COOK'S THEOREM

1 Cook's Theorem

Cook's theorem shows that the satisfiability problem is NP-complete. Without loss of generality, we assume that languages in NP are over the alphabet $\{0,1\}^*$. Lemma 1, useful for the proof, states that we can restrict the form of a computation of a NTM that accepts languages in NP.

Lemma 1 *If $L \in NP$, then L is accepted by a 1-tape NTM N with alphabet $\{0,1\}$ such that for some polynomial $p(n)$, the following properties hold.*

- *N's computation is composed of two phases, the guessing phase and the checking phase.*
- *In the guessing phase, N nondeterministically writes a string $\sqcup y$ directly after the input string, and in the checking phase, N behaves deterministically.*
- *N uses at most $p(n)$ tape cells, never moves its head to the left of w, and takes exactly $p(n)$ steps in the checking phase.*

$\boxed{p(n)}$

A Boolean formula f over variable set V is in *conjunctive normal form (CNF)* if

$$\boxed{f = \wedge_{i=1}^{m} \vee_{j=1}^{k_i} l_{i,j}}$$

for some values of m and k_i, $1 \leq i \leq m$, where literal $l_{i,j}$ is either x or $\bar{x}$ for some $x \in V$. For each i, the term $\vee_{j=1}^{k_i} l_{i,j}$ is called a clause of the formula. f is *satisfiable* if there exists a truth assignment to the variables in V that sets f to true. CNFSAT is the set of satisfiable Boolean formulas in CNF.

$(\bar{x} \vee \vee) \wedge (-$

Theorem 1 (Cook's Theorem) *CNFSAT is NP-complete.*

Proof sketch: It is not hard to show that CNFSAT $\in$ NP. To prove that CNFSAT is NP-complete, we show that for any language $L \in$ NP, $L <_m^p$ CNFSAT.

Let $L \in$ NP and let N be a NTM accepting L that satisfies the properties of Lemma 1. Let the transition function of N be δ. Let the states of N be $q_0, ..., q_r$. Let s_0, s_1, s_2 denote $0, 1, \sqcup$, respectively. Assume that the tape cells are numbered consecutively from the left end of the input, starting at 0. On input w of length n, we show how to construct a formula in CNF form f_w, which is satisfiable if and only if w is accepted by N. The variables of f_w are as follows:

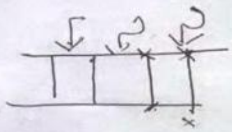
Variables	Range	Meaning
$Q[i,k]$	$0 \leq i \leq p(n)$ $0 \leq k \leq r$	At step i of the checking phase, the state of N is q_k.
$H[i,j]$	$0 \leq i \leq p(n)$ $0 \leq j \leq p(n)$	At step i of the checking phase, the head of N is on tape square j.
$S[i,j,l]$	$0 \leq i \leq p(n)$ $0 \leq j \leq p(n)$ $0 \leq l \leq 2$	At step i of the checking phase, the symbol in square j is s_l.

$p(n) \times p(n)$

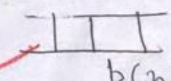
$\leq$ W

A computation of N naturally corresponds to an assignment of truth values to the variables. Other assignments to the variables may be meaningless. For example, an assignment with $Q[i,k] = Q[i,k'] =$ true, $k \neq k'$, would imply that N is in two different states at step i, which is impossible.

Our goal is to construct f_w so that it is satisfied only by assignments to the variables that correspond to accepting computations of N on w. The clauses of f_w are constructed to ensure that the following conditions are satisfied:



1. At each step i of the checking phase, N is in exactly 1 state.
2. At each step i, the head is on exactly one tape square.
3. At each step i, there is exactly 1 symbol in each tape square.
4. At step 0 of the checking phase, the state is the initial state of N in its checking phase, and the tape contents are $w \sqcup y$ for some y.
5. At step $p(n)$ of the checking phase, N is in an accepting state.



$p(n)$

6. The configuration of N at the $(i+1)$st step follows from that at the i^{th} step, by applying the transition function of N.

Consider condition 1. For each i, we have the following clause:

$$Q[i,0] \vee Q[i,1] \vee \ldots \vee Q[i,r].$$

This clause ensures that the machine is in at least 1 state at step i. We also need clauses to ensure that N is not both in state q_j and $q_{j'}$:

$$\overline{Q[i,j]} \vee \overline{Q[i,j']} \text{ for each } j \neq j', 0 \leq j, j' \leq r.$$

Conditions 2 and 3 are handled similarly. Conditions 4 and 5 are quite easy. Finally, consider condition 6. For each (i,j,k,l) we add clauses that ensure the following: If at step i, the tape head of N is pointing to the j^{th} tape cell, N is in state q_k, s_l is the symbol under the tape head, and $(q_k, s_l, q_{k'}, s_{l'}, X) \in \delta$, where $X \in \{\text{L}, \text{R}\}$ then at step $i+1$, the tape head is pointing to the $(j+y)^{th}$ tape cell where $y = 1$ if $X = \text{R}$ and $y = -1$ if $X = \text{L}$, N is in state $q_{k'}$ and the symbol in cell j is $s_{l'}$. The following clauses ensure this:

$$\overline{Q[i,k]} \vee \overline{H[i,j]} \vee \overline{S[i,j,l]} \vee Q[i+1,k']$$
$$\overline{Q[i,k]} \vee \overline{H[i,j]} \vee \overline{S[i,j,l]} \vee H[i+1,j+y]$$
$$\overline{Q[i,k]} \vee \overline{H[i,j]} \vee \overline{S[i,j,l]} \vee S[i+1,j,l']$$

All of the clauses for condition 1 to 6 can be computed in polynomial time (how many clases are there?). Moreover, w is accepted by N if and only if f_w is satisfiable. □

$Q[i,0] \vee Q[i,1] \vee \ldots \vee Q[i,r]$

$\overline{Q[i,j]} \vee \overline{Q[i,j']}$

T T

j

q_k

$j+y$

$\overline{Q[i,k]} \vee \overline{H[i,j]} \vee \overline{[S,i,j,l]} \vee [\longrightarrow]$

$Q[i,k] \vee$