{\text{Regularizer}} \quad (4)$$

- $x \in \mathbb{C}^P$: vectorized image, $y \in \mathbb{C}^m$: measurements $(m < P)$.
- A : fat sensing matrix, Ψ : transform. ℓ_0 "norm" counts non-zeros.
- CS with non-adaptive regularizer limited to low undersampling in imaging.

Compressed Sensing MRI

- Data - samples in k-space of spatial Fourier transform of object, acquired sequentially in time.
- Acquisition rate limited by MR physics, physiological constraints on RF energy deposition.
- CSMRI enables accurate recovery of images from far fewer measurements than # unknowns or Nyquist sampling.
- Two directions to improve CSMRI -
  - **better sparse modeling - TLMRI**
  - better choice of sampling pattern (F_u)

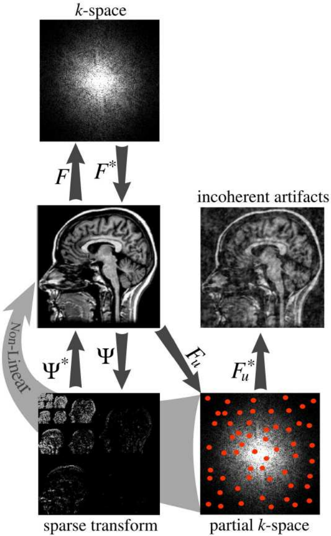


Fig. from Lustig et al. '07

Transform Blind Compressed Sensing Idea

- ~~*Could* use an image database to train the sparsifying transform~~
- Learn transform W to sparsify the *unknown image* x using only the undersampled data $y \approx Ax$
- $\Rightarrow$ **model adaptive to underlying image.**
- Use the learned transform W to perform compressed sensing reconstruction of the image x from undersampled data y

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- *Could* use a database to train the sparsifying transform
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Transform-based Blind Compressed Sensing (BCS)

$$(\text{P1}) \quad \min_{x,W,B} \overbrace{\sum_{j=1}^{N} \|WR_j x - b_j\|_2^2}^{\text{Sparsification Error}} + \nu \overbrace{\|Ax - y\|_2^2}^{\text{Data Fidelity}} + \lambda \overbrace{v(W)}^{\text{Regularizer}}$$

$$s.t. \quad \sum_{j=1}^{N} \|b_j\|_0 \leq s, \quad \|x\|_2 \leq C.$$

- (P1) learns $W \in \mathbb{C}^{n \times n}$, and reconstructs x, from only undersampled $y \Rightarrow$ **transform adaptive to underlying image.**
- $v(W) \triangleq -\log |\det W| + 0.5 \|W\|_F^2$ controls scaling and κ of W.
- $\|x\|_2 \leq C$ is an energy/range constraint. $C > 0$.

Block Coordinate Descent (BCD) Algorithm for (P1)

- Alternate the updating of W, B, and x.

- **Sparse Coding Step:** solve (P1) for B with fixed x, W.

$$\min_{B} \sum_{j=1}^{N} \|WR_j x - b_j\|_2^2 \quad s.t. \quad \sum_{j=1}^{N} \|b_j\|_0 \leq s. \tag{5}$$

  - **Cheap Solution:** Let $Z \in \mathbb{C}^{n \times N}$ be the matrix with $WR_j x$ as its columns. Solution $\hat{B} = H_s(Z)$ computed exactly by zeroing out all but the s largest magnitude coefficients in Z.

Block Coordinate Descent Algorithm for (P1)

- **Transform Update Step:** solve (P1) for W with fixed x, B.

$$\min_{W} \sum_{j=1}^{N} \|WR_j x - b_j\|_2^2 + 0.5\lambda \|W\|_F^2 - \lambda \log |\det W| \tag{6}$$

  - **Exact Closed-form solution involving SVD of a small matrix**

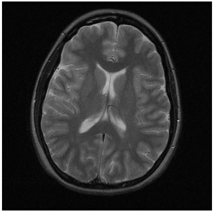
Block Coordinate Descent Algorithm for (P1)

- **Image Update Step:** solve (P1) for x with fixed W, B.

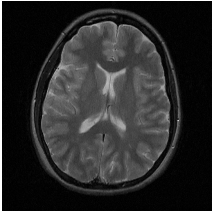
$$\min_x \sum_{j=1}^{N} \|WR_j x - b_j\|_2^2 + \nu \|Ax - y\|_2^2 \;\; s.t. \;\; \|x\|_2 \leq C. \qquad (7)$$

  - Standard least squares problem with ℓ_2 norm constraint. For MRI can be solved iteratively efficiently using CG+ FFT.

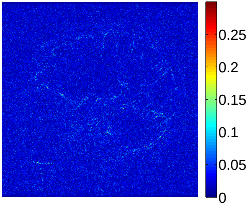
Example - 2D Cartesian 7x Undersampling



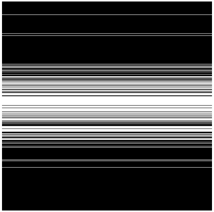
Reference



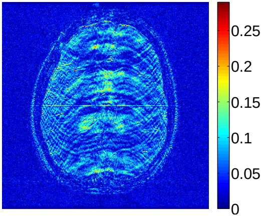
TLMRI (31 dB)



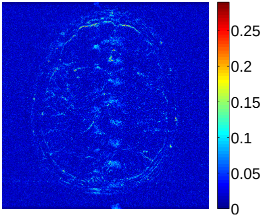
TLMRI Error



Sampling Mask

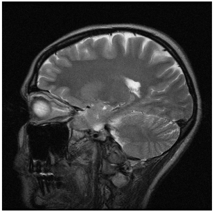


Sparse MRI Error (25.5dB)
Lustig et al, 2007



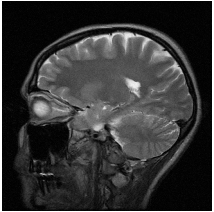
DLMRI Error (30.7 db)
Saiprasad & Bresler, 2011

Example - 2D random 5x Undersampling

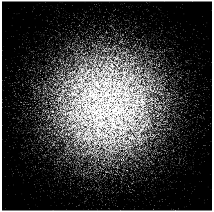


Reference

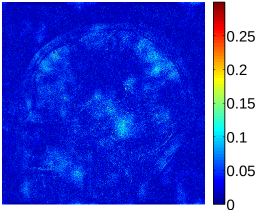
DLMRI (28.54 dB)



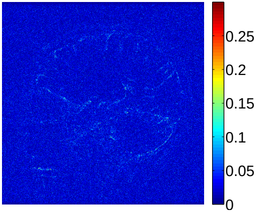
TLMRI (30.47 dB)



Sampling Mask



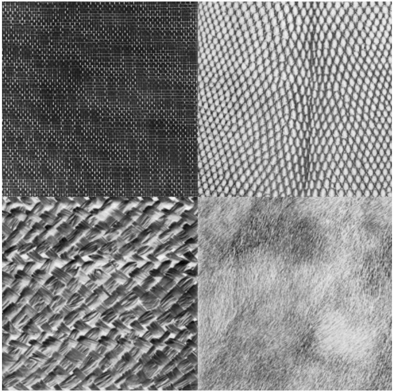
DLMRI Error



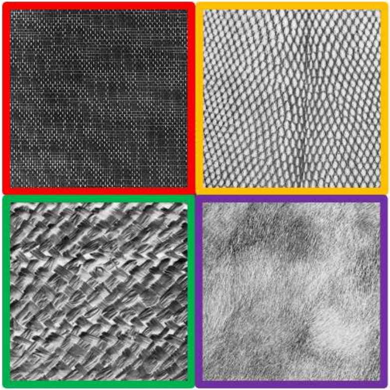
TLMRI Error

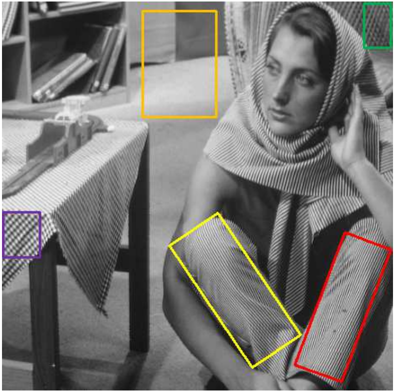
The More the Merrier?

- A single square transform is learned in the Basic TL Algorithm for all the data.
- But, natural images typically have diverse features or textures.



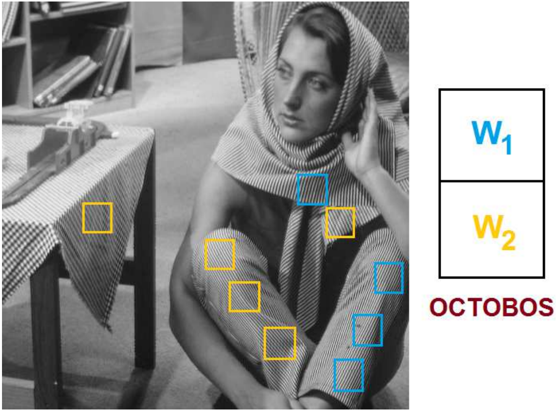
- **Union of transforms: one for each class of textures or features.**





OCTOBOS Learning Idea

- Group patches based on their match to a common transform.
- **Learn the transforms + cluster the data jointly**



OCTOBOS Learning Formulation

- **Goal:** jointly learn a union-of-transforms $\{W_k\}$ and cluster the data Y.

$$(\mathrm{P2}) \quad \min_{\{W_k, X_i, C_k\}} \overbrace{\sum_{k=1}^{K} \sum_{i \in C_k} \|W_k Y_i - X_i\|_2^2}^{\text{Sparsification Error}} + \overbrace{\sum_{k=1}^{K} \lambda_k \left(\|W_k\|_F^2 - \log |\det W_k| \right)}^{\text{Regularizer} = \sum_{k=1}^{K} \lambda_k v(W_k)}$$

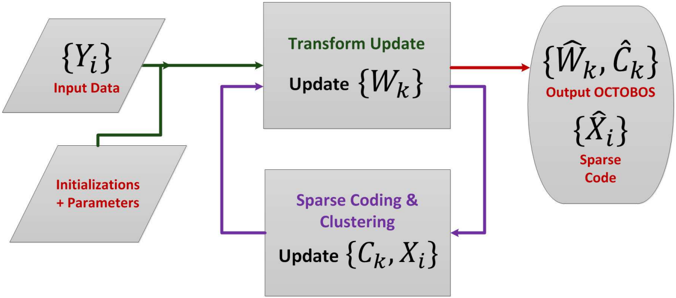
$$\text{s.t.} \quad \|X_i\|_0 \leq s \;\; \forall\, i, \quad \{C_k\}_{k=1}^{K} \in G$$

- C_k is the set of indices of signals in class k.
- G is the set of all possible partitions of $[1 : N]$ into K disjoint subsets.
- The regularizer controls the scaling and conditioning of the transforms

Alternating Minimization Algorithm for (P2)

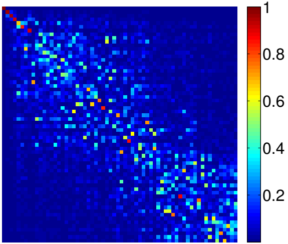
$$(\text{P2}) \quad \min_{\{W_k, X_i, C_k\}} \overbrace{\sum_{k=1}^{K} \sum_{i \in C_k} \|W_k Y_i - X_i\|_2^2}^{\text{Sparsification Error}} + \overbrace{\sum_{k=1}^{K} \lambda_0 \|Y_{C_k}\|_F^2 v(W_k)}^{\text{Regularizer}}$$

$$\text{s.t.} \quad \|X_i\|_0 \leq s \;\; \forall i, \quad \{C_k\}_{k=1}^{K} \in G$$

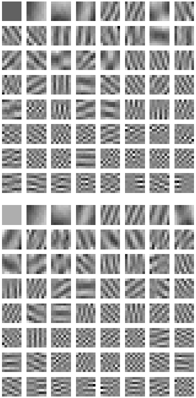


Visualization of Learned OCTOBOS

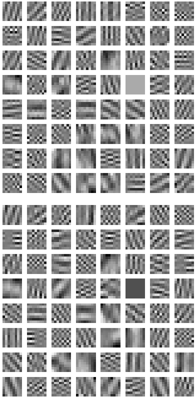
- The square blocks of a learnt OCTOBOS are **NOT** similar $\Rightarrow$ cluster-specific W_k.
- OCTOBOS **W** learned with different initializations appear different.
- **The W learned with different initializations sparsify equally well.**



Cross-gram matrix between W_1 and W_2 for KLT Init.



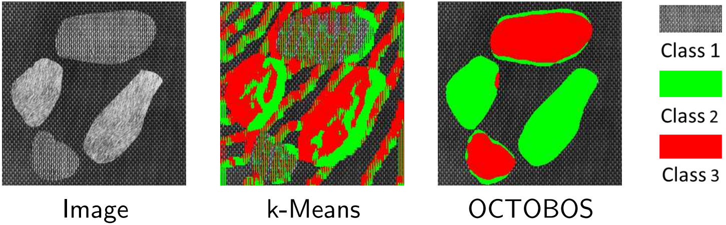
Random matrix Init.



KLT Init.

Example: Unsupervised Classification

- The overlapping image patches are first clustered by OCTOBOS learning
- Each image pixel is then classified by a majority vote among the patches that cover that pixel



Imaging:

Transform Blind Compressed Sensing with a Union of Transforms

UNITE-BCS: Union of Transforms Blind CS

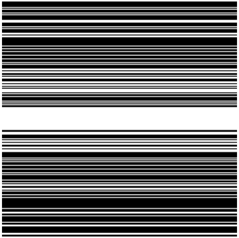
- **Goal:** learn union of transforms, reconstruct x, and cluster the patches of x, using only the undersampled y.
  - $\Rightarrow$ **model adaptive to underlying image.**

$$(\mathrm{P2}) \quad \min_{x, B, \{W_k, C_k\}} \nu \overbrace{\|Ax - y\|_2^2}^{\text{Data Fidelity}} + \overbrace{\sum_{k=1}^{K} \sum_{j \in C_k} \|W_k R_j x - b_j\|_2^2}^{\text{Sparsification Error}} + \eta^2 \overbrace{\sum_{j=1}^{N} \|b_j\|_0}^{\text{Sparsity Penalty}}$$

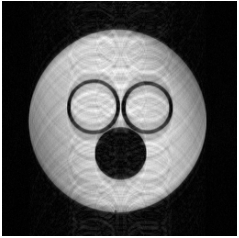
$$\text{s.t. } W_k^H W_k = I \ \forall\, k, \ \ \|x\|_2 \leq C.$$

- $R_j \in \mathbb{R}^{n \times P}$ extracts patches. $W_k \in \mathbb{C}^{n \times n}$ is a unitary cluster transform.
- $\|x\|_2 \leq C$ is an energy or range constraint. $B \triangleq [b_1 \mid b_2 \mid \ldots \mid b_N]$.
- Efficient alternating algorithm for (P2) with convergence guarantee

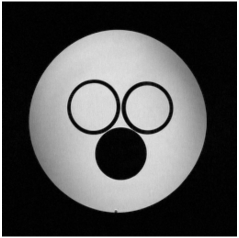
CS MRI Example - 2.5x Undersampling ($K = 3$)



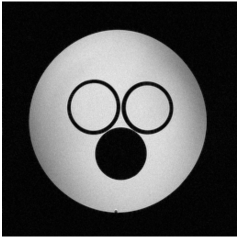
Sampling mask



Initial recon (24.9 dB)

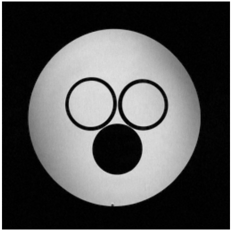


UNITE-MRI recon (37.3 dB)

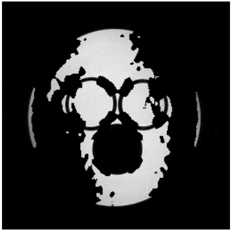


Reference

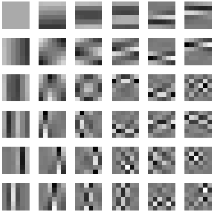
UNITE-MRI Clustering with $K = 3$ ($\eta = 0.07$, $\nu = 15.3$)

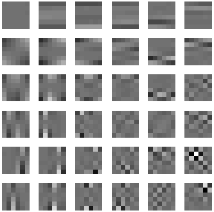

UNITE-MRI recon


Cluster 1

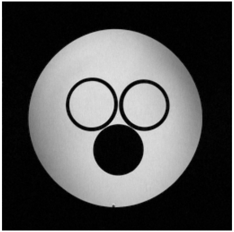

Cluster 2


Cluster 3

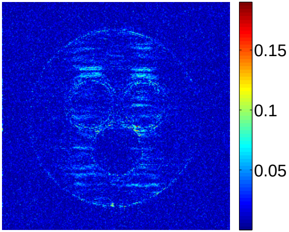

Real part of learned W for cluster 2


Imaginary part of learned W for cluster 2

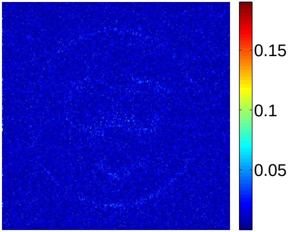
Reconstructions - Cartesian 2.5x Undersampling ($K = 16$)



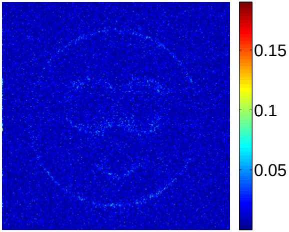
UNITE-MRI recon (37.4 dB)



PANO[11] error (34.8 dB)



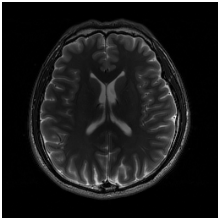
UNITE-MRI error



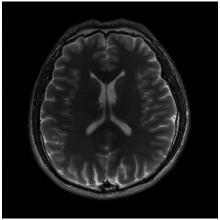
UTMRI ($K = 1$) error (37.2 dB)

[11] [Qu et al. '14]

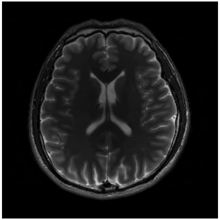
Example - Cartesian 2.5x Undersampling ($K = 16$)



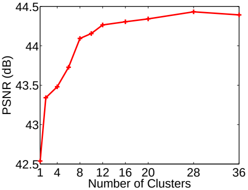
Reference



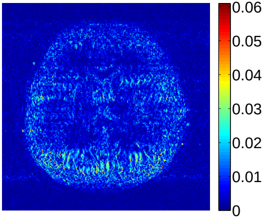
UTMRI (42.5 dB)



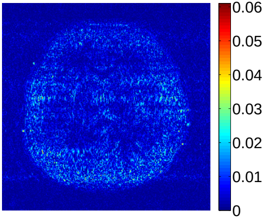
UNITE-MRI (44.3 dB)



PSNR vs. K



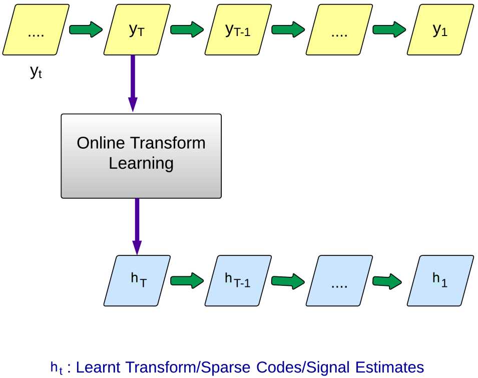
UTMRI Error



UNITE-MRI Error

Online Transform Learning for Dynamic Imaging and Big Data

Online Transform Learning



- Big data ⇒ large training sets ⇒ batch learning (using all data) is computationally expensive in time and memory.
- Streaming data ⇒ must be processed sequentially to limit latency.
- Online learning involves cheap computations and memory usage.

Online Transform Learning Formulation

- For $t = 1, 2, 3, \ldots,$ solve

$$\text{(P3)}\ \left\{\hat{W}_t, \hat{x}_t\right\} = \underset{W,\, x_t}{\arg\min}\ \frac{1}{t}\sum_{j=1}^{t}\left\{\|Wy_j - x_j\|_2^2 + \lambda_j v(W)\right\}$$
$$\text{s.t. } \|x_t\|_0 \leq s,\ x_j = \hat{x}_j,\ 1 \leq j \leq t-1.$$

- $\lambda_j = \lambda_0 \|y_j\|_2^2$. λ_0 controls condition number and scaling of $\hat{W}_t \in \mathbb{R}^{n\times n}$.
- Denoised image estimate $\hat{y}_t = \hat{W}_t^{-1}\hat{x}_t$ is computed efficiently.
- For non-stationary data, use forgetting factor $\rho \in [0, 1]$, to diminish the influence of old data.

$$\frac{1}{t}\sum_{j=1}^{t}\rho^{t-j}\left\{\|Wy_j - x_j\|_2^2 + \lambda_j v(W)\right\} \tag{12}$$

Online Transform Learning Algorithm

- **Sparse Coding** - solve for x_t in (P3) with fixed $W = \hat{W}_{t-1}$: **Cheap Solution:** $\hat{x}_t = H_s(Wy_t)$.

- **Transform Update:** solve for W in (P3) with $x_t = \hat{x}_t$. Cheap, closed-form update using SVD rank-1 update.
- **No matrix-matrix products.** Approx. error bounded, and cheaply monitored.

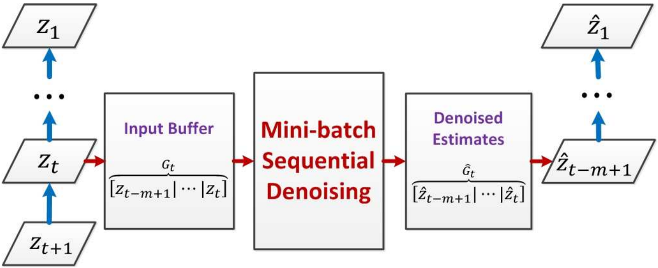
Online Transform Learning (OTL) Convergence Results

- **Assumption:** y_t are i.i.d. random samples from the sphere $S^n = \{y \in \mathbb{R}^n : \|y\|_2 = 1\}$.
- Consider the minimization of the expected learning cost:

$$g(W) = \mathbb{E}_y \left[\|Wy - H_s(Wy)\|_2^2 + \lambda_0 \|y\|_2^2 \, v(W) \right] \tag{13}$$

- Mild assumptions: Exact computations, Nondegenerate SVDs.
- **Main Result: $\hat{W}_t$ in OTL converges to the set of stationary points of $g(W)$ almost surely.** $\hat{W}_{t+1} - \hat{W}_t \sim O(1/t)$.

Online Video Denoising by 3D Transform Learning

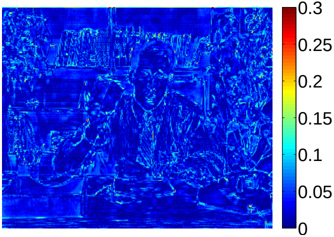


- z_t is a noisy video frame. $\hat{z}_t$ is its denoised version.
- G_t is a tensor with m frames formed using a sliding window scheme.
- Overlapping 3D patches in the G_t's are denoised sequentially.
- Denoised patches averaged at 3D locations to yield frame estimates.

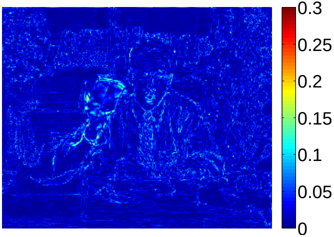
Video Denoising Example: Salesman

Noisy frame

VIDOLSAT (PSNR = 30.97 dB)



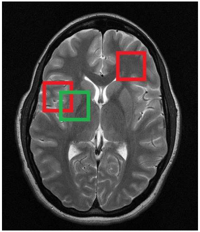
VBM4D[12] Error (PSNR = 27.20 dB)



VIDOLSAT Error

[12] [Maggioni et al. '12]

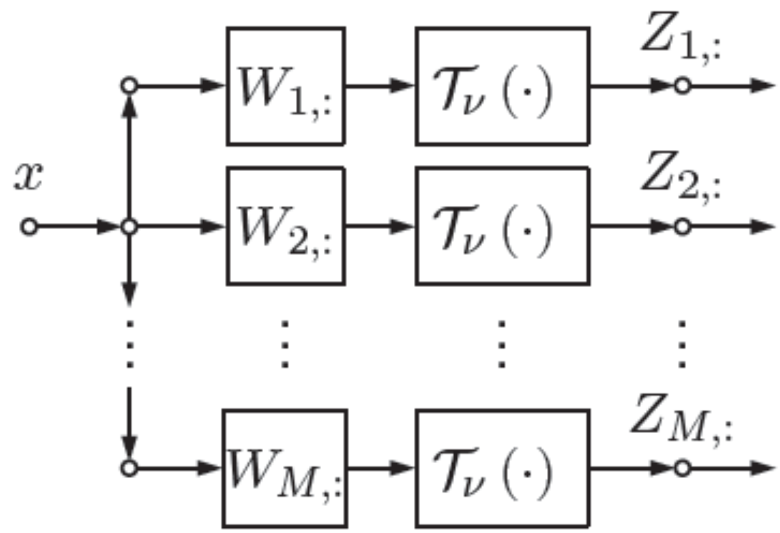
From Patches To Filter Banks



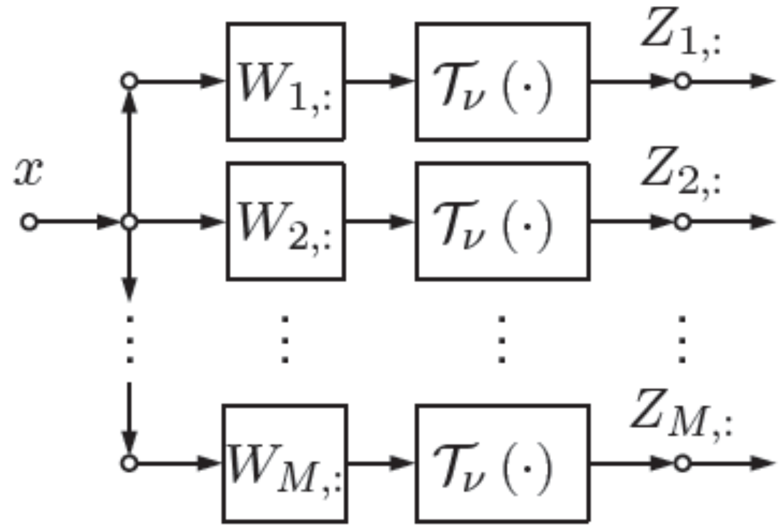
Patches of image

- $Y_j = R_j y$, $j = 1,..\text{N}$: jth image patch, vectorized.
- $Y = [\, Y_1 \,|\, Y_2 \,|\, ... \,|\, Y_N \,] \in \mathbb{R}^{n \times N}$: matrix of vectorized patches - training signals

Sparsifying Transforms as Filter Banks for Maximally Overlapping Patches

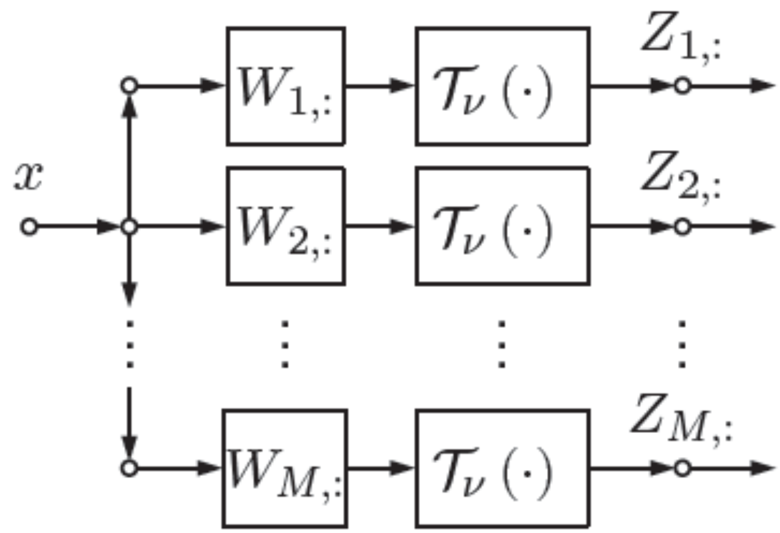


Sparsifying Transforms as Filter Banks for Maximally Overlapping Patches



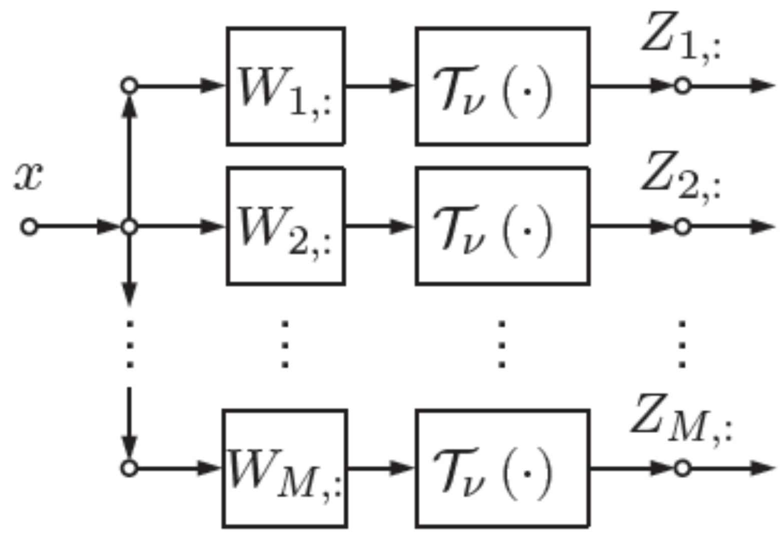
- $\operatorname{vec}(WX) = \mathcal{H}_W x$

Sparsifying Transforms as Filter Banks for Maximally Overlapping Patches



- $\text{vec}(WX) = \mathcal{H}_W x$
- Defn: $\mathcal{H}_W$ is perfect reconstruction (PR) if $\mathcal{H}_W$ left invertible (LI).

Sparsifying Transforms as Filter Banks for Maximally Overlapping Patches



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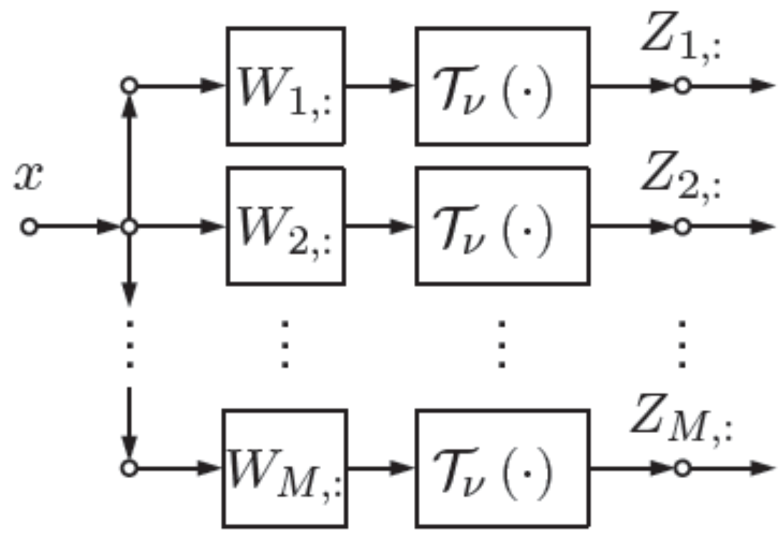

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  - Rows of W $\rightarrow$ channels of filter bank
  - W is LI $\Rightarrow$ $\mathcal{H}_W$ is PR

Sparsifying transforms as filter banks

- Take away: Existing transform learning algorithms learn perfect reconstruction filter banks!
- ... But, requiring W to be LI is stronger than requiring $\mathcal{H}_W$ to be PR!
- Two questions:
  1. Do we benefit by requiring $\mathcal{H}_W$ to be PR and relaxing the LI condition on W?
  2. Can we find an efficient algorithm to learn such an $\mathcal{H}_W$?

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Previous Work

- Connection between patch-based analysis operators and convolution previously known
- Convolution often used as a computational tool

The Key Property

- Each frequency must pass through at least one channel!

Diagonalization

$$C_W^H C_W = \Phi^H \operatorname{ddiag}\left(\left| \bar{\Phi} W^T \right|^2 \mathbf{1}_{N_c} \right) \Phi$$

Perfect Recovery Condition

$\mathcal{H}_W$ is PR $\Leftrightarrow$ each entry of $\left| \bar{\Phi} W^T \right|^2 \mathbf{1}_{N_c} > 0$

- Decouples the choice of number of channels N_c and patch size (support of transform) $K \times K$
- Especially attractive for high dimensional data

Learning a sparsifying filter bank

Learning Formulation

- Desiderata
  - Parameterize with few degrees of freedom
  - $\mathcal{H}_W x$ should be (approximately) sparse
  - $\mathcal{H}_W$ should be PR and well conditioned
  - No identically zero filters
  - No duplicated filters

Learning Formulation

- $\mathcal{H}_W x$ should be (approximately) sparse
- $\Longrightarrow WX$ should be (approximately) sparse

$$F(W, Z, x) \triangleq \frac{1}{2}\|WX - Z\|_F^2 + \nu\|Z\|_0$$

Learning Formulation

- $\mathcal{H}_W$ should be PR and well conditioned
- Let ζ_i be an eigenvalue of $\mathcal{H}_W^H \mathcal{H}_W$

$$
\begin{aligned}
\sum_{i=1}^{N^2} f(\zeta_i) &= \sum_{i=1}^{N^2} \frac{\zeta_i^2}{2} - \log {\zeta_i}^2 \\
&= 0.5 \sum_{i=1}^{N^2} \sum_{j=1}^{N_c} (\left|\bar{\Phi} W^T\right|^2)_{i,j} - \log \left(\sum_{j=1}^{N_c} (\left|\bar{\Phi} W^T\right|^2)_{i,j} \right)
\end{aligned}
$$

Learning Formulation

- No identically zero filters

$$-\beta \sum_{j=1}^{N_c} \log\left(\|W_{j,:}\|_2^2\right)$$

Learning Formulation

- $\mathcal{H}_W$ should be PR and well conditioned
- No identically zero filters

$$J_1(W) = 0.5 \sum_{i=1}^{N^2} \sum_{j=1}^{N_c} (\left|\bar{\Phi} W^T\right|^2)_{i,j} - \log\left(\sum_{j=1}^{N_c} (\left|\bar{\Phi} W^T\right|^2)_{i,j}\right) - \beta \sum_{j=1}^{N_c} \log\left(\|W_{j,:}\|_2^2\right)$$

Learning Formulation

- No duplicated filters

$$J_2(W) = \sum_{1 \le i < j \le N_c} - \log \left(1 - \left(\frac{\langle W_{i,:}, W_{j,:} \rangle}{\|W_{i,:}\|_2 \|W_{j,:}\|_2} \right)^2 \right)$$

Learning Formulation

$$\min_{W,Z} \frac{1}{2}\|WX - Z\|_F^2 + \alpha J_1(W) + \gamma J_2(W) + \nu\|Z\|_0$$

Alternating minimization:

- $Z^{k+1} = \arg\min_Z \frac{1}{2}\|W^k X - Z\|_F^2 + \nu\|Z\|_0$
- $W^{k+1} = \arg\min_W \frac{1}{2}\|WX - Z^{k+1}\|_F^2 + \alpha J_1(W) + \gamma J_2(W)$

Application to Magnetic Resonance Imaging

Imaging Model

- Imaging Model: Undersampled Fourier measurements

$$y = \Gamma\Phi x + e$$

- $x \in \mathbb{R}^{N^2}$: Input image
- $\Phi \in \mathbb{C}^{N^2 \times N^2}$: DFT matrix
- $\Gamma \in \mathbb{C}^{M \times N^2}$: Row selection matrix
- $e \in \mathbb{C}^{M}$: Zero mean Gaussian noise

Image Reconstruction - Transform Blind Compressed Sensing

$$\min_{x,\mathcal{H}_W,z} \frac{1}{2}\|y - \Gamma\Phi x\|_2^2 + \lambda \left(\frac{1}{2}\|\mathcal{H}_W x - z\|_2^2 + \nu\|z\|_0 + \alpha J_1(\mathcal{H}_W) + \gamma J_2(\mathcal{H}_W) \right)$$

- Data fidelity
- Transform learning
- Solve using alternating minimization

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Image Reconstruction - Transform Blind Compressed Sensing

Image Update

$$\min_{x} \frac{1}{2}\|y - \Gamma\Phi x\|^2 + \frac{\lambda}{2}\|\mathcal{H}_W x - z\|_2^2$$

Transform Update

$$\min_{W} \frac{1}{2}\|\mathcal{H}_W x - z\|_F^2 + \alpha J_1(W) + \gamma J_2(W)$$

Sparse Code Update

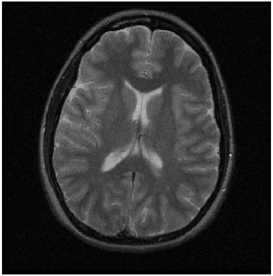
$$\min_{z} \frac{1}{2}\|\mathcal{H}_W x - z\|_2^2 + \nu\|z\|_0$$

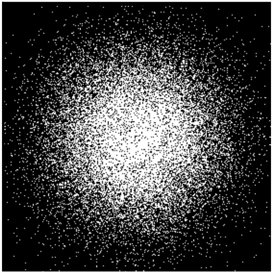
Experiments

- Synthetic MR data from magnitude image
- $\approx$ 5 fold undersampling
- Vary filter size $\&$ number of channels
- Compare against square patch-based transform learning:

$$\min_{W,x,Z} \frac{1}{2}\|y - \Gamma\Phi x\|_2^2 + \frac{\lambda}{2}\|WX - Z\| + \nu\|Z\|_0 + \alpha\|W\|_F^2 - \beta \log\det W$$

- Solved using alternating minimization
- Initialized with DCT matrix

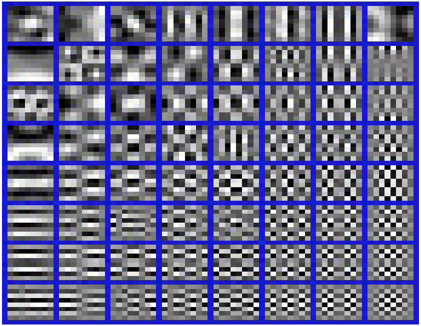


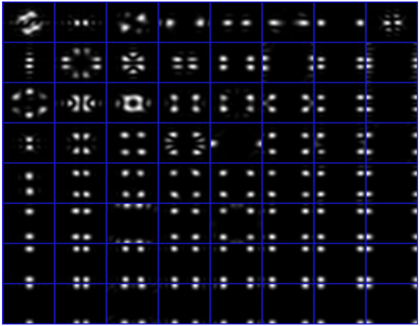


Reconstruction PSNR (dB)

σ / PSNR In	Filter Bank			Patch Based
	$N_c = 64$, $K = 8$	$N_c = 128$, $K = 8$	$N_c = 64$, $K = 12$	64×64
0 / 29.6	35.2	**35.2**	35.1	34.6
$\frac{10}{255}$ / 28.8	32.6	**32.7**	32.6	32.5
$\frac{20}{255}$ / 26.9	**31.6**	31.6	31.2	31.3

Learned filters 8×8





Conclusion

- New framework for learning filter bank sparsifying transforms
- Replace patch recovery conditions with image recovery
- Decouples number of channels from filter length
- Can outperform patch-based transform for MR reconstruction

Summary

- We introduced several data-driven sparsifying transform adaptation techniques.
- Proposed learning methods
  - are highly efficient and scalable
  - enjoy good theoretical and empirical convergence behavior
  - are highly effective in many applications
- Highly promising results obtained using transform learning in denoising and compressed sensing.
- Papers and software available for download at http://transformlearning.csl.illinois.edu

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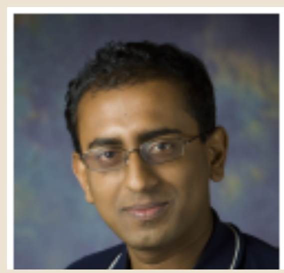
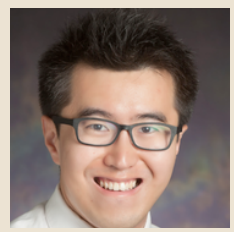
Thank You!

Work with

- Sairprasad Ravishankar
- Bihan Wen
- Luke Pfister





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