Catadioptric projective geometry

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Abstract

In this paper we formulate a novel unifying geometry of catadioptric imaging. We prove that all single viewpoint mirror-lens devices are equivalent to a composite mapping from space to sphere and then from sphere to plane. The second mapping is equivalent to stereographic projection in case of parabolic mirrors. Using this equivalence we observe that images of lines in space are mapped to great circles on the sphere and to conic sections on the catadioptric image plane. The composite mappings are paired with a duality principle which relates points to line projections[1].

1 Introduction

Recent technology advances in omnidirectional sensors have inspired many researchers to rethink the way images are acquired and analyzed. It is a recurring theme that access to new devices opens new avenues in addressing basic questions. Novel solutions are proposed for the long-standing problems of visual navigation and 3D-reconstruction. Sensors designed to imitate the omnidirectional systems of flies and bees, for example, are replacing the conventional TV-cameras improving substantially the visual competences of robots.

In this paper, we present a unifying theory for all catadioptric systems with a unique effective viewpoint. We prove that all cases of a mirror surface—parabolic, hyperbolic, elliptic, planar—with the appropriate lens—orthographic or perspective—can be modeled with a projection from the sphere to the plane where the projection center is on a sphere diameter and the plane perpendicular to it. Singular cases of this model are stereographic projection, i.e. projection from the north pole, and perspective projection, i.e. projection from the sphere center.

Given this unifying projection model we establish two kinds of duality: A duality among point projections and line projections and a duality among two sphere projections from two different centers. We show that 3D-lines are projected onto conics whose foci build also a conic. If the 3D-lines are concurrent then the focus of the foci-conic is the projection of the line pencil and therefore dual to the foci-conic. In case of perspective projection all conics are degenerated to lines and we have the well known projective duality between lines and points in P^2.

2 History and Related Work

The geometry of the reflective properties of surfaces has been known since the 3rd century BC. Though Menaechmus (380-320 BC) was the one who coined the term conic section, it was Dio-

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[1]Though the principles of this theory were already presented at ECCV-2000, the derivation and proofs here are new. A similar version was presented at Vision Interface 2001 and an extended version has been submitted only to a journal.

cles (240-180 BC) [28] in his monograph *On Burning Mirrors* who first discovered the reflective properties of the parabola. Diocles' contemporary Apollonius (262-190 BC) [1] wrote an excellent comprehensive treatise on conic sections. Archimedes (287-212 BC) studied conic sections in specific applications like the quadrature of the parabola. In the meantime between Menaechmus and Apollonius, conic sections were mentioned by Euclid (325-265 BC) and Aristaeus (370-300 BC).

In optics, wide-angle lenses were used to capture a large field of view. Such fish-eye lenses, introduced radial distortions which were hard to model explicitly since the lenses were complex combinations of individual lens elements. In computer vision, we find such fish-eye lenses in the work of Aggarwal [24] where the issues of calibration and resolution are addressed.

From a different perspective, results in projective geometry showed that the mapping between images of a purely rotating camera is an easy to compute collineation. Several groups built mosaicking systems [27, 23] based on purely rotating cameras. The results of such approaches were panoramic mosaics with high quality but more suitable for photographic purposes. Mosaicking approaches have two main weaknesses: a) simultaneous, multi-directional image acquisition is impossible and b) they can only be successful with stationary scenes. Moreover, versatile mechanical designs with two rotational degrees of freedom are necessary in order to simulate a hemispherical field of view. Thus, while such techniques were useful for entertainment applications they proved inadequate for dynamic reactive behaviors needed in navigation or for continuous alertness in surveillance and telepresence.

Reflective properties of surfaces have been extensively documented in books about telescopes and antennas [6]. The term "catadioptric" meaning the combination of refractive and reflective elements has long been used for telescopes. A panoramic field of view camera was first proposed by Rees [22]. Later on, Greguss [10] introduced a combination of multiple refractive and reflective elements preserving a single viewpoint. Nalwa [15] introduced a pyramidal mirror-lens system with planar mirror faces. Basu [25] used a conical design for pipe-line inspection. The immersive aspect of omnidirectional vision is innovative with its ultimate purpose being visualization. Fuchs et al. [21] constructed a dodecahedral mirror system facing on twelve cameras so that the entire system will have one effective viewpoint. Immersive applications necessitate a unique effective viewpoint which will be the viewpoint of the human observer. Moreover, only a unique effective viewpoint allows an image mapping to a virtual image plane of any orientation. Boult [4] introduced a remote reality system where the tracker gives the orientation of the observer's head which then determines the section of the panoramic image and the normal of the virtual plane the image will be warped to. Onoe et al. [19], also, used a hyperbolic mirror for tele-presence and they study the accuracy of visualization vs. warping time. Bogner et al. [3] used a spherical mirror for remote immersive sensing in space. Recently, Peleg et al [20] designed a camera for stereoscopic panoramic visualization based on the concept of a cauctic curve.

Nayar [16, 2, 17] and Drucker [7] gave an exhaustive classification of all catadioptric surfaces satisfying the unique viewpoint constraint.

Note that most of the above work addresses visualization and not necessarily navigation or vision-based control. Camera configurations covering a sphere have been used by Fermüller [8] to prove the superiority over planar imaging surfaces in structure from motion. Tasks like locomotion as well as mab building and localization can greatly benefit from omnidirectional sensing [32, 30, 31]. Srinivasan [5] and Hicks [12] realized that other constraints than the viewpoint uniqueness might be imposed on the construction of a mirror. Srinivasan [5] derived a mirror shape which yields an image where the radius is linearly varying with the elevation of the projected object. In this sense, the mirror acts as a computational sensor which directly produces an elevation map without any software calculation. Hicks [12] derived another mirror shape which preserves a similitude transformation of the plane perpendicular to the mirror's axis. This is equivalent to a software rectification of this plane when other mirrors are used. The reader is also referred to the recent review by Yagi [29] as well as to the proceedings of the IEEE Workshop on Omnidirectional Vision 2000. Further information on omnidirectional systems can be found in `http://www.cis.upenn.edu/~kostas/omni.html`.

3 Central Catadioptric Projections

Catadioptric projections are a subset of a general type of projection. In a central catadioptric projection, a point is first projected to a conic from one of the foci and then this point is projected to an image plane from the second focus (see Figure 1). We could instead choose different points from which to project and different surfaces to intersect but these configurations may not induce optical projections which coincide with the abstract projections.

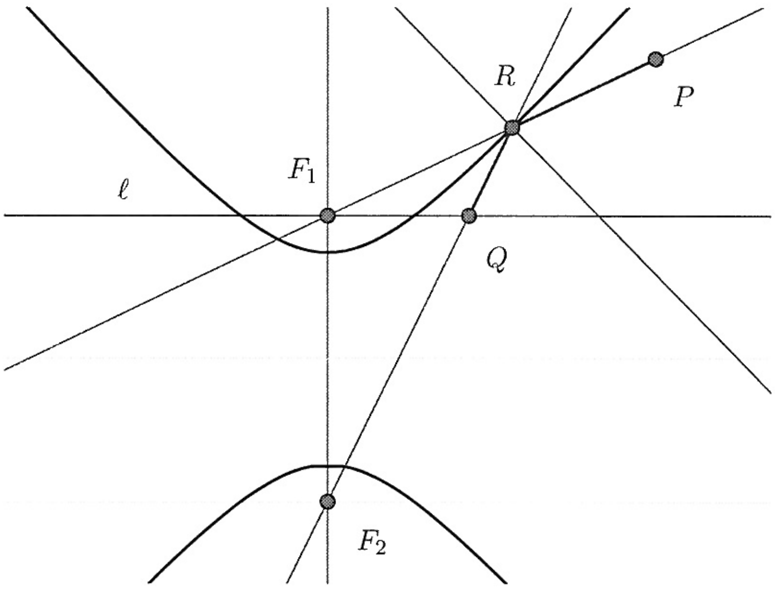

Figure 1: In general a conic reflects any ray of light incident with one of its foci (here F_1) to a ray of light incident with its other focus (F_2). Central catadioptric devices utilize this property and achieve a single effective viewpoint at one of the foci of a conic (F_1).

We use the notation $A \vee B$ to mean the line joining points A and B, and $l \wedge m$ to mean the point lying on both lines l and m. For notational convenience we have overloaded these operators to include quadratics, so that $l \vee q$, where l is a line and say q is a conic, to mean the two points of intersection of the line with the conic. Finally, when the intersection is a pair, we distribute over other applications of $\vee$ and $\wedge$, i.e. $A \vee (l \wedge q)$ is the pair $(A \vee P_1, A \vee P_2)$, where $P_{1,2}$ are points obtained from the intersection of l and q.

Definition of a quadratic projection. Let c be a conic, let A and B be two arbitrary points, and let ℓ be any line not containing B. Assume that c is non-degenerate and that B does not lie on ℓ. Choose a point P. The intersection of a line and a quadric is two, possibly imaginary, points, so let R_1 and R_2 be the intersection of c with AP, imaginary or not. Then R_1 is one of the projections of the point P to the conic c, R_2 is the second. Now project the R_i's to the line ℓ from point B. Let Q_i

be the intersection of BR_i with ℓ. The Q_i's are the quadratic projections of the point P to the line ℓ. We call this map $q(c, A, B, \ell) : P^2(\mathbb{R}) \longrightarrow \pi_\ell$ where π_ℓ is the projective line induced on the line ℓ in which points such as Q_1 and Q_2 are identified. We may write the map as

$$P \overset{q(c,A,B,\ell)}{\longrightarrow} (((P \vee A) \wedge c) \vee B) \wedge \ell \,.$$

In the three dimensional case, the conic becomes a quadric surface and ℓ is replaced by a plane, inducing a projective plane. Note that any map $q(c, A, B, \ell)$ has a single effective viewpoint at A, at least in the abstract sense; this may not correspond to any optical projection, involving c or otherwise.

We are interested in the quadratic mappings $q(c, F_1, F_2, \ell)$ where c is a conic whose foci are F_1 and F_2 and ℓ is a line through F_1 and perpendicular to F_1F_2 (Fig. 1). These are general catadioptric projections, for in the cross-section, first a point in space is projected to the mirror, a conic section, from the focus F_1. Then this point is projected to the image line from the point F_2, the second focus. We wish to find c' and B, where c' is a circle centered at F_1, such that

$$q(c, F_1, F_2, \ell) = q(c', F_1, B, \ell) \,,$$

up to scale.

Theorem 1. ***Projective Equivalence.*** Let $q(c, F_1, F_2, \ell)$ be a catadioptric projection, F_i are the foci of c and ℓ is the image line. There exists a circle c' centered at F_1 and a point B such that

$$q(c, F_1, F_2, \ell) = q(c', F_1, B, \ell) \,.$$

Proof: We will prove the theorem by deriving the general catadioptric formula $q(c, F_1, F_2, \ell)$, then deriving the spherical projection formula $q(c', F_1, B, \ell)$, equating them and solving for the radius of the circle and the point B. We will see that the parameters c' and B are independent of the choice of the point to project.

Step 1: Derivation of $q(c, F_1, F_2, \ell)$. Assume that F_1 is $(0, 0, 1)$ and that the quadratic form of c, in terms of its eccentricity, is as follows:

$$\mathbf{Q}_\epsilon = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 - 4\epsilon^2 & -4\epsilon \\ 0 & -4\epsilon & -4 \end{pmatrix} \,.$$

Then $F_2 = (0, -2\epsilon, \epsilon^2 - 1)$, so that when $\epsilon = 1$, $F_2 = (0, 1, 0)$ is the point at infinity on the axis of the parabola. This conic has a latus rectum of 2. The latus rectum is the length of the line segment created by the two points of intersection of the conic c and line ℓ.

We now find the projection $q(c, F_1, F_2, \ell)$ of P. First, the points R_1 and R_2 in $(P \vee F_1) \wedge c$, being the intersection of a line and a conic, may be expressed as

$$R_i = F_1 + \theta_i P \,,$$

for some $\theta_1, \theta_2 \in \mathbb{C}$, where these θ_i are roots of a quadratic equation. We obtain the quadratic equation from the condition that R_i lies on the conic,

$$\begin{aligned} 0 &= R_i \mathbf{Q}_\epsilon R_i^T \\ &= (F_1 + \theta_i P)\mathbf{Q}_\epsilon (F_1 + \theta_i P)^T. \end{aligned}$$

We solve for θ and after variable substitution we obtain

$$\theta_i = \frac{1}{(-1)^i\sqrt{x^2+y^2} - \epsilon y - w}, \tag{1}$$

when $P = (x, y, w)$. So the points

$$R_i = \begin{pmatrix} \frac{x}{(-1)^i\sqrt{x^2+y^2}-\epsilon y-w}, \\ \frac{y}{(-1)^i\sqrt{x^2+y^2}-\epsilon y-w}, \\ 1+\frac{w}{(-1)^i\sqrt{x^2+y^2}-\epsilon y-w} \end{pmatrix}. \tag{2}$$

Next we project the R_i to the line $\ell = [0, 1, 0]$ from the point F_2. This transformation is expressed as the matrix

$$\mathbf{T}_\epsilon = \begin{pmatrix} -2\epsilon & 0 & 0 \\ 0 & 0 & 1-\epsilon^2 \\ 0 & 0 & -2\epsilon \end{pmatrix}.$$

The projected points Q_i are then given by

$$q(c, F_1, F_2, \ell) = \left(2\epsilon x\ ,\ 0\ ,\ -(1+\epsilon^2)y + 2(-1)^i\epsilon\sqrt{x^2+y^2}\right). \tag{3}$$

Step 2: Derivation of $q(c', A, B, \ell)$. Now find the spherical projection, or in the cross-section, the projection to the circle. Let c' be a circle centered at F_1 whose radius is r. The points R_i', which are the intersections of the line F_1P with this circle, may be found without difficulty due to the simplicity of the circle, all that is necessary is essentially a normalization. In particular

$$R_i' = \left(rx, ry, (-1)^i\sqrt{x^2+y^2}\right).$$

Now we must determine the projection of the points R_i' to the image line ℓ. The projection is just a perspective transformation from the unknown point B. By symmetry the point B lies on the line F_1F_2, we therefore parameterize B with l, writing $B = (0, l, 1)$. Then the matrix projecting a point to the line ℓ from B may be expressed as

$$\mathbf{U}_l = \begin{pmatrix} l & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & l \end{pmatrix}.$$

And thus,

$$\begin{aligned} q(c', F_1, B, \ell) &= R_i'\mathbf{U}_l \\ &= \left(lrx, 0, -ry + l(-1)^i\sqrt{x^2+y^2}\right) \end{aligned} \tag{4}$$

Step 3: For what B and radius of c' is $q(c, F_1, F_2, \ell) = q(c', A, B, \ell)$? If r and l can be chosen independently of x, y, and w such that equations (3) and (4) are equal (up to a scale, remember that we work in homogeneous coordinates), then we have shown that the two projections are equivalent. This is indeed the case, and if we choose

$$\begin{aligned} l &= \frac{2\epsilon}{1+\epsilon^2}, \\ r &= 1, \end{aligned}$$

then substituting in (4) gives

$$\left(\frac{2\epsilon x}{1+\epsilon^2}, 0, -y + \frac{2(-1)^i \epsilon\sqrt{x^2+y^2}}{1+\epsilon^2}\right).$$

Multiply this by $1+\epsilon^2$ and we obtain

$$\left(2\epsilon x, 0, -(1+\epsilon^2)y + 2(-1)^i\epsilon\sqrt{x^2+y^2}\right),$$

which is the same as (3). Therefore

$$\begin{aligned} & q(c, F_1, F_2, \ell) \propto q(c', F_1, B, \ell) \\ = \; & (1+\epsilon^2) q\left(\odot(F_1;1), F_1, \left(0, \frac{2\epsilon}{1+\epsilon^2}, 1\right), \ell\right). \end{aligned}$$

□

Extension to three dimensions. We can now extend the definition to three dimensions. We assume that c is rotationally symmetric about the z-axis having only two foci F_1 and F_2 with coordinates $(0,0,0,1)$ and $(0,0,-2\epsilon,\epsilon^2-1)$, and also that the plane p has coordinates $[0,0,1,0]$. Then $q(c, F_1, F_2, \ell)$ reads

$$(1+\epsilon^2) q\left(\odot(F_1;1), F_1, \left(0, 0, \frac{2\epsilon}{1+\epsilon^2}, 1\right), p\right),$$

where $\odot(F_1;1)$ is the sphere centered at F_1 with a radius of 1.

Including the perspective case and scale parameter. In the current definition we are unable to represent different scale mirrors or perspective projection. The scale is fixed by the radius of the sphere and the latus rectum, and in order to represent perspective projection we would need F_1 equal to F_2, but if this were the case, then F_2 would lie on the plane p, giving a degenerate projection. We could let the radius of the sphere vary. Although this would allow a change of scale, the parameterization of the point B would change and we would still not be able to represent perspective projection. A translation of the plane, however, allows for a scaling parameter as well as moving p away from F_1 so that perspective projection is possible. First, note that $q(c', F_1, B, p)$ can be written as

$$q(c', F_1, B, [0,0,1,m]) \begin{pmatrix} \frac{l+m}{l} & 0 & 0 & 0 \\ 0 & \frac{l+m}{l} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

on the right hand side the image plane is at $z=-m$, and thereby induces a scale of $\frac{l+m}{l}$ on the image coordinates. Now let the latus rectum be $4p$, this induces a scale in image coordinates of $2p$; placing the plane at $z=-l(2p-1)$ achieves the same scale, for then the scale is

$$\frac{l+m}{l} = \frac{l+l(2p-1)}{l} = 2p.$$

Therefore if c is a quadric with latus rectum $4p$, foci at $F_1=(0,0,0,1)$ and $F_2=(0,0,-4p\epsilon,\epsilon^2-1)$, then

$$\begin{aligned} & q(c, F_1, F_2, \ell)\,\mathbf{P} = \\ & \quad q\left(\odot(F_1;1), F_1, \frac{2\epsilon}{1+\epsilon^2}, \left[0,0,1,\frac{2\epsilon(2p-1)}{1+\epsilon^2}\right]\right)\mathbf{P}, \end{aligned}$$

where when

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

premultiplying by $\mathbf{P}$ has the effect of disregarding the y-component, i.e. the horizontal placement of the image plane. We then write $s_{l,m}$ for a spherical projection

$$s_{l,m} = q(\odot(F_1; 1), F_1, (0,0,l,1), [0,0,1,m])\,\mathbf{P}$$

representing some central catadioptric projection. The projection of a point (x, y, z, w) in projective space are in general the two points

$$\left((l+m)x, (l+m)y, -z \pm l\sqrt{x^2+y^2+z^2}\right).$$

In the case above, for a conic of eccentricity ϵ with latus rectum $4p$, the projection is represented by

$$s_{\frac{2\epsilon}{1+\epsilon^2}, \frac{2\epsilon(2p-1)}{1+\epsilon^2}}.$$

Summary. *All non-degenerate central catadioptric projections are equivalent to a central projection of the spherical representation of the projective plane to a plane.* All such projections can be represented with the single map $s_{l,m}$, where the parameter l is a function of the eccentricity of the conic and m is a function of its scale. We enumerate the possible central catadioptric sensors:

1. $0 < \epsilon < 1$. Elliptic projection is also equivalent to the composition of normalization and central projection. The distance of the second projection center from the center of a unit sphere is $\frac{2\epsilon}{1+\epsilon^2}$. Given an ellipse c of eccentricity ϵ whose latus rectum is $4p$, and whose foci are $F_1 = (0,0,0,1)$ and $F_2 = (0,0,4p\epsilon,\epsilon^2-1)$, we have

$$q(c, F_1, F_2, p) = s_{\frac{2\epsilon}{1+\epsilon^2}, \frac{2\epsilon(2p-1)}{1+\epsilon^2}}.$$

2. $\epsilon = 1$. Parabolic projection is equivalent to stereographic projection. We have the equivalence up to scale in homogeneous coordinates,

$$q(c, F_1, F_2, p) = s_{1,2p-1}\,\mathbf{P},$$

   where c is a parabola with latus rectum $4p$ and foci $F_1 = (0,0,0,1)$ and $F_2 = (0,0,1,0)$.

3. $\epsilon > 1$. Hyperbolic projection is equivalent to the composition of normalization to the sphere followed by central projection to a plane from a point between the north pole and the sphere's center. In a unit sphere, the distance of this point from the center is $\frac{2\epsilon}{1+\epsilon^2}$. The plane is perpendicular to the line through this point and the center, and its distance from the center is determined by the length of the latus rectum. In particular a hyperbola c of eccentricity ϵ whose latus rectum is $4p$, and whose foci are $F_1 = (0,0,0,1)$ and $F_2 = (0,0,4p\epsilon,\epsilon^2-1)$, we have

$$q(c, F_1, F_2, p) = s_{\frac{2\epsilon}{1+\epsilon^2}, \frac{2\epsilon(2p-1)}{1+\epsilon^2}}.$$

4. $\epsilon \to \infty$. Perspective projection with focus $F_1 = (0,0,0,1)$ and image plane $z = -f$, i.e. focal length f, is equivalent to $s_{0,f}$.

Corollaries.

1. Parabolic projection is conformal. The angles between great circles on the spherical representation of the projective plane are preserved in the parabolic projective plane. For example, the horizons of two perpendicular planes are two orthogonal circles. This is because stereographic projection is conformal [18].
2. Conformal maps on the sphere project to conformal maps in the parabolic projective plane. In particular, pure rotations of space preserve the angles between great circles, and thus rotations of space preserve angles in the parabolic projective plane.
3. Catadioptric projections with reciprocal eccentricities are projectively equivalent; such projections have the same representation, for if $\epsilon = {\epsilon'}^{-1}$, then

$$l' = \frac{2\epsilon'}{1+\epsilon'^2} = \frac{2\frac{1}{\epsilon}}{1+\left(\frac{1}{\epsilon}\right)^2} = \frac{2\epsilon}{1+\epsilon^2} = l\,.$$

This implies that any elliptic catadioptric device is projectively equivalent to a hyperbolic projection. We therefore need only consider one of these cases, and we arbitrarily choose to refer to such projections as hyperbolic.

4 Images of Lines

It is now trivial to see that the image of a line in the general case is a conic: First the projection of a line in space to the sphere is great circle. There is a cone through the second center of projection and this great circle as in Figure 2. The intersection of this cone with the image plane is the line image and is obviously a conic.

The intersection of any great circle with the equator are two points antipodal on the equator. The projection of these two points are, in the hyperbolic and parabolic cases, two points antipodal on the fronto-parallel horizon. Thus, the intersection of any line image and the fronto-parallel horizon are two points antipodal on the fronto-parallel horizon.

The proof for the projection of an arbitrary line lying on the plane $n_x x + n_y y + n_z z = 0$ through the composite mapping with parameters (l, m) is lengthy but straightforward. The image of the line is a conic with foci

$$f_{i=1,2} = \left((l+m)n_x, (l+m)n_y, (-1)^i\sqrt{1-l^2} - n_z\right) \tag{5}$$

and semi-axes:

$$a = \left|\frac{l(l+m)n_z}{l^2 - n_x^2 - n_y^2}\right|$$
$$b = \left|\frac{l+m}{\sqrt{l^2 - n_x^2 - n_y^2}}\right|,$$

Notice that the foci are collinear with the image center, and thus the major axis contains the image center.

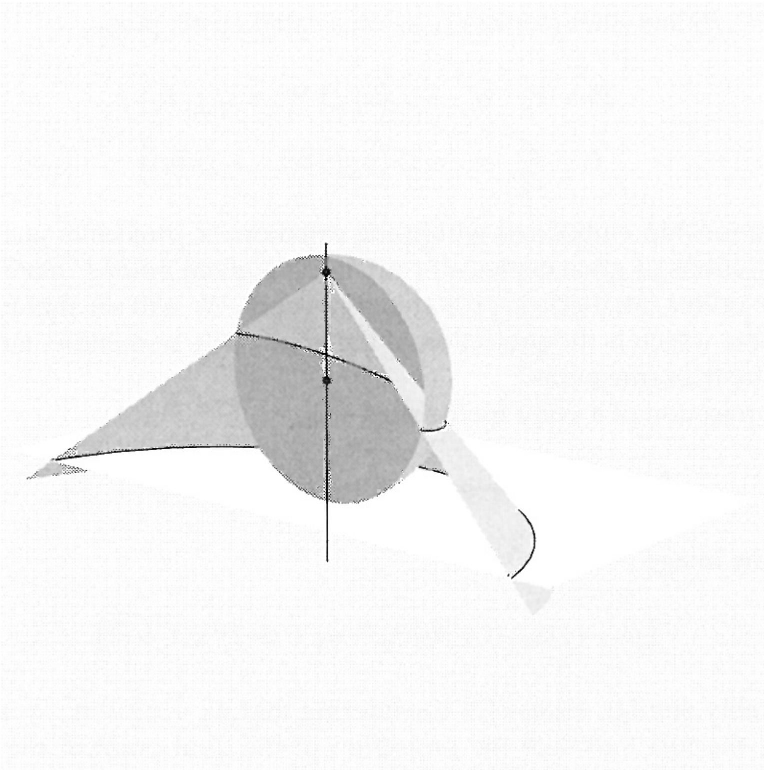

Figure 2: The projection of a line to the sphere is a great circle; the projection of the great circle is obtained from the intersection of the image plane with a cone containing the great circle and whose vertex is the point of projection.

5 Duality

In standard projective geometry there is a one to one correspondence with points and lines of a projective plane. On the sphere, a representation of the projective plane, the correspondence is between a great circle and its poles. We write the dual great circle of a point P as $\tilde{P}$ and the dual point of a great circle ℓ as $\tilde{\ell}$.

An example of their usage is in the following. Suppose we have two points P_1 and P_2 on the sphere and we wish to determine the great circle ℓ between them. We take the dual great circles of the two points, $\tilde{P}_1$ and $\tilde{P}_2$. They must intersect in a pair of points which are antipodal and represented by Q. Taking the dual Q gives the very great circle through the two original points, that is $\ell = \tilde{Q}$. This is because the dual great circle $\tilde{P}$ of any point P on the great circle is a great circle containing the point Q. So intersecting any two yields the point Q.

We call $P_1 \vee P_2$ the great circle between points P_1 and P_2 and $\ell_1 \wedge \ell_2$ the intersection of the great circles ℓ_1 and ℓ_2. We express the fact above in the equations

$$\begin{aligned} P_1 \vee P_2 &= \widetilde{\tilde{P}_1 \wedge \tilde{P}_2}\,, \\ \ell_1 \wedge \ell_2 &= \widetilde{\tilde{\ell}_1 \vee \tilde{\ell}_2}\,. \end{aligned}$$

The operators $\wedge$ and $\vee$ can be used on the catadioptric projective plane as well, in particular we define for points P_1, P_2 and lines (conics) ℓ_1, ℓ_2 on a catadioptric plane,

$$\begin{aligned} P_1 \vee P_2 &= s_{l,m}\left(s_{l,m}^{-1}(P_1) \vee s_{l,m}^{-1}(P_2)\right) \\ \ell_1 \wedge \ell_2 &= s_{l,m}\left(s_{l,m}^{-1}(\ell_1) \wedge s_{l,m}^{-1}(\ell_2)\right) \end{aligned}$$

Is there such a relationship embedded within the catadioptric projective plane? What properties do the sets of projections of great circles all containing a given point P have? We have seen that the image of a line under catadioptric projection is a conic. We will see that foci of coincident line images lie on a conic which is the projection of the great circle perpendicular to them all; though it is not projected by the same point.

Consider the projection of a point by the map $s_{l,m}$,

$$\left((l+m)x, (l+m)y, -z + l(-1)^i\sqrt{x^2+y^2+z^2}\right)$$

and the foci of a line image,

$$\left((l+m)n_x, (l+m)n_y, -n_z + (-1)^i\sqrt{1-l^2}\right).$$

They look remarkably similar, especially considering that $n_x^2 + n_y^2 + n_z^2 = 1$. Remembering the point-line duality, the foci look like the projection of the dual point of the great circle, i.e. its normal. We will see that this is more than a coincidence.

Lemma 2. Let ℓ be a line of a catadioptric projective plane $\pi_{l,m}$ which is the projection of a great circle whose normal is $\hat{n}$. The foci pair of ℓ is the projection of the point $\hat{n}$ by $s_{l',m'}$ where l' and m' satisfy

$$\begin{aligned} l+m &= l'+m', \\ l^2+l'^2 &= 1. \end{aligned}$$

Proof: The foci of the line are

$$\left((l+m)n_x, (l+m)n_y, -n_z + (-1)^i\sqrt{1-l^2}\right).$$

If

$$\begin{aligned} l' &= \sqrt{1-l^2}, \\ m' &= l+m-\sqrt{1-l^2} \end{aligned}$$

then the foci can be rewritten

$$\left((l'+m')n_x, (l'+m')n_y, -n_z + (-1)^i l'\sqrt{n_x^2+n_y^2+n_z^2}\right).$$

This is projection of the point $(n_x, n_y, n_z, 1)$ by $s_{l',m'}$. Conversely, if a point P is projected to a point pair in a catadioptric projective plane $\pi_{l,m}$, this point pair is the foci pair of a line image of a projective plane $\pi_{\sqrt{1-l^2},l+m-\sqrt{1-l^2}}$. □

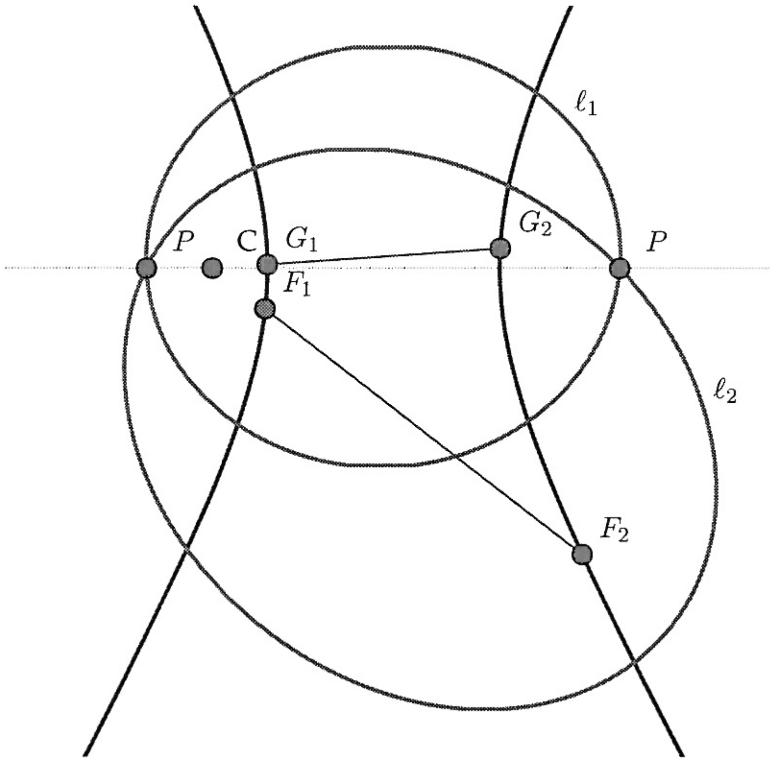

Figure 3: The two ellipses ℓ_1 and ℓ_2 are projections of two lines in space containing the point P. Their foci F_1, F_2, and G_1, G_2 respectively lie on a hyperbola containing the foci of *all* ellipses through P. The foci of this hyperbola are the points in P. The point C is the image center.

Lemma 3. Let $\{\ell_k\}$ be a set of line images all of which intersect a point P, i.e. for all k, $P \in \ell_k$. Then the locus of foci of the line images lie on a conic c whose foci are the same as the points in P (see figure 3).

Proof: Assume that P is the projection of the point $\hat{n} = (n_x, n_y, n_z)$ on the sphere. Also assume that the lines ℓ_k are images of great circles whose plane's normals are $\hat{m}_k$. Because of rotational symmetry, we may assume without loss of generality that $n_y = 0$. This implies that for some θ_k that

$$\{[\hat{m}_k]\} = \{[-n_z \sin\theta_k, \cos\theta_k, n_x \sin\theta_k]\}\,.$$

Then the foci of the ℓ_k are,

$$\begin{aligned} f_i^k &= ((l+m)n_z \sin\theta_k, (l+m)\cos\theta_k, \\ &\quad (-1)^i\sqrt{1-l^2} - n_x \sin\theta_k)\ . \end{aligned}$$

But these are the pair of points in the projection of $\hat{m}_k$ by

$$s_{\sqrt{1-l^2},l+m-\sqrt{1-l^2}}\,.$$

Therefore this point is in the image of the line $\hat{n}$ by this same projection. Its foci are

$$f_i = \left((l+m)n_x, 0, (-1)^i l - n_z\right)\,,$$

which is the projection of $\hat{n}$ by $s_{l,m}$. □

We use these lemmas to prove the following duality theorem.

Theorem 2. ***Duality.*** If $\pi_{l,m} = (\Pi_1, \Lambda_1)$ and $\pi_{l',m'} = (\Pi_2, \Lambda_2)$ are two catadioptric planes such that

$$l^2 + l'^2 = 1 \quad \text{and} \quad l + m = l' + m',$$

then $f_{l,m}$, which gives the foci of a line image in the context of some catadioptric plane $\pi_{l,m}$, maps as follows,

$$f_{l,m} : \Lambda_1 \rightarrow \Pi_2 \,,$$
$$f_{l',m'} : \Lambda_2 \rightarrow \Pi_1 \,,$$

and their inverse mappings exist. In addition, incidence relationships are preserved by $f_{l,m}$:

$$\begin{aligned} P_1 \vee P_2 &= f_{l,m}^{-1}\left(f_{l',m'}^{-1}(P_1) \wedge f_{l',m'}^{-1}(P_2)\right) \,, \\ \ell_1 \wedge \ell_2 &= f_{l,m}\left(f_{l',m'}(\ell_1) \vee f_{l',m'}(\ell_2)\right) \,, \end{aligned}$$

where $P_1, P_2 \in \Pi_1$ and $\ell_1, \ell_2 \in \Lambda_2$.

We call the projective planes, $\pi_{l,m}$ and $\pi_{l',m'}$, dual catadioptric projective planes.

Proof: We have already shown the first part of the theorem in Lemma 2. It only remains to show that incidence relationships are preserved. This follows from Lemma 3 and the fact that incidence relationships are already known to be preserved on the sphere by the mapping taking antipodal points to great circles and vice versa. □

Corollaries.

1. Perspective projection ($l = 0$) is dual to parabolic projection ($l' = 1$). This means that the parabolic projection of a line is a circle whose center is the perspective projection of the normal of the plane containing the line. It also implies that the parabolic projection of a pair of antipodal points are two points whose perpendicular bisector is the projection of the great circle dual to the antipodal point pair.

2. A catadioptric projection with a mirror of eccentricity ϵ is dual to a catadioptric projection with mirror eccentricities $\left|\frac{1-\epsilon}{1+\epsilon}\right|$ and $\left|\frac{1+\epsilon}{1-\epsilon}\right|$.

3. A catadioptric projection with eccentricity $\pm 1 + \sqrt{2}$ is self-dual ($l = \frac{1}{\sqrt{2}}$). In this case the foci of a projected great circle are exactly the projections of the dual points.

6 Conclusion

We presented a novel theory on the geometry of central panoramic or catadioptric vision systems. In particular, we proved:

- Every single viewpoint catadioptric system is equivalent to central projection to a sphere followed by projection from a point on the sphere's axis. In particular we prove that parabolic projection is equivalent to stereographic projection, and is therefore conformal.

- Using this equivalence we observe that images of lines in space are mapped to great circles on the sphere and to conic sections on the catadioptric image plane.
- To every catadioptric projection there is a dual one. dual projections map poles of great circles on the sphere to the foci of the conic sections corresponding to the great circles of the poles.

This theory helps in elegantly formulating the problems of uncalibrated recovery of structure from a single or multiple views. For the single view we proved [9] that calibration of a catadioptric projection is possible with only two lines, and in general three. In the parabolic case, calibration is performed by intersecting spheres whose equators are line images. The perspective case proves to be the only one not providing the sufficient constraints for such a calibration. The natural next step is to extend this theory to multiple catadioptric views as well as a study of robustness of scene recovery using the principles described herein.

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