The Remarkable Inverse Distance-Squared Law

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Abstract – Numerous fundamental physical laws depend on inverse distance as distance squared. The reason for why this distance must be PRECISELY SQUARED is examined.

I. Physical Laws That Depend on Inverse Separation Distance Squared

A large number of physical laws depend on inverse distance squared as $1/R^2$: NOT $1/R^{1.999}$, NOT $1/R^{2.001}$, etc. There is a reason why this precise squared integer power of the distance in these laws is required. This reason will be explained.

Perhaps the most famous inverse distance-squared law is the law of gravity where the force exerted on one body by the presence of a nearby body varies as the product of the masses of the two bodies and as the inverse of the square of the distance R between them: $1/R^2$. Another of the inverse distance-squared laws in electromagnetics is that of Coulomb's law for the vector force exerted by one stationary point charge on another nearby stationary point charge [1]:

$$\mathbf{F} = \frac{1}{4\pi\,\varepsilon_0}\frac{Q_1\,Q_2}{R^2}\mathbf{a}_R$$

as illustrated in Fig. 1 where $\mathbf{a}_R$ is a unit vector on a line between the charges and pointing away from the charges if the charges have the same sign.

The electric field produced by a stationary point charge is obtained by dividing out the second charge in Coulomb's law which remains an inverse distance-squared law as shown in Fig. 2:

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{Q}{4\pi\,\varepsilon_0 R^2}\mathbf{a}_R$$

The corresponding law for determining the magnetic field due to a linear, DC differential current element is the Biot-Savart law [1]:

$$d\mathbf{B} = \frac{\mu_0\,I}{4\pi\,R^2}d\mathbf{l} \times \mathbf{a}_R$$

which is illustrated in Fig. 3.

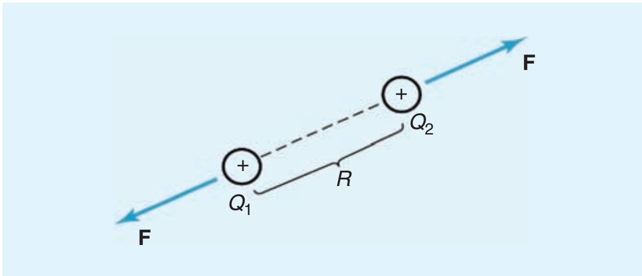

Fig. 1. Coulomb's law for two stationary point charges.

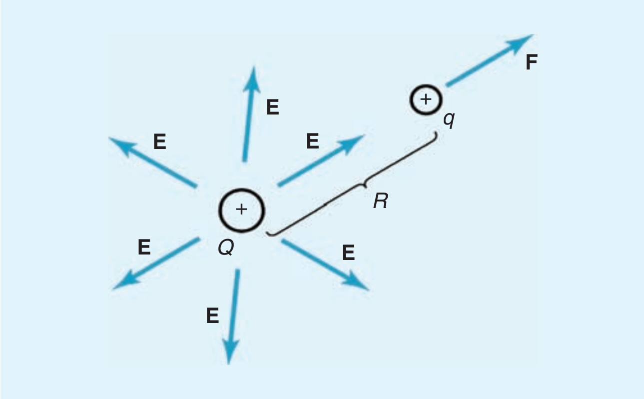

Fig. 2. Electric field of a point charge.

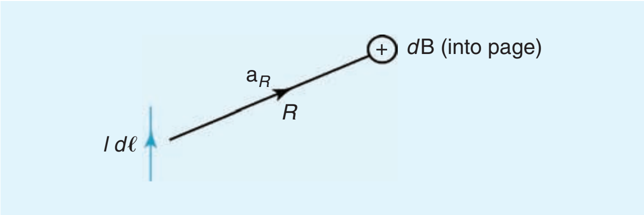

Fig. 3. The Biot-Savart law.

Note that this is also an inverse distance-squared law. The differential magnetic field, $d\mathbf{B}$, is perpendicular to the plane containing the differential current and the vector *from* the current *to* the point where the field is to be determined. Its direction is determined according to the right-hand rule. Hence the magnetic field forms closed loops in the circumferential direction around the current, I, that produced it.

II. Vector Mathematical Principles

The key to why the inverse distance, $1/R^2$, in the many physical laws must be precisely squared and not some other approximately square law power of distance such as $1/R^{1.999}$ or $1/R^{2.001}$ depends on the vector mathematics involved. All vector problems require a coordinate system. A useful coordinate system in electromagnetics problems is the spherical coordinate system shown in Fig. 4 which also shows the differential surfaces [1]. Each differential surface is perpendicular to the coordinate axis. For example, ds_r is a differential surface that is perpendicular to the radial distance from the origin of the coordinate system, r.

The physical electromagnetics laws are frequently used in integrals over a *closed* surface s for the purposes of determining the *net flux* of the law *out of (leaving) the closed surface* (much like light flux through a window) as

$$\text{Flux of the Law } \textit{out of} \text{ the closed surface } s = \oint\!\!\oint_{s} \textbf{Law} \bullet d\mathbf{s}$$

This is referred to as a *surface integral* and represents the summation of the products of the differential surfaces, ds, and the components of the vector Law that are *perpendicular to (leaving) the closed surface*. Note in Fig. 4 that the differential surface through which the flux of the Law penetrates and *flows away from the origin of the coordinate system* is

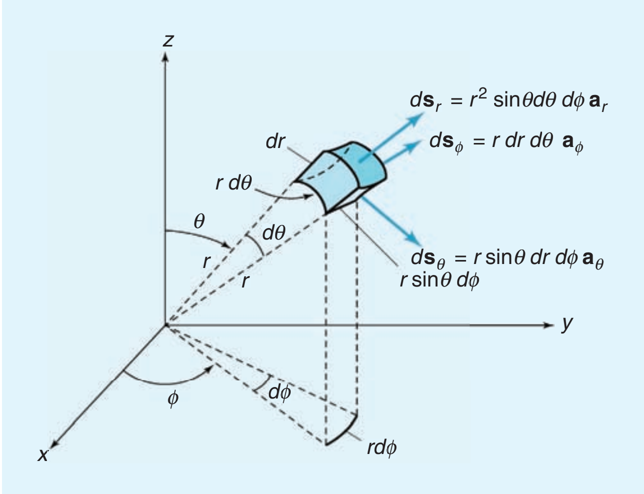

Fig. 4. The differential surfaces in a spherical coordinate system.

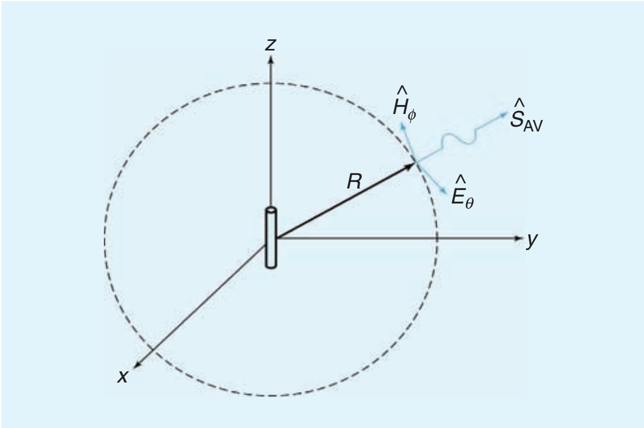

Fig. 5. Computation of the radiated power from an antenna.

$$ds_r = r^2 \sin\theta\; d\theta\; d\phi$$

which involves the *square of the distance from the origin of the coordinate system*. Observe that neither of the other differential surfaces, ds_ϕ and ds_θ, are distance squared.

III. Examples Where The Inverse Square Distance Must Be Present In The Law or The Result Would Make No Sense

An important example that shows the need for the inverse distance-squared law is the computation of the total average power radiated into space by an antenna as shown in Fig. 5.

Surround the antenna by a closed, spherical surface of radius R. The total power radiated by the antenna is the power *out of* the closed surface. The time-varying radiated electric field $\left({}^{\text{V}}\!/_{\text{m}}\right)$ and radiated magnetic field $\left({}^{\text{A}}\!/_{\text{m}}\right)$ in the *far field* of the antenna both depend on the distance from the origin of the coordinate system as $1/R$ [1]. Hence the *power density* in the radiated wave is dependent on $1/R^2$:

$$S_{AS} \propto \frac{1}{R^2} \quad \frac{\text{W}}{\text{m}^2}$$

and is directed in the radial direction. The surface integral for computing the *total* average power radiating into space from the antenna through *any closed surface that encloses the antenna* becomes:

$$P_{AV} = \oint\!\!\oint_S \mathbf{S}_{AV} \bullet d\mathbf{s}$$
$$= \oint\!\!\oint_S S_{AV} \underbrace{R^2 \sin\theta \, d\theta \, d\phi}_{ds_r}$$

Therefore the $1/R^2$ in S_{AV} and the R^2 in the differential surface *cancel* and the integrand is *independent* of R. Hence the total power radiated into space (never to return),P_{AV}, would be a constant independent of the size or shape of the closed surface! If this cancellation of R^2 were not the case, the integrand of the integral would be a function of R and *we could change the radius of the closed surface thereby changing the total power radiated from the antenna.* But the total average power radiated into space by an antenna *must be a constant*! Hence the absence of the complete cancellation of R would make absolutely NO SENSE! In order for the power of R^2 in the differential surface, ds_r to cancel, the power of R^2 in S_{AV} must be precisely 2.000000· · ·. Powers of approximately 2 such as 1.999 and 2.001 in S_{AV} will not work since these will not cancel with the R^2 in the differential surface, and this fact is a vector algebra property and is not approximatable.

There are several other cases that show this dependence on the inverse-square law. First consider Gauss' law for the electric field [1]. Gauss' law provides that *if we perform a surface integral of the electric field over a closed surface that surrounds some charge, we would obtain as the result the net positive charge contained within that closed surface irrespective of the shape of the closed surface so long as the surface is closed*:

$$\oint\!\!\oint_S \mathbf{D} \bullet d\mathbf{s} = Q_{\text{enclosed}}$$

as illustrated in Fig. 6 where $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and ε_0 is the permittivity of free space. Electric field lines that begin on a positive charge must end on a corresponding negative charge as illustrated in Fig. 6. Hence electric field lines that begin on positive charge within the closed surface must terminate on corresponding negative charge which exists either within the closed surface or outside it. If the negative charge exists within the closed surface, the associated field line does not penetrate the closed surface. If the associated negative charge exists outside the closed surface, the associated field line must *penetrate* the closed surface.

For example, consider a point charge shown in Fig. 7. Placing a sphere of radius r around the charge, the net flux out of the closed surface is obtained with a surface integral as

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 r^2} \underbrace{r^2 \sin\theta \, d\theta \, d\phi}_{ds_r}$$
$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi = Q$$

Hence the $1/r^2$ in **D** (or **E**) *cancels* the r^2 in ds and the result would be *independent of the size of the enclosing sphere which makes sense*. If this cancellation did not occur, i.e., the integrand did not contain $1/r^2$ in **D** (or **E**) and r^2 in $d\mathbf{s}$, the flux of the result (the charge contained within the sphere, Q) *would be different for different sizes of the sphere* (dependent on r) which does not make sense!

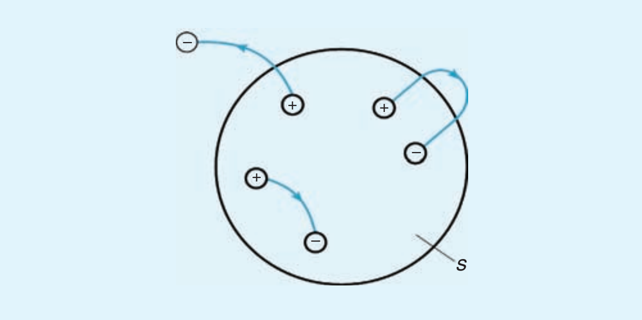

Fig. 6. Gauss' law for the electric field.

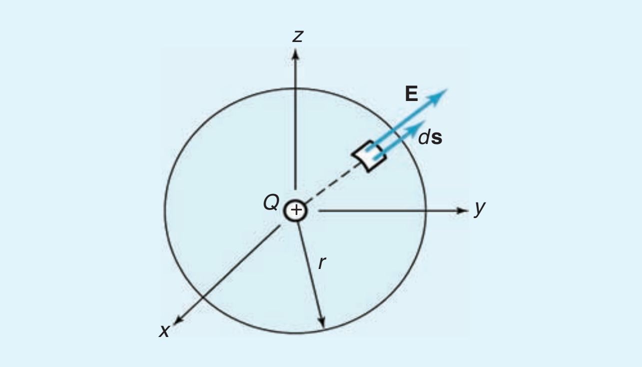

Fig. 7. Electric field of a point charge.

The corresponding law for the magnetic field is Gauss' law for the magnetic field as illustrated in Fig. 8 [1]. Gauss' law provides that a surface integral of the magnetic field over any closed surface yields a result of zero:

$$\oint\!\!\oint_S \mathbf{B} \bullet d\mathbf{s} = 0$$

Hence all magnetic field lines must form *closed loops* or, in other words, unlike the electric field due to a stationary charge, there are no known sources or sinks of the magnetic field. As we saw earlier, the DC magnetic field **B** depends on distance as $1/R^2$ as does the electric field both of which completely cancel the R^2

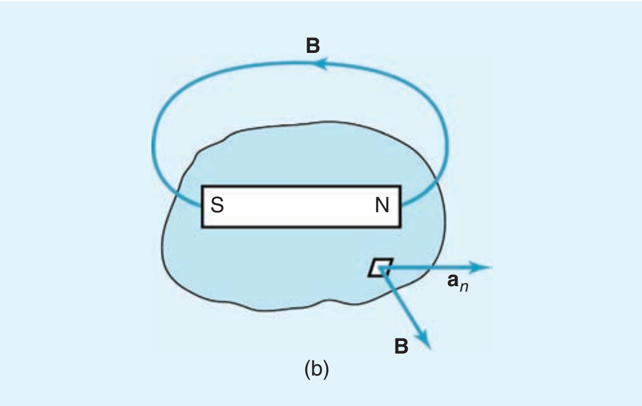

Fig. 8. Gauss' law for the magnetic field.

in ds_r. The result would be *independent of the size or shape of the enclosing surface which makes sense*.

If the law being considered did not contain the inverse squared-distance, $1/R^2$, it would not cancel the R^2 in the differential surface ds_r on the closed surface and the result would not be *independent* of the size of the closed surface which would again make no sense. A power of R of other than exactly two, $2.000000\cdots$, such as 1.999 or 2.001 would also not provide cancellation and would also not make any sense.

IV. Summary

Since I started studying science and, in particular electromagnetics, some 48 years ago I was always profoundly impressed at how the physical laws and their mathematical formulations are so precise and none are arbitrary. These are examples of that observation. It also seems to indicate that the creation of this universe was not accidental or random but was the result of planning by some higher power.

Reference

[1] C.R. Paul, *Electromagnetics for Engineers: with applications to digital systems and electromagnetic interference*, John Wiley, NY, 2004.

Biography

Clayton R. Paul *received the B.S. degree from The Citadel, Charleston, SC, in 1963, the M.S. degree from Georgia Institute of Technology, Atlanta, GA, in 1964, and the Ph.D. degree from Purdue University, Lafayette, IN, in 1970, all in Electrical Engineering. He is an Emeritus Professor of Electrical Engineering at the University of Kentucky where he was a member of the faculty in the Department of Electrical Engineering for 27 years retiring in 1998. Since 1998 he has been the Sam Nunn Eminent Professor of Aerospace Systems Engineering and a Professor of Electrical and Computer Engineering in the Department of Electrical and Computer Engineering at Mercer University in Macon, GA. He has published numerous papers on the results of his research in the Electromagnetic Compatibility (EMC) of electronic systems and given numerous invited presentations. He has also published 18 textbooks and Chapters in four handbooks. Dr. Paul is a Life Fellow of the Institute of Electrical and Electronics Engineers (IEEE) and is an Honorary Life Member of the IEEE EMC Society. He was awarded the IEEE Electromagnetics Award in 2005 and the IEEE Undergraduate Teaching Award in 2007.* **EMC**

Introducing Kye Yak See (SM'02)

Dr. See obtained his B. Eng from the National University of Singapore in 1986. From 1986 to 1994, he held various senior technical and management positions in the electronic industries in Singapore, the United Kingdom and Hong Kong. In 1994, he was awarded a scholarship by the Nanyang Technological University (NTU) to pursue his Ph.D. at Imperial College, United Kingdom. He joined NTU as a faculty member after obtaining his Ph.D. in 1997.

He is currently an Associate Professor of the School of Electrical and Electronic Engineering. He also holds the concurrent appointments of Deputy Head of the Division of Circuits and Systems and Director of Electromagnetic Effects Research Laboratory (EMERL).

He has co-authored three books and has published close to 90 refereed international journal and conference publications in the areas of EMC, signal integrity and computational electromagnetics. He is a senior member of IEEE, the founding chair of the IEEE EMC Society Singapore Chapter and a member of the Technical Committee on EMC. He was the Organizing Committee Chair for the 2006 EMC Zurich Symposium and the 2008 Asia Pacific EMC Conference. He was also one of the invited international speakers for the "Global EMC University" at the 2007 and 2008 IEEE International EMC Symposiums in the USA.

He looks forward to meeting authors of potential practical papers for this Newsletter at the EMC Europe 2011 Symposium over September 26–30.