Nonholonomic Motion Planning based on Nonholonomic Spheres

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Abstract

In this paper an algorithm of motion planning using extensively nonholonomic spheres of small radii is presented. Iteratively, a direction of motion from the current state towards a goal state is expressed in Cartesian coordinates. Then, the direction is transformed into Ph. Hall coordinates. The coordinates when mapped into nonholonomic spheres of small radii determine controls to steer the system. Convergence of the algorithm is guaranteed if only the system is (small time) locally controllable. The detailed description of the algorithm is given and tests on the unicycle mobile robot shown.

1 Introduction

Nonholonomic systems, those constrained by non-integrable equations, are frequently encountered in robotics while studying a motion of free-floating robots, underwater vessels, wheeled vehicles, planning a grasp or designing underactuated manipulators. Motion constraints of the systems given in the Pfaff form can be transformed into equivalent driftless nonholonomic system:

$$\dot{q} = \sum_{i=1}^{m} g_i(q)\, u_i = G(q)\, u, \qquad (1)$$

where $\dim q = n > m = \dim u$, q is a state vector, u is a control vector, $g_i(q)$ are analytic vector fields associated with the system (1) and called generators. A minimal requirement for a nonholonomic system is its controllability. According to Chow's theorem [2], the driftless system (1) is controllable (and also small time locally controllable) if its Lie algebra, $LA(G)$ (spanned by generators) spans the state space everywhere, i.e. $\forall_q \operatorname{rank} LA(G)(q) = n$. Obviously, to be controllable, the system must be nonlinear (when $g_i(q) = const.$, $\operatorname{rank} LA(G) \leq m < n$). Nonlinearity of the system makes global considerations fairy complicated. Even locally, around a given point in the state space, some non-trivial problems can be encountered while planning a motion in a desired direction. For local motion planning of nonholonomic systems it is desirable to distinguish energy affordable directions of motion from those consuming a lot of energy. The energy-based metric-like function

$$d(q_0, q_T) = \min_{\substack{u(\cdot)|\, t\, \in\, [0,\, T],\\ \dot{q}\, =\, G(q)u,\; q(0)\, =\, q_0,\; q(T)\, =\, q_T}} \int_0^T ||u(t)||^2\, dt, \qquad (2)$$

defines controls that generate the energy optimal trajectory $q(t)$ joining two states q_0 and q_T. In Eq. (2), T denotes the time of completing a motion. Computation of the distance $d(q_0, q_T)$ and the optimal trajectory (energy optimal curve) is a standard task of optimal control although its analytic solution is hardly ever possible. A small radius attainability sphere (nonholonomic sphere), centered at the state q_0, defines a set of locations, reached by the system (1), as distant as possible from q_0, when the energy of motion $\int_{t=0}^{T} ||u(t)||^2 dt = E$ is fixed and small. Fortunately, the shape of small radius attainability spheres (q_0 is close to q_T) of nonholonomic systems (1), in the metric-like function introduced by Eq. (2), has been determined in the paper [5]. Nonholonomic spheres of small radii are expressed in Ph. Hall coordinates and therefore, are independent on each particular system. For two input nonholonomic systems with three dimensional state space, the spheres can be generated even analytically. For other systems they are computed with the use of numeric procedures. A nonholonomic sphere is composed of a set of its radii and each radius determines controls. A little bit informally, the nonholonomic radius anchored at q_0 is determined by the farthest point in the given direction reached by trajectory initialized at q_0 and subordinated to the equations of motion (1) when the energy of controls is small and fixed.

In robotic literature some work has been done in optimizing energy expenditure on controls of nonholonomic systems. Since Brockett's work [1] it is known that sinusoidal controls are optimal for steering the Brockett's integrator. Some results on optimizing the energy expenditure in the basic Newton algorithm of nonholonomic motion planning [7] are also known [6].

This paper is organized as follows. In Section 2 the construction of small radius nonholonomic spheres is recalled and some basic terminology introduced. In Section 3 algorithm of motion planning with the use of nonholonomic spheres is presented. Section 4 provides extensions of the algorithm to the case of more controls and high dimensional state spaces. Section 5 concludes the paper.

2 Nonholonomic spheres

Although the construction of nonholonomic spheres is valid for general driftless nonholonomic systems, for computational reasons it will be presented for two input systems, $m = 2$, with three dimensional state space, $n = 3$

$$\dot{q} = X(q)\,u + Y(q)\,v. \tag{3}$$

Instead of considering a particular system $\dot{q} = g_1(q)u_1 + g_2(q)u_2$, with X substituted by g_1 and Y by g_2, we will describe attainability spheres in a coordinate frame given by the very first $n = 3$ elements of Ph. Hall basis of the free Lie algebra spanned by X and Y. This coordinate system will be referred to as Ph. Hall coordinate frame and vectors expressed in that frame are described by Ph. Hall coordinates. It is reasonable to assume that the Ph. Hall frame is orthogonal. Ph. Hall basis begins with elements $X, Y, [X,Y], [X,[X,Y]], [Y,[X,Y]], \ldots$ Here above $[\cdot,\cdot]$ denotes the Lie bracket of vector fields. For two vector fields Z, V, the Lie bracket defines another field

$$[V, Z] = \frac{\partial Z}{\partial q}V - \frac{\partial V}{\partial q}Z.$$

It is natural to define a measure of complexity of vector fields

degree(X) = 1, if X is a generator
degree($[X,Y]$) = degree(X) + degree(Y),
if $[X,Y]$ is a compound vector field.

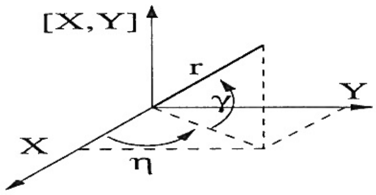

Figure 1: Ph. Hall coordinate frame and meaning of spherical coordinates r, γ, η.

When nonholonomic spheres of small radii are needed for a particular system then Ph. Hall elements are evaluated for this system at a given point. The resulting spheres are continuously deformed spheres in system-independent Ph. Hall coordinates. This observation validates usefulness of investigating spheres in Ph. Hall coordinate frame. In order to produce points on spheres with controls, the generalized Campbell-Baker-Hausdorff-Dynkin (GCBHD) formula is used, [3]. The formula describes locally trajectories of a non-autonomous system of differential equations $\dot{q} = F(t)(q)$. The trajectory $q(t)$ for small t initialized at q_0 is given by the formula

$$\begin{aligned} q(t) \simeq q_0 \quad &+\beta_X(t)X(q_0) + \beta_Y(t)Y(q_0) \\ &+\beta_{[X,Y]}(t)[X,Y](q_0) + \ldots \end{aligned} \tag{4}$$

where control-dependent coefficients-Ph. Hall coordinates $\beta_X, \beta_Y, \beta_{[X,Y]}, \cdots$ are as follows

$$\begin{aligned} &\beta_X(t) = \textstyle\int_0^t u(s)ds, \qquad \beta_Y(t) = \int_0^t v(s)ds, \\ &\beta_{[X,Y]}(t) = \tfrac{1}{2}\textstyle\int_0^t \int_0^{s_2}((u(s_1)\,v(s_2) - u(s_2)\,v(s_1))\,ds_1 ds_2 \\ &\beta_{[X,[X,Y]]}(t) = \ldots \qquad \beta_{[Y,[X,Y]]}(t) = \ldots \end{aligned} \tag{5}$$

The time t should be small enough that the remainder of Lie series composed of vector fields of higher degrees be negligible. The time dependent coefficient multiplying any vector field Z is proportional to $t^{degree(Z)}$ therefore there exists t small enough to suppress the influence of higher degree vector fields. In further analysis, $t = 1$ to fix the time horizon. The remainder of Lie series is suppressed by small amplitudes of u, v, cf. Eq. (5). Selecting sinusoidal controls as admissible class of control makes it possible to reformulate the set of functional equations (5) as a set of algebraic equations involving parameters of the sinusoids (amplitudes, frequencies, and phase shifts). Now, when a given direction of motion in Ph. Hall frame is given (as a set of coefficients β), controls generating the direction can be computed by solving optimization task in the parameter space composed of parameters of controls. Sections of three spheres with diverse energy radii E are presented in Fig. 2. Due to symmetry of the spheres, it is enough to present only the semi-half plane $([X,[X,Y])$. Fig. 2 confirms that the motion in directions corresponding to higher degree vector fields (here $[X,Y]$) are much more energy expensive than motions in directions determined by lower degree vector fields (here X,Y). As can be seen in Fig. 2, nonholonomic spheres are not convex.

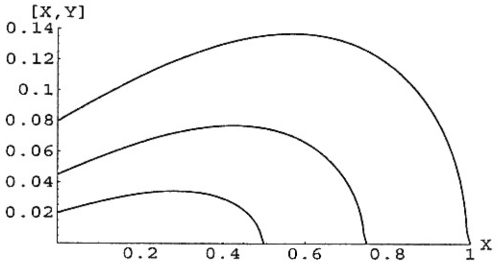

Figure 2: The section of nonholonomic spheres with the energy radiuses $E = 0.5, 0.75, 1$ along the plane $(X,[X,Y])$.

3 Motion planning with the use of nonholonomic spheres

By switching on either controls u, v or their linear combination it is possible, at a given state q_0, to generate motion in the plane spanned by the vectors $X(q_0)$, $Y(q_0)$; still not enough to span the state space R^3. Therefore, vector fields of the second, and higher degree should supplement the generators. To span R^3 it suffices to take the vector field $[X,Y]$ if only the system (3) is maximally nonholonomic. In nonholonomic motion planning some elements of construction of nonholonomic spheres can be utilized. In fact, the basic task is to move, locally, the state in the direction towards the goal state. It means that a radius of a particular nonholonomic sphere should be generated with controls. As the nonholonomic spheres are defined in Ph. Hall coordinate frame, the real direction of motion towards the goal should be transformed into the frame. For controllable systems with $n = 3$ and $m = 2$, the algorithm of motion planning with the use of nonholonomic spheres is implemented as a sequence of the following steps:

Step 1. Read the initial state q_0 and the goal one q_d. Analytically, compute the vector field $[X,Y]$. The initial state becomes the current state q_c.

Step 2. In the current state, evaluate vector fields $X, Y, [X,Y]$ that results in the vectors $X(q_c)$, $Y(q_c)$, $[X,Y](q_c)$ forming a coordinate frame for R^3. This coordinate frame is a projection of the Ph. Hall frame into R^3 at q_c.

Step 3. Calculate the direction towards the goal state $q_d - q_c$, and express the direction in the coordinate frame

$$q_d - q_c = \beta_X X(q_c) + \beta_Y Y(q_c) + \beta_{[X,Y]}[X,Y](q_c). \tag{6}$$

Find spherical coordinates corresponding to the vector of β-s, cf. Fig. 1

$$Spherical((\beta_X, \beta_Y, \beta_{[X,Y]})) = (\eta, \gamma, r). \tag{7}$$

Step 4. Using one dimensional optimization with the energy radius parameter E varied, compute the optimal value of $E^\star$ according to the following equation

$$||k_{q_c,T}(u_{E^\star}) - q_d|| = \min_{E>0} ||k_{q_c,T}(u_E) - q_d||, \tag{8}$$

where $||\cdot||$ is the Euclidean norm and $k_{q_c,T}(u_E)$ denotes the state reached by the system (3) initialized at q_c and steered with controls determined by the values η, γ, E. The angles η, γ are determined by Eqn. (7), while E sets the value of r. Additionally,

$$\frac{\langle k_{q_c,T}(u_E) - q_c, q_d - q_c\rangle}{||(k_{q_c,T}(u_E) - q_c|| \cdot ||q_d - q_c||} < \alpha. \tag{9}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product and α is the given angle accuracy of motion towards the goal state.

Step 5. Move the current state q_c to the state $k_{q_c,T}(u_{E^\star})$, that becomes the new current state.

Step 6. Check the stop condition. If $||q_c - q_d|| < \epsilon$, the algorithm stops and the final trajectory is obtained. Here ϵ denotes accuracy of reaching the goal state. Otherwise the algorithm continues with Step 2.

The algorithm can be implemented in the reverse direction when the trajectory is searched to join the initial point q_d and the goal one q_0. When the trajectory is found, controls should be modified to restore the right direction of motion.

The algorithm can also be modified to be more flexible in generating the desired direction of motion. Formula (8) says that for fixed values of E, η, γ, the

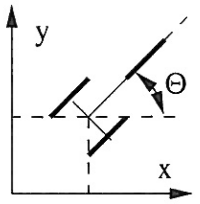

Figure 3: The unicycle robot.

radius r of the nonholonomic sphere is determined as well as controls generating the spherical coordinates (η, γ, r). As formula (4) provides only approximation of the desired direction of motion (the vector fields of degrees higher than two were neglected), then after computing controls it is required to check whether the resulting state approaches the goal state or not. According to theoretical consideration for small values of E the desired direction can be realized as accurately as required, because the contribution of the higher degree vector fields to resulting motion is negligible. Consequently, the convergence property of the algorithm is preserved. However, from the computational perspective it is reasonable to increase the value of the energy radius E, even at the price of worsening quality of generating the desired direction (defined by the angles η, γ), if only the goal state is approached. In this case, the auxiliary condition (9) is not checked.

4 Simulation results

Several tests were performed to verify the efficiency of the method of motion planning based on nonholonomic spheres. As an example, the simple unicycle robot was selected. The unicycle is described by the kinematic equations

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 = X\,u_1 + Y\,u_2. \tag{10}$$

In Eq. (10) x, y denote coordinates of the unicycle on the plane, while the angle θ describes its orientation, cf. Fig. 3. The control u_1 is the linear velocity of the vehicle while the control u_2 is used to change its orientation. It can be checked easily that the vector fields X, Y, given by Eqn. (10) together with $[X, Y] = (\sin\theta, -\cos\theta, 0)^T$ span the state space everywhere.

The aim of the first test was to plan a trajectory between the initial state $q_0 = (20, 10, 0°)^T$ and the goal state $q_d = (0, 0, 0°)^T$. The goal state was reached when the distance from the final state to the current state dropped below $\epsilon = 0.01$. Resulting controls were piecewise continuous, cf. Figs. 4, 5, with jumps when each iteration of the algorithm was completed. When the direction towards the goal state was not to be generated very precisely, Fig. 4, the number of iterations to complete the algorithm decreases and the resulting path is smoother than for the case of very accurate motion towards the goal, Fig. 5.

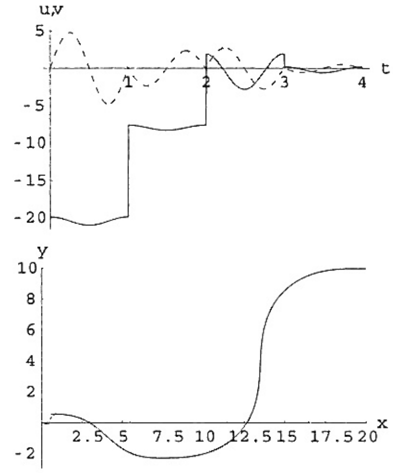

Figure 4: Controls and resulting path for the unicycle with a rough generation of the direction towards the goal.

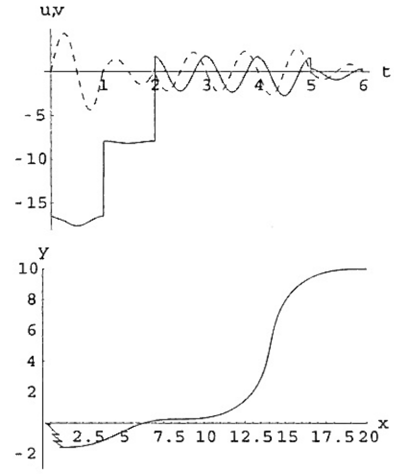

Figure 5: Controls and resulting path for the unicycle when the direction towards the goal state was generated very precisely.

iter	approximate		exact	
	x	y	x	y
1	7.44	-2.27	8.53	0.27
2	0.36	0.59	0.81	-1.55
3	0.20	-0.04	0.73	-1.16
4	0.04	0.00	0.58	-0.71
5			0.27	-0.07
6			-0.01	0.00

Table 1: Positions of the unicycle in the $X-Y$ plane after consecutive iterations of the algorithm with approximate and exact motion in a prescribed direction. The angle $\theta = 0$ after each iteration.

As nonholonomic spheres display rotational symmetry with respect to the axis $[X, Y]$, controls generating the motion along this axis are not determined uniquely. In fact the controls are parameterized with the phase shift $\psi \in [0, 2\pi]$ and described by the expression

$$\begin{aligned} u(s) &= E\,\sin(2\pi s + \psi + \pi/2), \\ v(s) &= E\,\sin(2\pi s + \psi), \quad s \in [0,1]. \end{aligned} \tag{11}$$

To verify impact of the phase shift parameter ψ on the resulting trajectory, a trajectory of the unicycle robot was planned initialized the state $q_0 = (0,1,0)^T$ and aimed at reaching the point $q_d = (0,0,0)^T$. As can be checked, the direction towards the goal is given by the vector $-[X,Y](q_0)$. The presented algorithm of motion planning was run for only one iteration with varied energy radii E. Resulting paths are presented in Fig. 6 for $\psi = 0°$, and in Fig. 7 for $\psi = 100°$. As can be seen, by varying the phase shift, long and very precise motions towards the goal can be generated.

5 Extensions

Till now, only the task of steering system with two inputs and three dimensional state space was considered in detail. The presented algorithm can be extended to the case including more controls and high dimensional state spaces. Generally, more controls make the motion planning simpler. With three controls that influence generators X, Y, Z, it is possible to steer a six dimensional state space with vector fields up to the second degree: $X, Y, Z, [X,Y], [X,Z], [Y,Z]$. When there are four controls, the dimension of steerable state spaces increases to 10. When n increases, vector fields of higher than the second degree should be involved. Although it is possible to construct nonholonomic spheres spanned by two generators and ex-

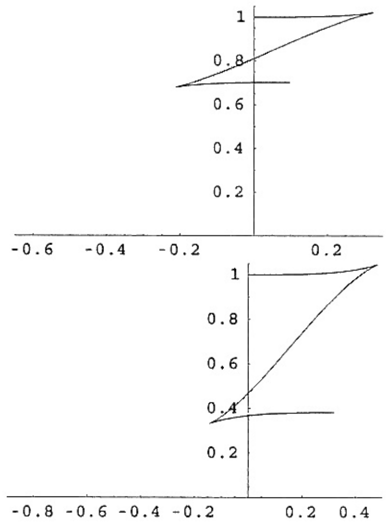

Figure 6: Motion in the direction $[X, Y]$ for the phase shift $\psi = 0°$ and energy radius $E = 2$ (above), and $E = 3$ (below).

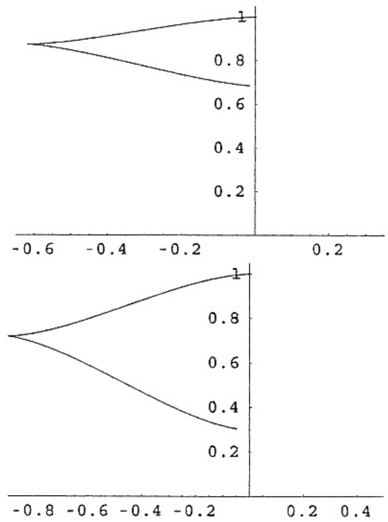

Figure 7: Motion in the direction $[X, Y]$ for the phase shift $\psi = 100°$ and energy radius $E = 2$ (above), and $E = 3$ (below).

pressed in Ph. Hall frame of dimension $n > 3$ (for example in the Ph. Hall frame composed of vector fields X, Y, $[X,Y]$, $[X,[X,Y]]$, $[Y,[X,Y]]$), it cannot be done analytically. As motion in each iteration of the algorithm is planned locally, the computational time of determining controls that steer the system (3) in the desired direction should be as short as possible. For the case with $n > 3$ a hybrid solution is proposed. When a motion in the direction described by the coordinates say β_X, β_Y, $\beta_{[X,Y]}$, $\beta_{[X,[X,Y]]}$, $\beta_{[Y,[X,Y]]}$ is required, it can be decomposed into three stage motion. In the first stage the coordinates $\beta_X, \beta_Y, \beta_{[X,Y]}$ are generated, then the coordinate $\beta_{[X,[X,Y]]}$, and, finally, $\beta_{[Y,[X,Y]]}$. The first stage motion is performed with the use of the presented algorithm of motion planning. In the paper [4] it has been shown that the generation of vector fields from Ph. Hall basis is very simple. The motion in the $[X,[X,Y]]$ direction is obtained with controls

$$u(s) = \sin(2\pi s + \pi/4) \quad v(s) = \sin(2 \cdot 2\pi s), \qquad (12)$$

while in the $[Y,[X,Y]]$ direction with controls

$$u(s) = \sin(2 \cdot 2\pi s) \quad v(s) = \sin(2\pi s + \pi/4), \qquad (13)$$

with $s \in [0,1]$. Consequently, two latter stages use controls derived from Eqns. (12), (13).

6 Conclusions

In this paper an algorithm of motion planning based on nonholonomic spheres has been introduced. Locally, the algorithm enables to optimize energy expenditure on controls. For the two input driftless nonholonomic systems with three dimensional state space, the algorithm uses only analytic formulas for steering, therefore it is very fast. For general driftless nonholonomic systems, some extensions have been proposed aimed at reducing the computational complexity of the motion planning. The algorithm is quite general as it requires only controllability of the steered driftless system.

Acknowledgments

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