Optimal Design of Parallel Manipulators

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Abstract

In this paper, two optimization methods on the design of a spatial 3 DOF parallel manipulator and a 6 DOF Stewart platform are performed. The objective function of the first method maximizes the total volume of manipulator workspace. The second method optimizes the total volume of well-conditioned workspace by maximizing a global condition index. The global condition index is a performance index to show the manipulator is how far from singularity. The results of both methods are compared to show which one is the better design. Monte Carlo method is used to perform integration numerically for both optimization methods.

1. Introduction

Parallel manipulators have been under increasing developments over the last few years from theoretical point of view and practical applications.

Optimal design of parallel manipulators, which is one of the important issue in designing parallel manipulators, means to obtain the geometry of the manipulator such that it fulfills a set of constraints. These constraints are a specified workspace, best accuracy over the workspace, minimum joint speeds and joint forces for a given Cartesian space, having maximum stiffness in all directions in the specified workspace, isotropic design and static balancing of the manipulator, and having the largest possible workspace for the manipulator. There are few investigators who have addressed the optimal design of parallel manipulators. Most of them have been considered only one or two of these constraints. Merlet[1][2] has determined the minimum joint speeds and joint forces for a given Cartesian space for a classical Gough type parallel manipulator with 6-DOF. Gosselin et. al[3][4] have studied the spherical 3-DOF parallel manipulator for determining the maximal workspace with the consideration of singularities. Rastegar and Perel[5], Alciatore and Ng[6] and Stamper et al.[7] applied the Monte Carlo method to determine the maximum workspace volume. Park and Brockett[8] and Gosselin and Angeles[9] performed the global performance index to determine the best accuracy over the workspace. This method is based on the integration of the reciprocal of the condition number over the entire workspace volume.

In this paper, two optimization approaches are performed for two types of parallel manipulators, i.e., a spatial 3DOF introduced in [10] and a 6DOF Stewart platform introduced in [11]. The first approach is based on having the largest possible workspace volume for the manipulators and the second approach is based on having the best accuracy over the well-conditioned workspace by maximizing the global condition index. Both methods are compared with each other to show which one is a better optimal design.

2. Description of the parallel manipulators

The two types of parallel manipulators are used in this paper. The first one is a 3 DOF parallel manipulator as it is shown in Fig. 1.

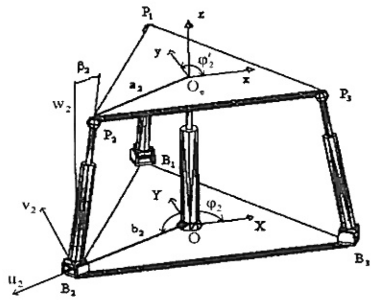

Fig. 1. 3 DOF manipulator

Three legs connect the moving platform to the base platform. Each leg consists of two parts, where connected to each other by a prismatic joint. The lower part and the base platform are connected by three universal joints at B_i and the upper parts of legs and the moving platform are connected by three spherical joints at P_i. A reference frame (XYZ) is attached to the fixed base at point O, located at the center of the circumferential circle of fixed platform. For each leg, another coordinate system $(uvw)_i$ is attached to the fixed base at B_i, such that u_i is in direction of b_i and at an angle φ_i from the X-axis as shown in Fig. 1. The angle φ_i for i^{th} leg defines the relative angular position of the legs. A moving frame (xyz) is attached to the moving platform at point O_e, located at the center of the circumferential circle of the moving platform. The same as the relative angular position of the base, the angle between a_i and x-axis is defined φ_i'. Here, we assume $\varphi_i' = \varphi_i$. A vertical leg connects moving platform to base platform at their center of circumferential circles and is connected to the moving platform by a universal joint and to the base platform by a prismatic joint. The central leg enables moving platform to have a translational motion along the Z-axis, i.e., heave and two orientations, i.e., roll about the x-axis and pitch

The second one is a 6 DOF parallel manipulator, as it is depicted in Fig. 2.

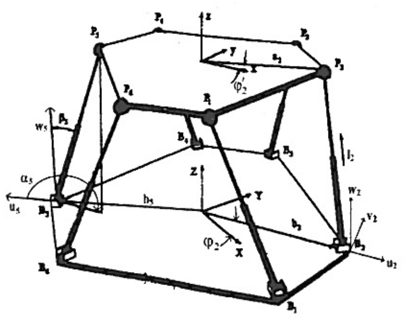

Fig. 2. 6 DOF manipulator

Six legs connect the moving platform to the base platform. Each leg consists of two parts, where connected to each other by a prismatic joint. The lower part and the base platform are connected by six universal joints at B_i and the upper parts of legs and the moving platform are connected by six spherical joints at P_i. A reference frame (XYZ) is attached to the fixed base at point O, located at the center of the circumferential circle of fixed platform. For each leg, another coordinate system $(uvw)_i$ is attached to the fixed base at B_i, such that u_i is in direction of b_i and at an angle φ_i from the X-axis as shown in Fig. 2. The angle φ_i for i^{th} leg defines the relative angular position of the legs. A moving frame (xyz) is attached to the moving platform at point O_e, located at the center of the circumferential circle of the moving platform. The same as the relative angular position of the base, the angle between a_i and x-axis is defined φ_i'. We assume φ_i and φ_i' as

$$\varphi_2 = \varphi_1 + \lambda_1 ,\ \varphi_4 = \varphi_3 + \lambda_1 ,\ \varphi_6 = \varphi_5 + \lambda_1$$

$$\varphi_1' = \varphi_1 ,\ \varphi_2' = \varphi_1' + \lambda_2 ,\ \varphi_3' = \varphi_3$$

$$\varphi_4' = \varphi_3' + \lambda_2 ,\ \varphi_5' = \varphi_5 ,\ \varphi_6' = \varphi_5' + \lambda_2$$

λ_1 and λ_2 are two variables that determine the differences between φ_i and φ_{i+1} as well as φ_i' and φ_{i+1}', i=1, 3, 5, respectively that will consider as design variables.

This manipulator has 6 DOF, three transitional motion along x, y and z axes and three orientations about x, y and z axes, i.e., roll φ, pitch ψ, and yaw θ, respectively.

3. Manipulators inverse kinematics

The inverse kinematics solution will be used during the determination of the workspace volume. The objective of the inverse kinematics is to determine length of legs and angles α_i and β_i for each leg.

As shown in Fig. 3, α_i is the angle between projection of i^{th} leg on $\mathbf{u}_i$-$\mathbf{v}_i$ plane and $\mathbf{u}_i$ and β_i is the angle between i^{th} leg and $\mathbf{w}_i$. Note that $i = 1,2,3$ for 3 DOF manipulator and $i = 1,2,\ldots,6$ for 6 DOF manipulator.

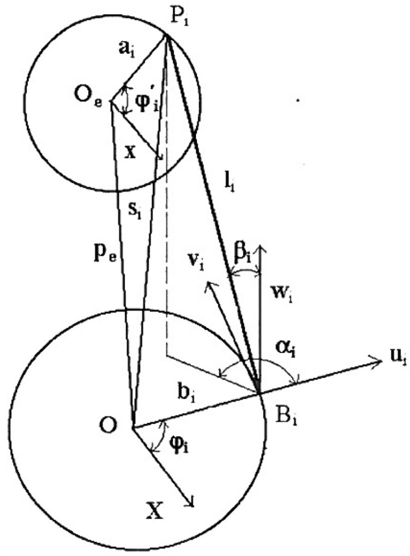

Fig. 3. i^{th} closed loop of both parallel manipulators

It can be readily shown From Fig. 3:

$$\mathbf{b}_i + \mathbf{l}_i - \mathbf{a}_i - \mathbf{p}_e = \mathbf{0} \tag{1}$$

where all vectors are expressed in reference frame as follows:

$$\mathbf{b}_i = [r\cos\varphi_i \quad r\sin\varphi_i \quad 0]^T \tag{2}$$

$$\mathbf{l}_i = \mathbf{R}_i \mathbf{l}_i' \tag{3}$$

$$\mathbf{a}_i = \mathbf{R}\mathbf{a}_i' \tag{4}$$

Here,

$$\mathbf{a}_i' = [c\cos\varphi_i' \quad c\sin\varphi_i' \quad 0]^T \tag{5}$$

$$\mathbf{l}_i' = [q_i \sin\beta_i \cos\alpha_i \quad q_i \sin\beta_i \sin\alpha_i \quad q_i \cos\beta_i]^T \tag{6}$$

$$\mathbf{R}_i = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

in which, r and c are the radii of circumferential circles of the base and moving platform, respectively. Moreover, q_i is the length of i^{th} leg and $\mathbf{R}$ is the rotation matrix of the moving platform with respect to base platform and can be written for 3 DOF manipulator and 6 DOF manipulator as

$$\mathbf{R} = \mathbf{R}_{roll} . \mathbf{R}_{pitch} \tag{8}$$

$$\mathbf{R} = \mathbf{R}_{roll} . \mathbf{R}_{pitch} . \mathbf{R}_{yaw} \tag{9}$$

where

$$\mathbf{R}_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \qquad (10)$$

$$\mathbf{R}_{picth} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \qquad (11)$$

$$\mathbf{R}_{yaw} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (12)$$

while, $\mathbf{p}_e$, the position vector of point O_e, is written for 3 DOF manipulator as $[0 \ \ 0 \ \ h]^T$ and for 6 DOF manipulator as $[x \ \ y \ \ z]^T$. The vector $\mathbf{l}_i$ can be written from eq. (1) as

$$\mathbf{l}_i = \mathbf{p}_e + \mathbf{a}_i - \mathbf{b}_i \qquad (13)$$

The length of i[th] leg is written as

$$q_i^2 = \|\mathbf{l}_i\|^2 = \mathbf{l}_i^{\mathrm{T}}.\mathbf{l}_i \qquad (14)$$

and the angle β_i is derived as

$$\beta_i = \arccos\left(\frac{\mathbf{l}_i(3)}{q_i}\right) \qquad (15)$$

where $\mathbf{l}_i(3)$ is the third component of vector $\mathbf{l}_i$.

It may be noted that there is no need to compute angle α_i since the problem does not force any limitation on it.

4. Workspace volume of the manipulators

The manipulator workspace volume, W, is numerically approximated using Monte Carlo method, as outlined by the following steps:

Step 1: A large number of points, n_{total}, are selected randomly in possible workspace. Possible workspace for 3 DOF is:

$$-\pi \le \varphi \le \pi, -\pi \le \psi \le \pi, H_{\min} \le h \le H_{\max}$$

And for 6 DOF is:

$$-\pi \le \varphi \le \pi, -\pi \le \psi \le \pi, -\pi \le \theta \le \pi$$

$$-r_s \le x \le r_s, -r_s \le y \le r_s, \ z_{\min} \le z \le z_{\max}$$

Where r_s=min(r, c) where r and c are as defined in eqs. (2) and (5).

Step 2: Each point of n_{total} points is considered to determine q_i, β_i and $\mathbf{s}_i = \mathbf{b}_i + \mathbf{l}_i$. Also to check if it falls within the manipulator workspace. This is accomplished by solving inverse kinematics problem for each leg as described by equations (13), (14), (15) in which if $q_{i\min} \le q_i \le q_{i\max}$ and $0° < \beta_i \le 90°$, the point is within the manipulator workspace.

Step 3: The workspace volume for 3DOF manipulator is defined as

$$W = \pi c^2 (h_{\max} - h_{\min}) \cdot \frac{n_i}{n_{total}} \qquad (16)$$

and for 6 DOF manipulator is written as

$$W = \pi r_{\max}^2 (h_{\max} - h_{\min}) \cdot \frac{n_i}{n_{total}} \qquad (17)$$

where n_i is the number of points that fall within the workspace obtained from step 2. h_{max}, h_{min} and r_{max} are defined by the following procedure

$$hmax = \max[\max(\mathbf{s}_i(3))] \ , \ \ hmin = \min[\min(\mathbf{s}_i(3))]$$

$$xmax = \max[|\max(\mathbf{s}_i(1))|] \ , \ \ xmin = \min[\min(\mathbf{s}_i(1))]$$

$$ymax = \max[|\max(\mathbf{s}_i(2))|] \ , \ \ ymin = \min[\min(\mathbf{s}_i(2))]$$

$$r_{max} = \max[x\max \ \ |x\min| \ \ y\max \ \ |y\min|] \qquad (18)$$

where $\mathbf{s}_i$ is the position vector of p_i with respect to O that is shown in Fig. 3. Note that, we define manipulator workspace as a space that contains three points of moving platform. These points are P_1, P_2, P_3 for 3 DOF manipulator and P_1, P_3, P_5 for 6 DOF manipulator. Therefore, i=1,2,3 for 3 DOF manipulator and i=1,3,5 for 6 DOF manipulator for the above mentioned procedure. It may be noted that having 3 points of vertices of the moving platform of 6 DOF manipulator, the other 3 points can be readily obtained.

5. Workspace volume optimization method

The objective of total workspace optimization is to determine the manipulator design variables such that results the largest total manipulator workspace. The design variables considered are r, c, φ_2, φ_3 for 3 DOF manipulator and r, c, φ_3, φ_5, λ_1, λ_2 for 6 DOF manipulator. Where leg no. 1 is assumed to align with the X-axis so $\varphi_1 = 0$ for both 3 and 6 DOF manipulators.

In order to bound the solution and to ensure a practical realization, the objective function is subject to the following constraints:

- All variables must be positive.
- r and c is not exceed one.
- Each leg must have an angular separation of at least $5°$ from the other legs.

Here, we assume:

$$H_{\max} = 2, H_{\min} = 1, q_{\max} = 2, q_{\min} = 1$$

$$z_{\max} = 1, \ z_{\min} = 1 \qquad (19)$$

Given this problem formulation, the optimization is computed using the Matlab optimization toolbox and performed the following results:

a. For 3 DOF manipulator r= .1, c=1, φ_2=5° and φ_3=355°.
b. For 6 DOF manipulator r=1, c=1, φ_3=10°, φ_5=350°, λ_1=5°, λ_2=5°.

Plots of the manipulator workspace with these design variables are shown for 3 DOF manipulator and 6 DOF manipulator in Fig. 4 and Fig. 5, respectively. Note that the optimization routine drove a value at the edge of the allowable design variables.

These solution results are theoretically acceptable for 3 DOF manipulator. However, with these results we can not connect vertical leg to base and moving platforms. For providing this constraint, we apply following constraints on design variables:

- $180^\circ \le \varphi_3 \le 180^\circ + \varphi_2$
- $5^\circ \le \varphi_2 \le 175^\circ$

The above constraints provide at least, the center of circumferential circle of moving platform to be located on one side of moving platform triangle.

Optimization routine produce the following results $r = .1$, $c= 1$, $\varphi_2=5^\circ$, $\varphi_3=180^\circ$ or $\varphi_2=5^\circ$, $\varphi_3=185^\circ$ or $\varphi_2=175^\circ$, $\varphi_3=355^\circ$.

A plot of the manipulator workspace with these new design variables is shown in Fig. 6 (for $\varphi_2=175^\circ$, $\varphi_3=355^\circ$).

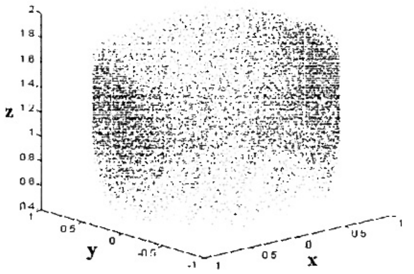

Fig. 4 Workspace of 3 DOF manipulator for max. workspace study

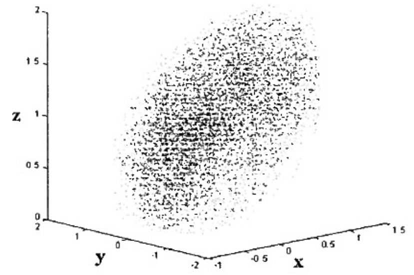

Fig. 5. Workspace of 6 DOF manipulator for max. workspace study

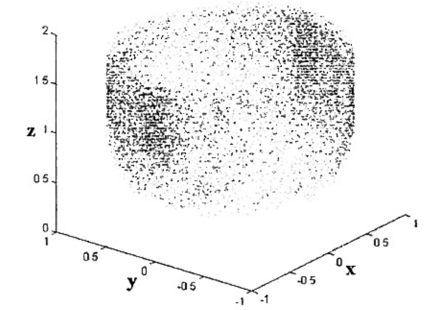

Fig. 6. Workspace of 3 DOF manipulator for max. workspace study with new parameters

6. Well-conditioned workspace volume optimization method

The global condition index over the entire workspace is defined for the manipulator as

$$\eta = \int_W \frac{1}{k} dW \tag{20}$$

where k is the condition number of the Jacobian of the manipulator at a given position in the workspace. The condition number of the Jacobian is defined as:

$$k = \frac{\sigma_{max}}{\sigma_{min}} \tag{21}$$

where σ_{max} and σ_{min} are the maximum and minimum singular value of the Jacobian[12]. The kinematics condition number, k, is varied between 0 and ∞ and thus we use $\frac{1}{k}$ in our study to limit its variations between 0 and 1. The Jacobian matrix maps the output velocity, i.e., the moving platform velocity to input velocity, i.e., actuated joint velocities. The Jacobian matrix for the manipulators at hand can be determined by time differentiating eq. (14) as

$$\mathbf{B}\dot{\mathbf{q}} = \mathbf{A}\mathbf{t} \Rightarrow \dot{\mathbf{q}} = \mathbf{B}^{-1}\mathbf{A}\mathbf{t} \Rightarrow \mathbf{J} = \mathbf{B}^{-1}\mathbf{A} \tag{22}$$

where **A** and **B** are defined for 3 DOF manipulator as

$$\mathbf{B} = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} \dot{\varphi} \\ \dot{\psi} \\ \dot{h} \end{bmatrix}, \dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{23}$$

where the elements of **A**, i.e., a_{ij} can be written as

a_{i1}=-rc sin(φ_i)cos(φ_i)cos(φ)sin(ψ)+rc sin(φ)-
rc cos^2(φ_i)sin(φ)+hc cos(φ_i)sin(φ)sin(ψ)+hc sin(φ_i)cos(φ)
a_{i2}=rc cos^2(φ_i) sin(ψ)-rc sin(φ_i)cos(φ_i)sin(φ)cos(ψ)-
hc cos(φ_i)cos(φ)cos(ψ)
a_{i3}=h-c cos(φ_i)sin(ψ)cos(φ)+c sin(φ_i)sin(φ) (24)

where, i=1,2,3 in above equations.

and **A** and **B** can be defined for 6 DOF manipulator as

$$\mathbf{B} = diag(q_1, q_2, q_3, q_4, q_5, q_6)$$

$$\mathbf{A} = \begin{bmatrix} (\mathbf{a}_1 \times \mathbf{l}_1)^T & \mathbf{l}_1^T \\ (\mathbf{a}_2 \times \mathbf{l}_2)^T & \mathbf{l}_2^T \\ (\mathbf{a}_3 \times \mathbf{l}_3)^T & \mathbf{l}_3^T \\ (\mathbf{a}_4 \times \mathbf{l}_4)^T & \mathbf{l}_4^T \\ (\mathbf{a}_5 \times \mathbf{l}_5)^T & \mathbf{l}_5^T \\ (\mathbf{a}_6 \times \mathbf{l}_6)^T & \mathbf{l}_6^T \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} \boldsymbol{\omega}^T & \dot{\mathbf{p}}_e^T \end{bmatrix}^T$$

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \dot{q}_4 & \dot{q}_5 & \dot{q}_6 \end{bmatrix}^T \tag{25}$$

here, ω is angular velocity of moving platform of the manipulator.

Upon computing the Jacobian matrix for both manipulators and substitution the results that obtained into eq. (20), the global condition index determined. However, analytical computation of eq. (20) is too cumbersome and thus we perform a numerical solution method such as Monte Carlo as follows:

the global condition index, η, is determined for 3 DOF manipulator as

$$\eta = \pi c^2 (h_{\max} - h_{\min}) \cdot \frac{KCI}{n_{total}} \quad \textbf{(26)}$$

and for 6 DOF manipulator as

$$\eta = \pi r_{\max}^2 (h_{\max} - h_{\min}) \cdot \frac{KCI}{n_{total}} \quad \textbf{(27)}$$

where KCI is the kinematics condition index which is the sum of the reciprocal of the condition number of the each point that falls within workspace, i.e., $KCI = \sum_{i=1}^{n_i} \frac{1}{k_i}$, where n_i is the number of points that fall within the workspace. All other parameter are as defined in eqs. (16) and (17). The global condition index is a performance index to show the manipulator is how far from singularity. Therefore, the objective of this section is to determine a well-conditioned workspace volume, which has the highest global condition index.

New design variable, L, is added to the same set of design variables that were used during the total workspace optimization and this new set is used for the well-conditioned workspace optimization. L is the characteristic length of the manipulator that is used to homogenize the elements of Jacobian matrix to have a same dimension, because the condition number should have no dimension.

The objective function is also subjected to the constraints as were used during the total workspace optimization. The well-conditioned workspace optimization produced the following results:

a. r=0.1, c=0.7, $\varphi_2=120^\circ$, $\varphi_3=240^\circ$, L=.5 for 3 DOF manipulator
b. r=1, c=0.5, $\varphi_3=120^\circ$, $\varphi_5=240^\circ$, $\lambda_1=115^\circ$, $\lambda_2=5^\circ$, L=.5 or $\lambda_1=5^\circ$, $\lambda_2=115^\circ$ for 6 DOF manipulator

When 100,000 points were used for Monte Carlo method. Plots of the manipulator workspaces with these design variables are shown for 3 DOF manipulator in Fig. 7 and for 6 DOF manipulator in Fig. 8.

7. Discussion on results

As it is shown for 3 DOF manipulator in Fig. 4 and 7 and for 6 DOF manipulator in Fig. 5 and 8, workspace volume is reduced when we optimize manipulators for global condition index.

Also, when we used first method, maximum workspace, to optimize the manipulator the condition number is reduced very much over the entire workspace and so the workspace become ill-conditioned, as it is depicted for 3 DOF manipulator in Fig 9 and 10 and for 6 DOF in Fig. 11 and 12. It can be concluded from the results that the second optimization method, i.e., a well-conditioned workspace volume that has the highest global condition index is a better optimal design.

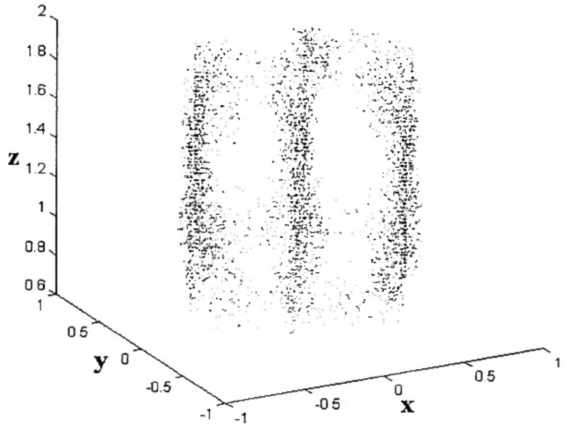

Fig. 7. Workspace of 3 DOF manipulator for well-cond. workspace study

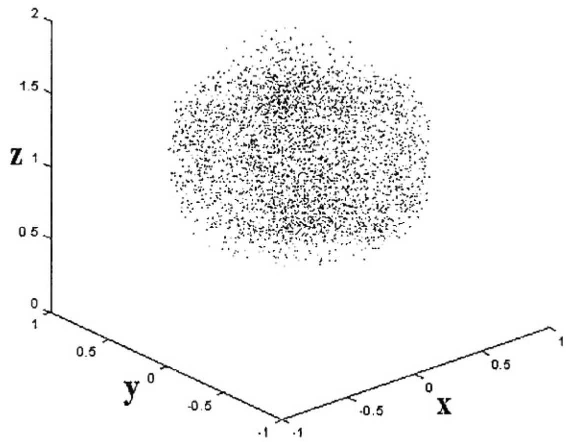

Fig. 8. Workspace of 6 DOF manipulator for well-cond. workspace study

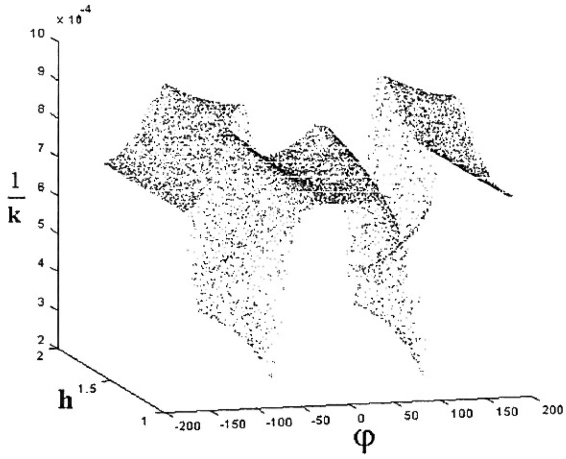

Fig. 9. (1/k) of 3DOF manipulator for max. workspace study in ψ=0 plane

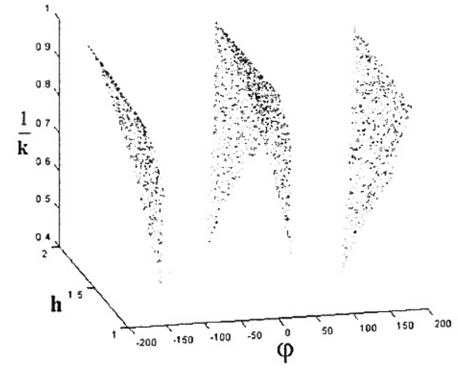

Fig. 10. (1/k) of 3DOF manipulator for well-cond. workspace study in ψ=0 plane

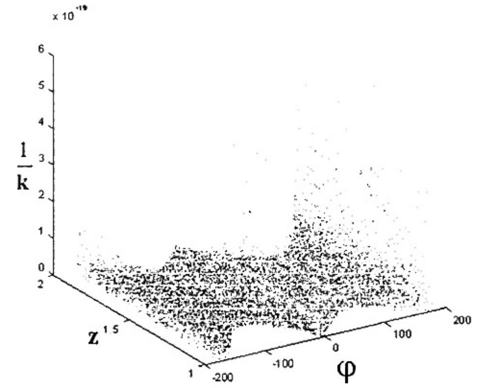

Fig. 11. (1/k) of 6DOF manipulator for max. workspace study in ψ=0, θ=0, x=0, y=0 plane

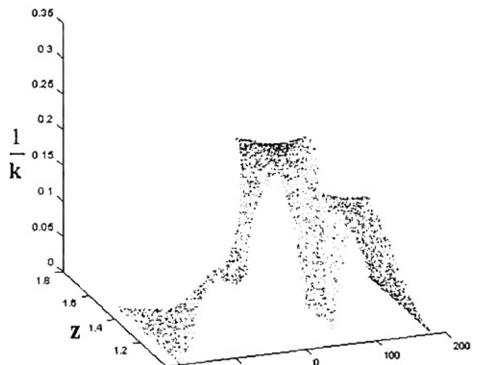

Fig. 12. (1/k) of 6DOF manipulator for well-cond. workspace study in ψ=0, θ=0, x=0, y=0 plane

8. Conclusion

Two optimization methods on the design of two kinds of parallel manipulators were performed. The first method was based on maximizing the total volume of manipulator workspace. While the second method optimized the total volume of well- conditioned workspace by maximizing a global condition index. The results of optimizations of both methods were shown graphically. It can be concluded from the results that second method is better for optimal design of manipulators.

Acknowledgment

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