Recent Advances in Verification and Analysis of Hybrid Systems

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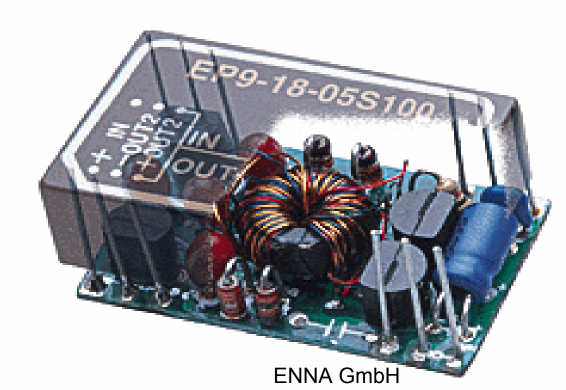
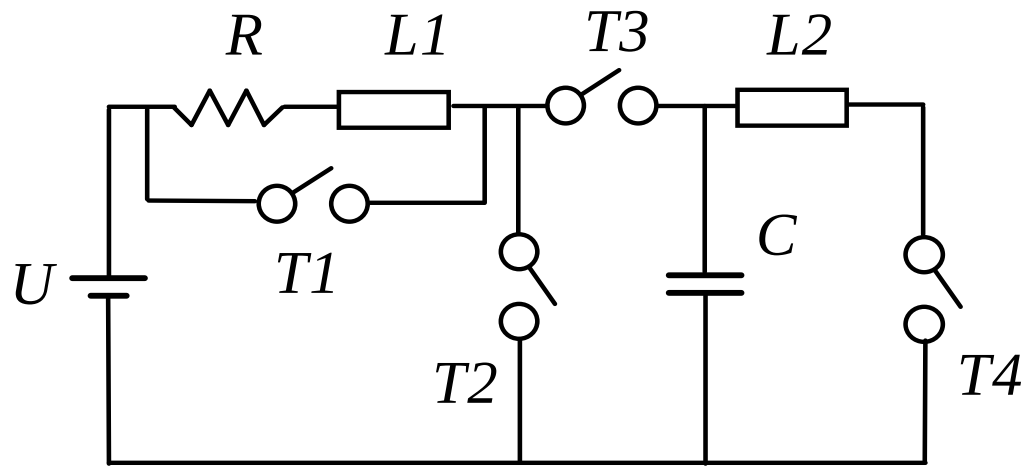
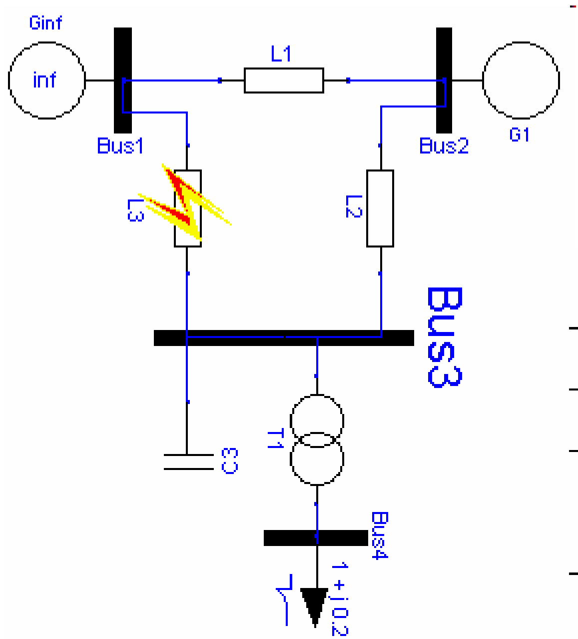
Hybrid Systems: Examples

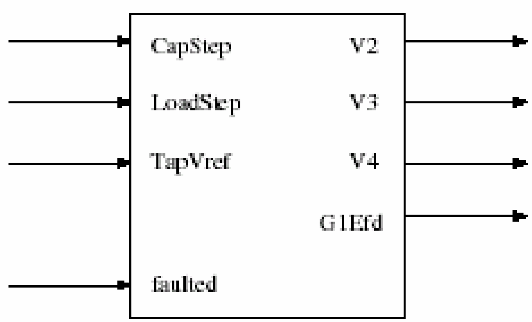
- Systems with commutations: electrical circuits

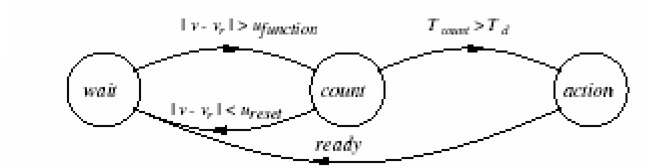




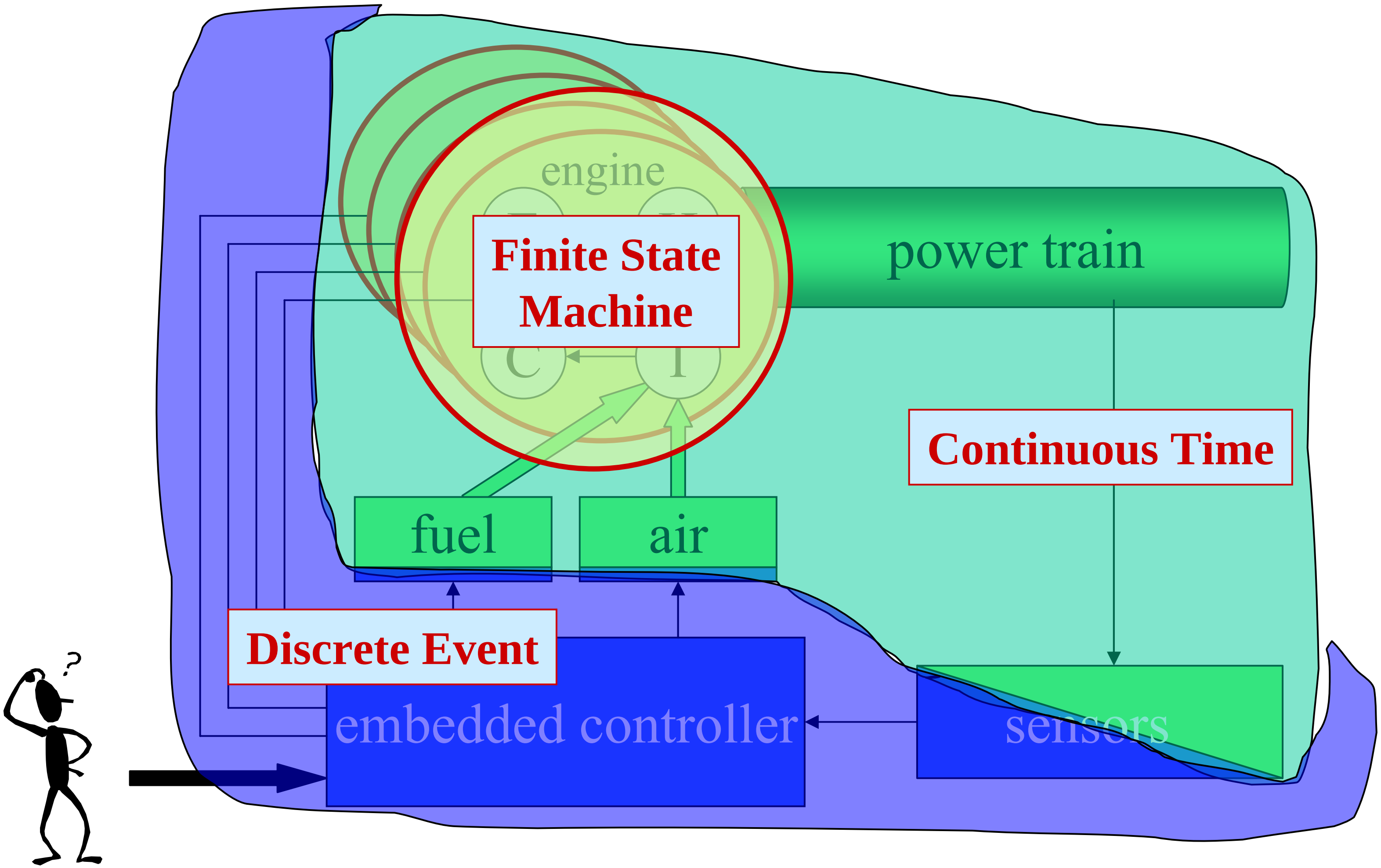
- Electric Networks: manage & optimize system configuration through discrete connections/disconnections of parts of the net to regulate electrical energy





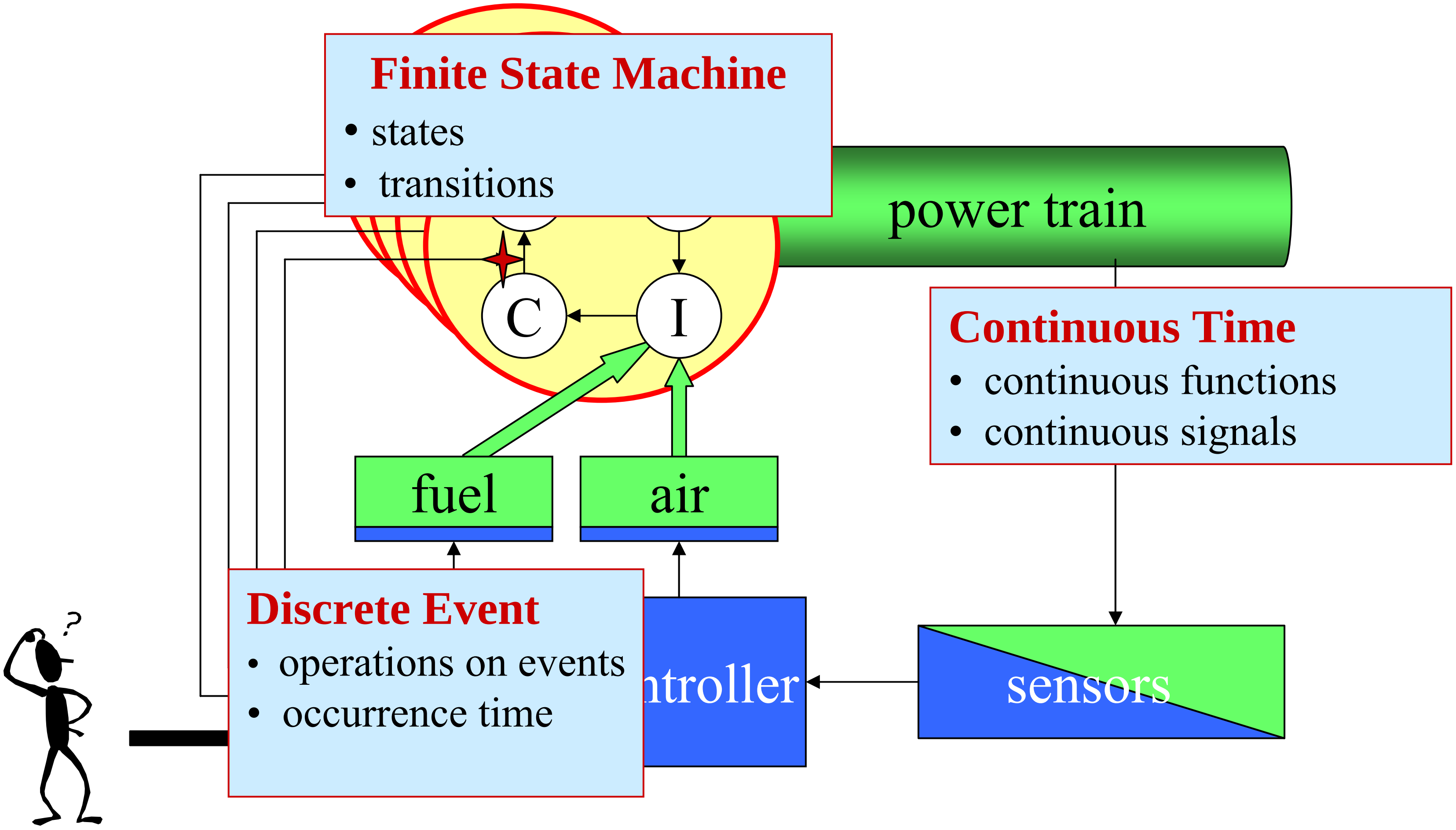


The Heterogeneity of Systems

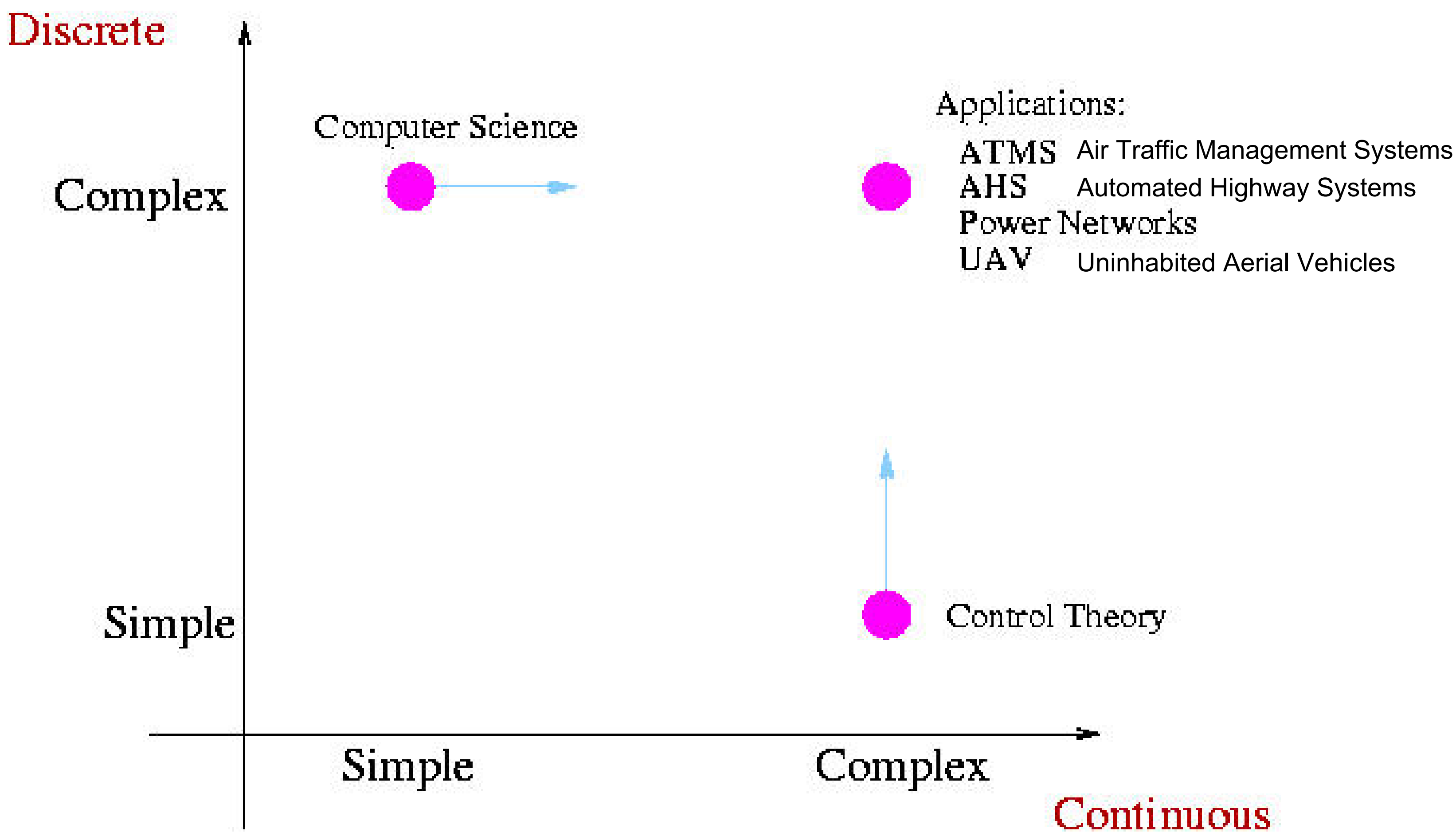


An Engine Control System

Models of Computation



Different Approaches



- Hybrid Systems: Dynamical systems with interacting continuous and discrete dynamics

Research Issues in Hybrid Systems

- Modeling & Simulation
  - classify discrete phenomena, existence and uniqueness of execution, Zeno
  - composition and abstraction operations
- Analysis & Verification
  - avoid or attain forbidden states: algorithmic or deductive methods, abstraction
  - stability, Lyapunov techniques, LMI techniques
- Controller Synthesis
  - optimal control, hierarchical control, supervisory control, safety specifications, control mode switching
  - algorithmic synthesis, synthesis based on HJB

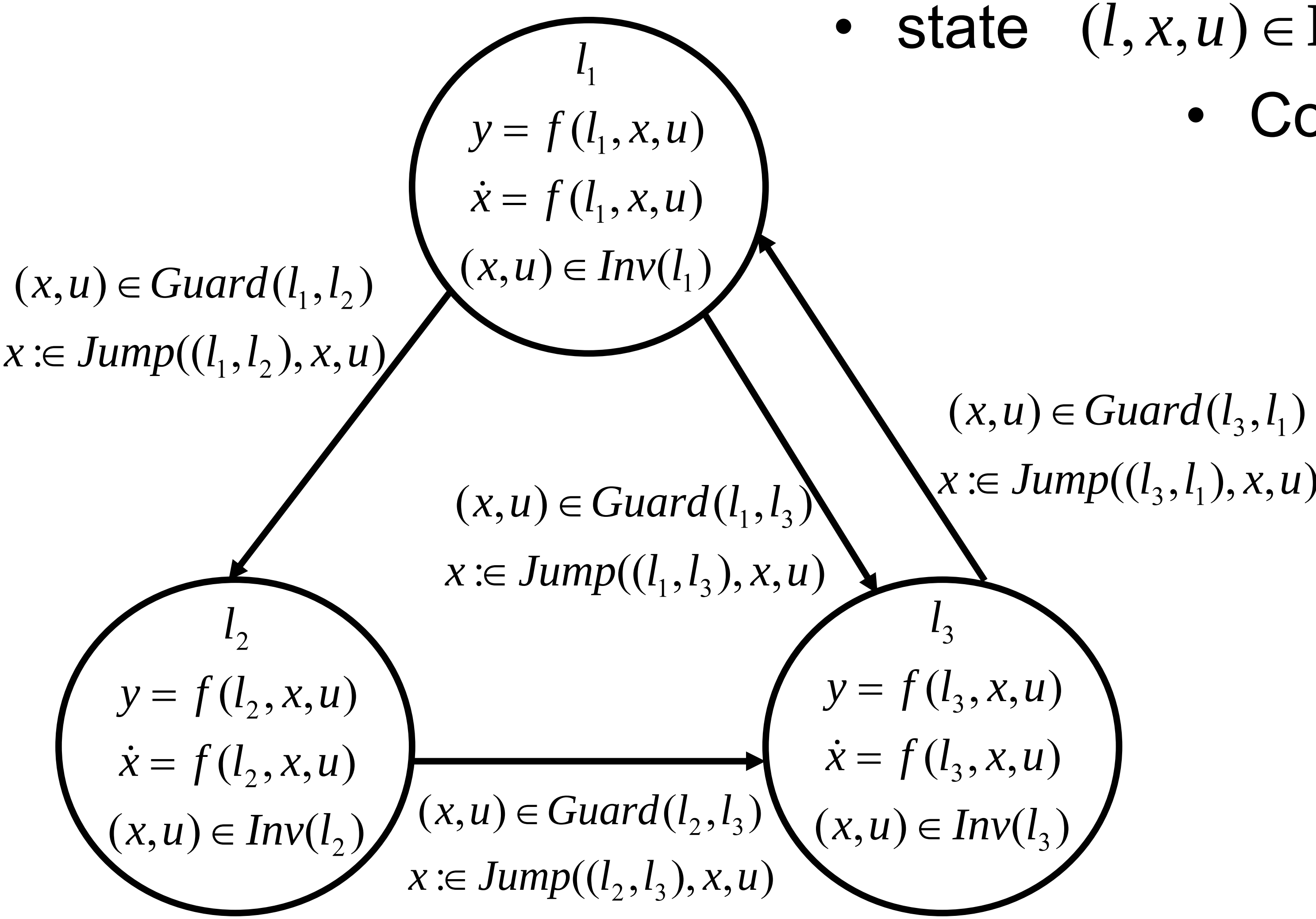
- IFAC Technical Committee on Discrete Event and Hybrid Systems
  - IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'03 in France, ADHS'06 in Italy, ADHS'09 in Zaragoza – Spain)
- IEEE WG Hybrid Systems
- Nonlinear Analysis: Hybrid Systems (International Journal, Elsevier)
- National groups, NOE, European and International projects, Annual Workshop on Hybrid Systems

Outline

- **Safety verification and reachability**
  - Hybrid automaton
- Abstraction
  - Conserve hybrid nature of the system
  - Discrete-Event abstraction
- Characterizing reachable space
- Reachable space computation (overapproximation)

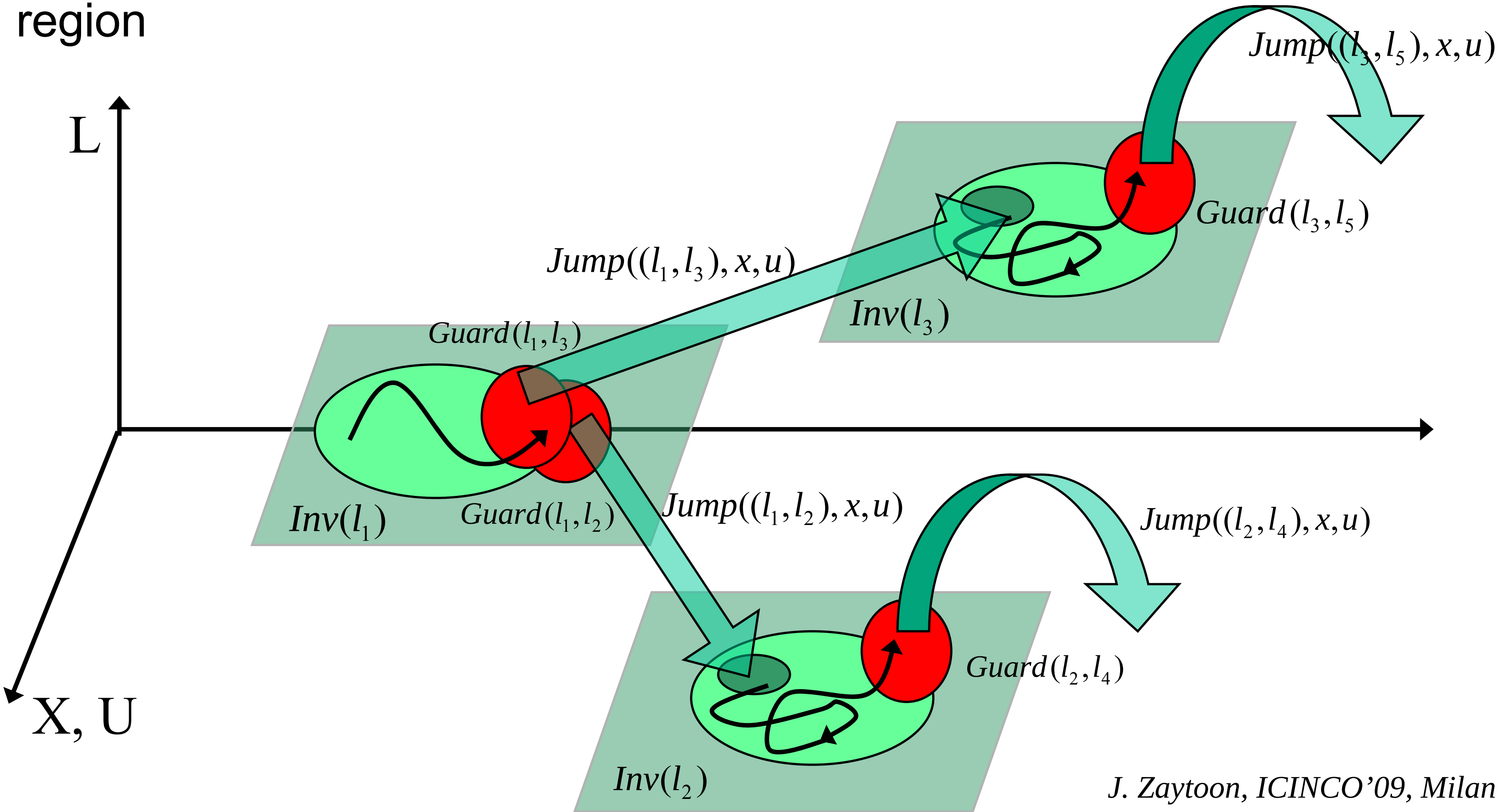
Hybrid Automaton

- *< L, X , U, INV, F, E, Guard, Jump, l_0, x_0 , u_0 >*
  - state $(l, x, u) \in \mathbf{L} \times \mathbf{X} \times \mathbf{U}$
    - Composition



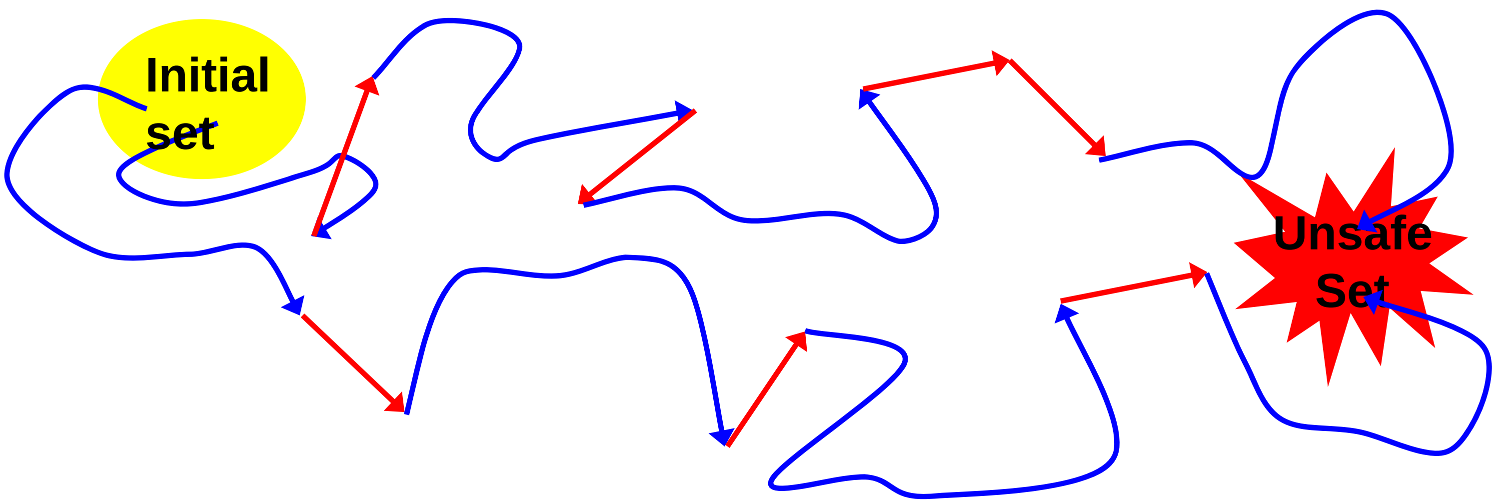
Reachable Sets

- Execution: Admissible trajectories described by a succession of continuous & discrete evolutions
- State can advance by progression of time in the current location or by an instantaneous transition to a new location
- Continuous & discrete successors (predecessors) for a point or a region



Algorithmic Verification: Safety verification

- Since the state space of HS implicitly includes time, many properties of HS can be expressed as reachability properties
- Safety properties (is the system dangerous to itself or to its environment): Verify, trough reachability computation, that for any initial condition, the hybrid state can never enter some unsafe region
- Decidability is a central issue in algorithmic analysis because of the uncountability of the hybrid state space



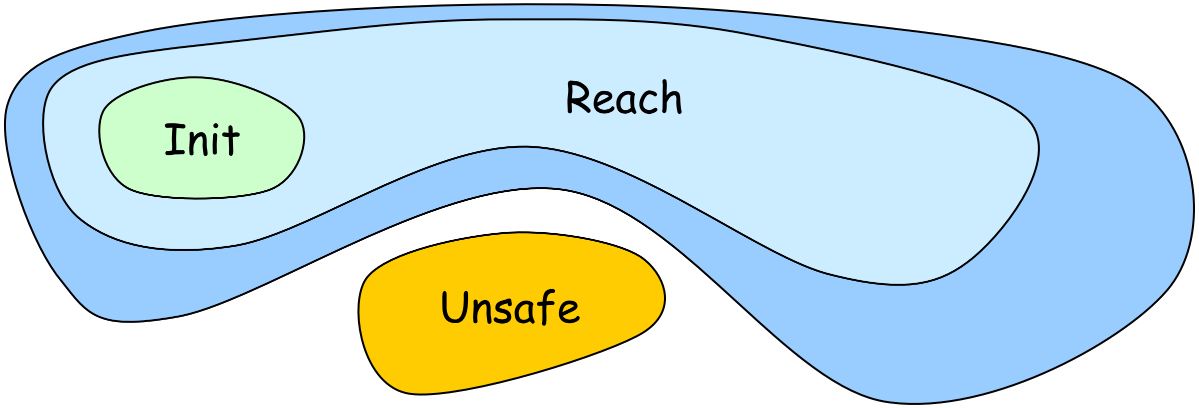
Hybrid Reachability based Verification

- Computation of the reachable set: *starting at Init, determine the limit of the series of regions defined by*

$$R_i = Succ_C(Init)$$

$$R_{i+1} = R_i \bigcup Succ_C(Succ_D(R_i))$$

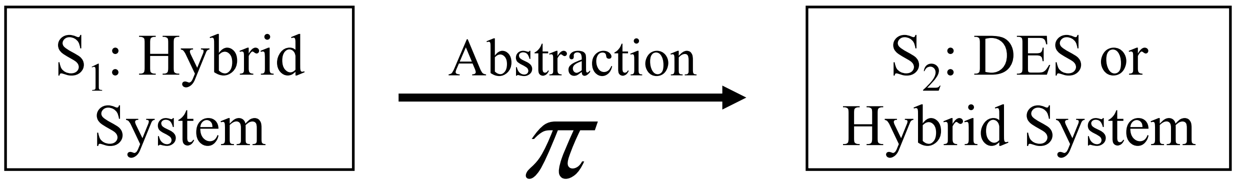
- exactly for some very simple classes of systems: *Piecewise constant differential inclusions, some linear systems*
- approximately for other classes: over-approximation algorithms, set-based simulation



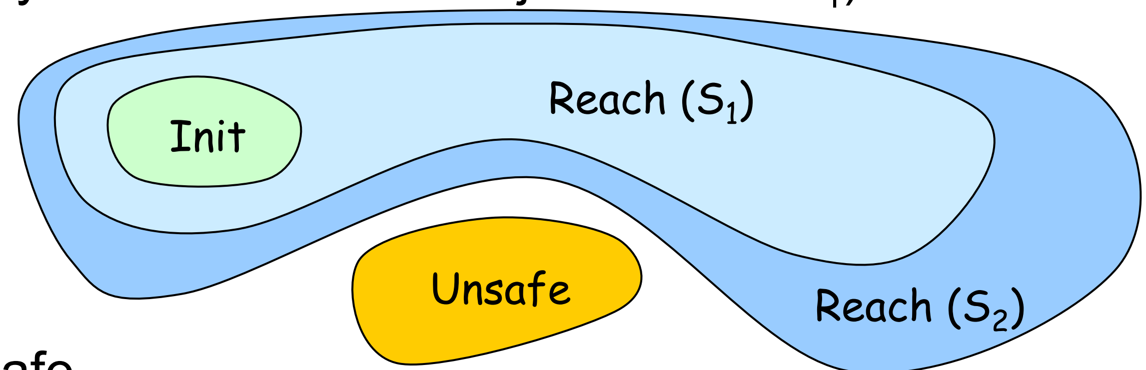
Outline

- Verification and reachability
- **Abstraction**
- Characterizing reachable space
- Reachable space computation

Abstraction



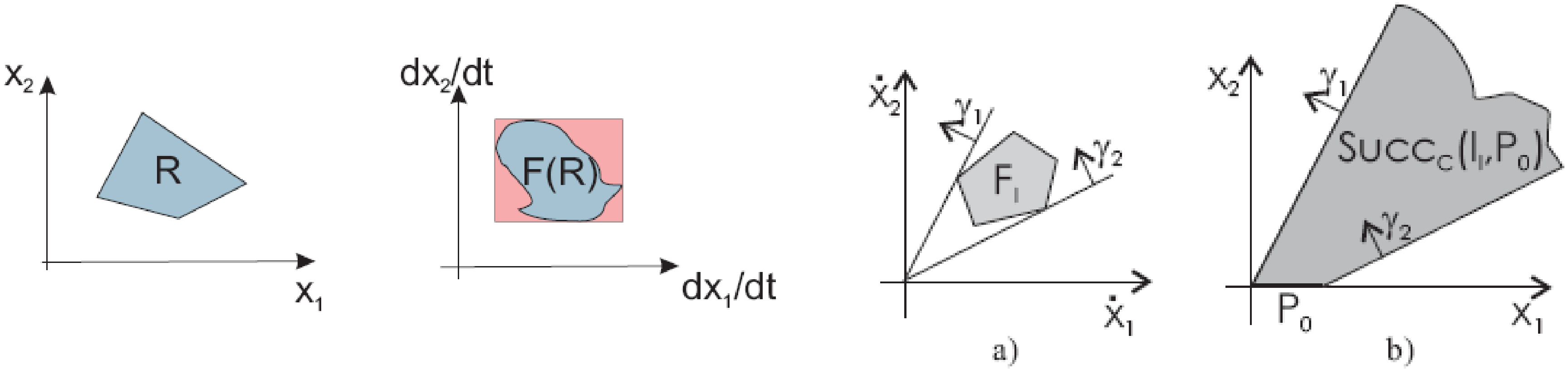
- S_2 is an abstraction of S_1 iff the image of each trajectory of S_1 is also a trajectory of S_2 (but some executions in S_2, introduced by the abstraction process, may not be related to trajectories in S_1)



- If S_2 is safe then S_1 is safe

- Linear differential inclusion abstraction
- Discrete avent abstraction

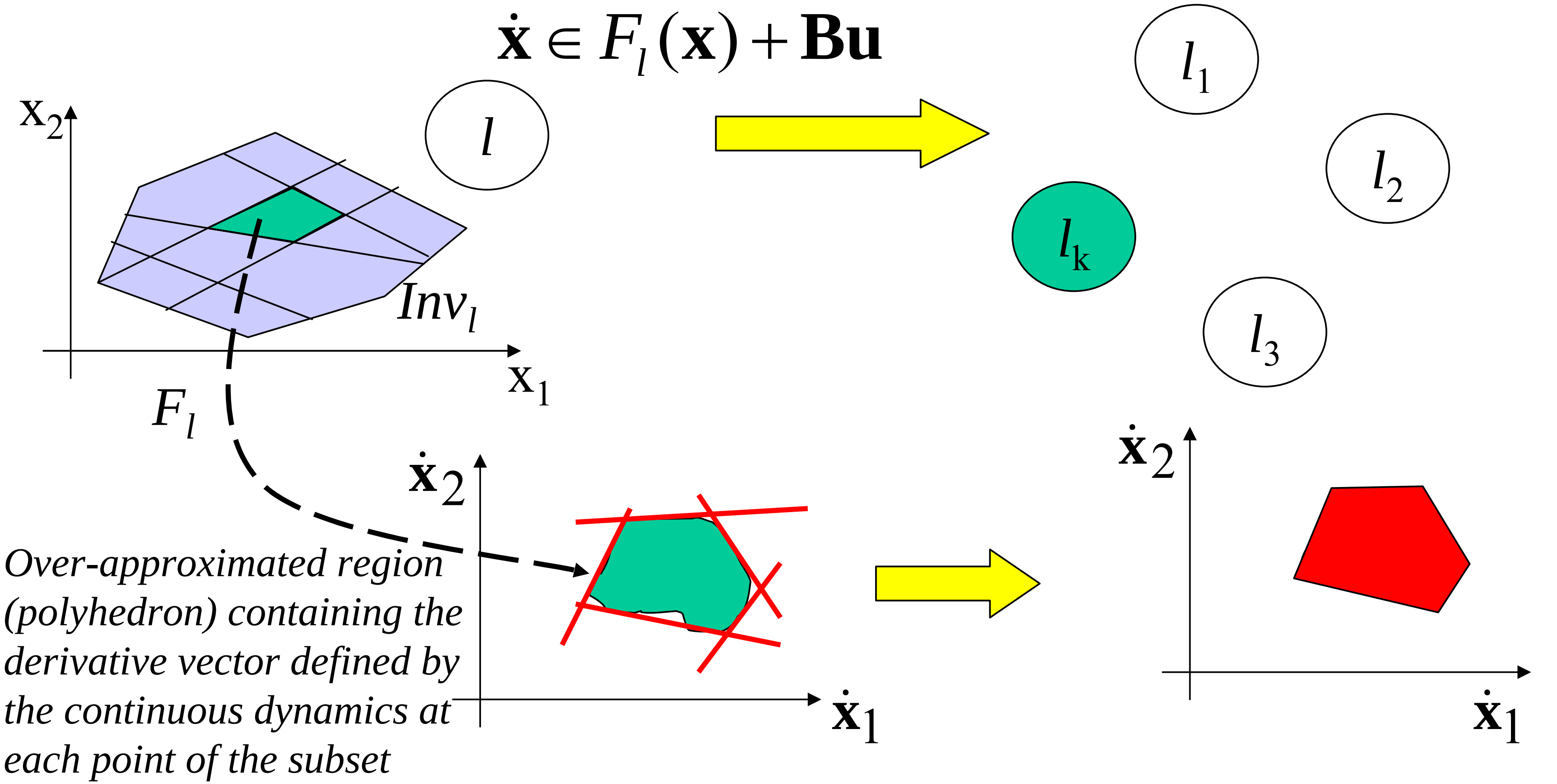
Linear differential inclusion abstraction or hybredization (Henzinger et al., 98; Frehse, 05; Lefebvre,Gueguen & Zaytoon, 06):



- Approximation of complex continuous dynamics by simpler hybrid dynamics
- Calculate differential inclusion that includes the derivative vector defined by the continuous dynamics at each point of the invariant of a location
- Use the differential inclusion (derivative vectors Υ_1 and Υ_2) to compute the reachable space from P_0
- The resulting abstraction (resulting HLA) is generally too coarse, and hence the overapproximated reachable space does not allow us to conclude for safety verification

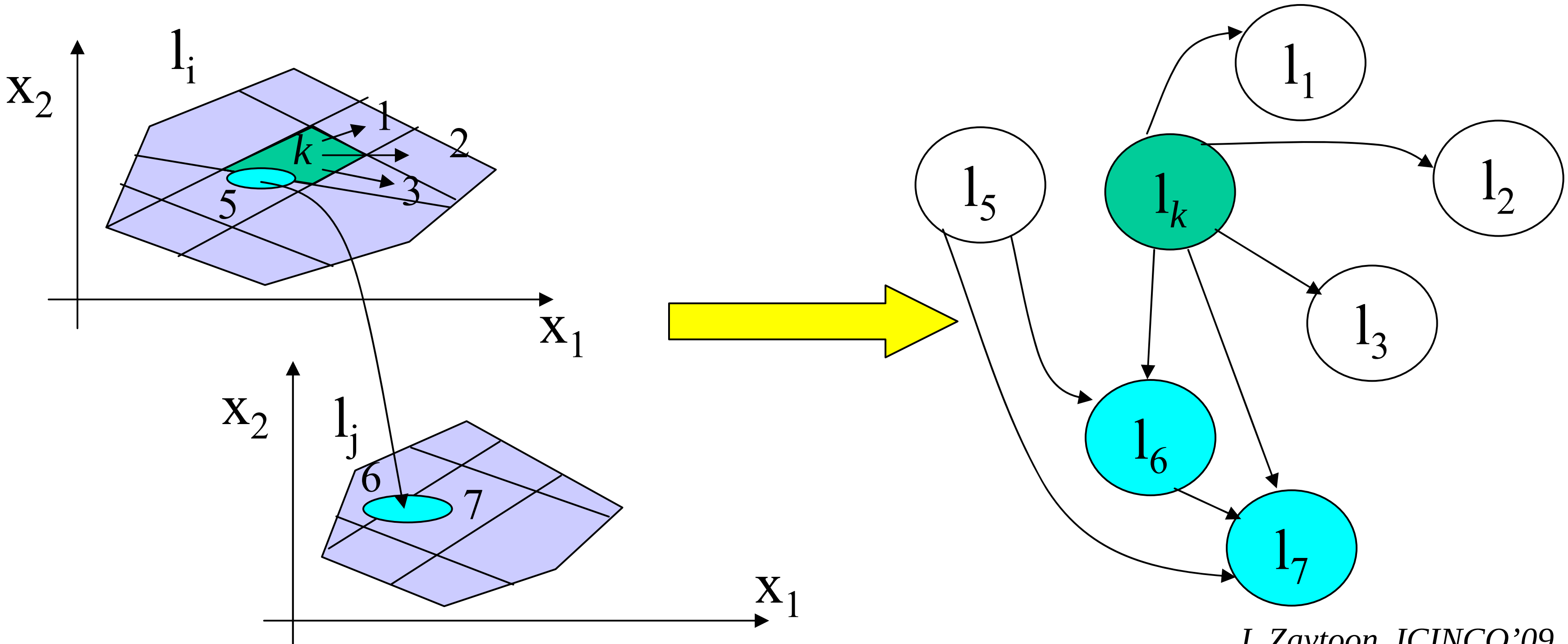
Hybredization: Refine the abstraction

– Partition the invariant of a location into n subsets and replace the location with n locations whose reachable spaces are over-approximations of the corresponding subset region



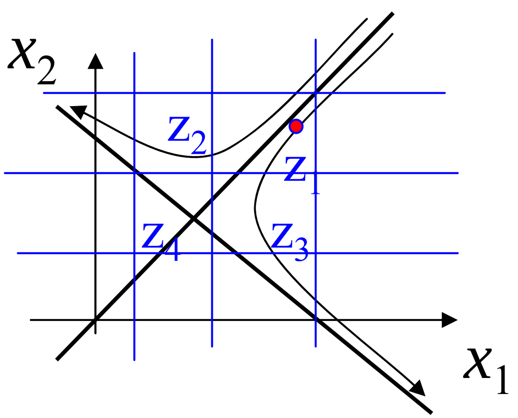
Linear differential inclusion abstraction

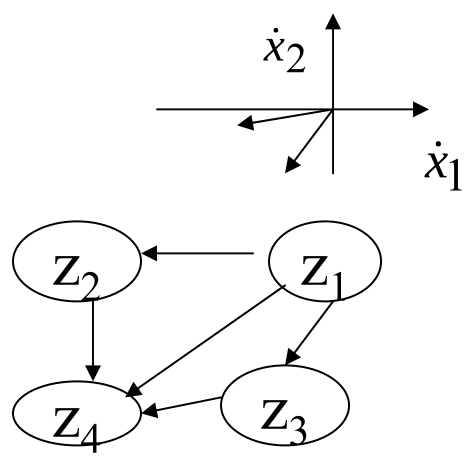
- Include a transition between two sub-locations of a location if there exists a continuous trajectory crossing the boundary between the corresponding elements of the partition
- For each $e(l_i \rightarrow l_j)$, include a transition from each sub-location of l_i intersecting *Guard*(e) to each sub-location of l_j intersecting *Jump*(e)
- Then calculate reachability using the resulting abstraction

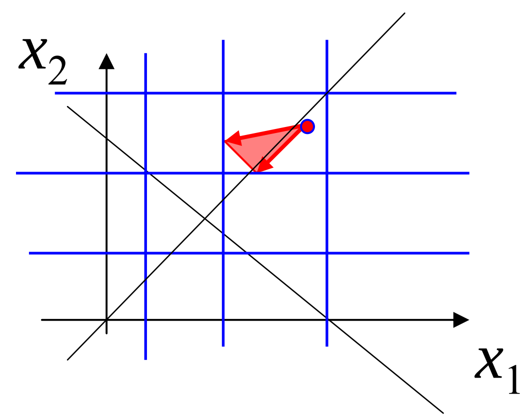


Reachability

- Refine abstractions if resulting regions are too coarse
- No guarantee that this abstraction will eventually allow to conclude
- Difficulty: determine a pertinent criteria to refine the partition to improve the efficiency of reachability calculation
  - Continuous dynamics can be used to determine the regions defining the partition of the state space (tradeoff: precision of abstraction vs. simplicity of calculation)

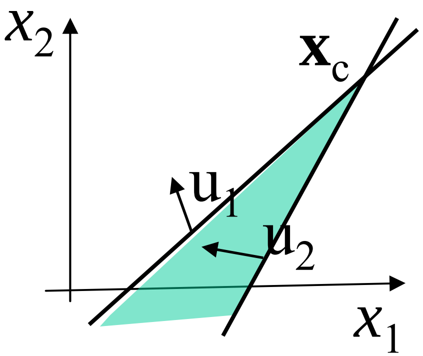


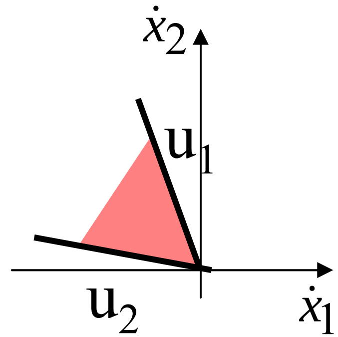




Linear differential inclusion abstraction: Lefebvre, Guéguen, Zaytoon

- Simple case: Affine planar systems: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathrm{b}$





$$\begin{matrix} w_1{}^t(\mathbf{x}-\mathbf{x_c}) \leq 0 \\ w_2{}^t(\mathbf{x}-\mathbf{x_c}) \geq 0 \end{matrix} \Rightarrow \begin{matrix} v_1{}^t\dot{x} \leq 0 \\ v_2{}^t\dot{x} \leq 0 \end{matrix}$$

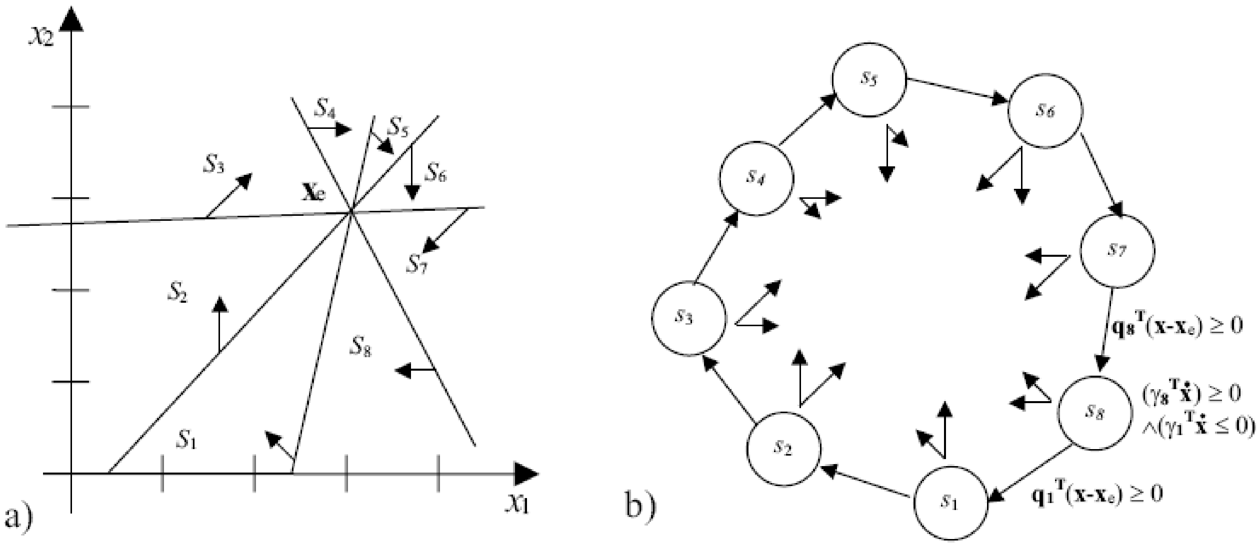
- Half lines defined by the equilibrium point are very useful in specifying the partition: at all points of this line, the derivative vector is collinear to a unique vector and, so, the trajectories cross the half-line in the same direction, leading to a very simple structure for the abstraction
- The derivative vector of each point between 2 such half lines, is included in the convex hull of the 2 vectors characterizing the boarder lines, and this defines the differential inclusion of the abstraction

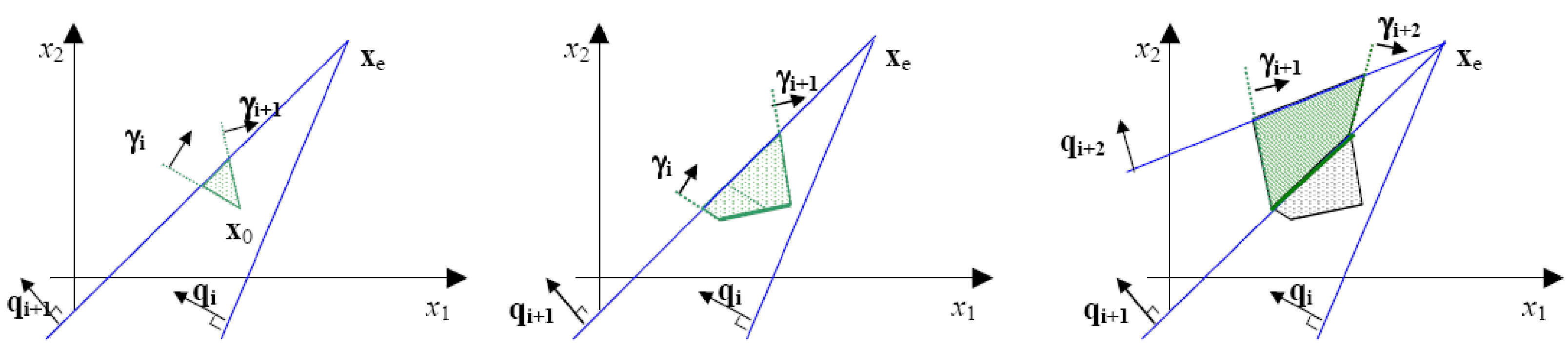
Linear differential inclusion abstraction: Lefebvre, Guéguen, Zaytoon

- Resulting HA for a partition of 8 elements:

  - continuous dynamics in each location given by the differential inclusion representing the border line of the corresponding region

  - transition guards given by the border lines

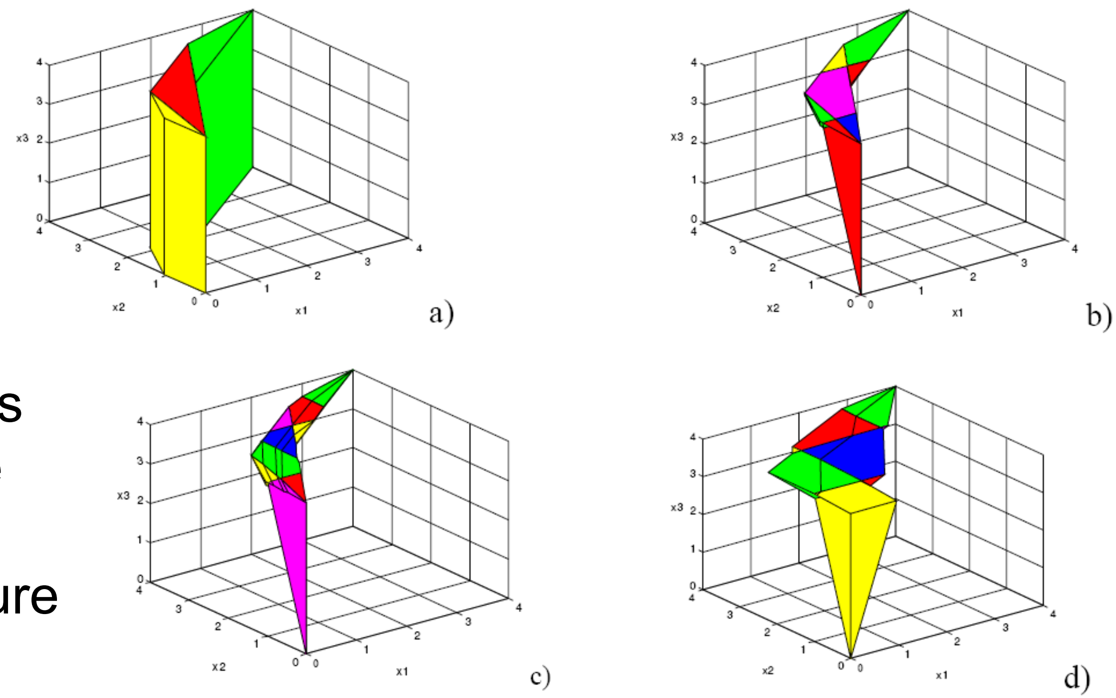




Affine systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathrm{b}$

$$H = \{x \mid \mathbf{q}^T x = k\}, \text{ where } \exists \gamma \text{ s.t. } \mathbf{q} = \mathbf{A}^{\mathrm{T}}\gamma \text{ , k} = -\gamma^{\mathrm{T}} b$$

- For higher dimension affine systems, it is possible to consider families of hyperplanes with certain constraints s.t. all trajectories cross the hyperplanes in the same direction, leading to a very simple transition structure for the abstraction



- Extension to systems defined by: $\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}\mathbf{u}$

  U : space of continuous inputs is a polytope (Nasri et al., 06)

Discrete Event Abstraction: Alur et al. 03, Chutinan & Krogh 03, Tiwari & Khanna 04, Ratschan & She 05, Blouin et al. 03, Kloetzer & Belta 06

$$\boxed{\text{HS}} \xrightarrow{\text{Abstraction } \pi} \boxed{\text{DES}}$$

- Construction
  - partition of state space (consider specific regions: guard, invariants, R_{init}, R_{unsafe}, and other regions linked to the property or sometimes their borders)
  - associate an abstract discrete-state to each element of the partition
  - Calculate the transitions: constraint to satisfy

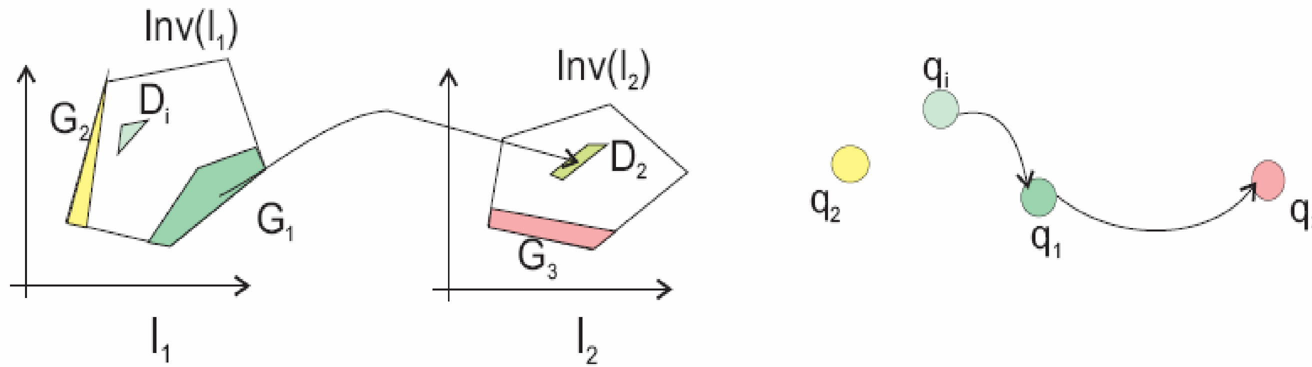
$$(l_k, \mathbf{x}_k) \in \mathrm{Re}\,ach(l_n, \mathbf{x}_n) \Rightarrow \pi((l_k, \mathbf{x}_k)) \in Succ(\pi((l_n, \mathbf{x}_n)))$$

$$R_{unsafe} \cap \mathrm{Re}\,ach(R_{init}) = \phi \quad \Longleftarrow \quad q_{unsafe} \notin Succ(q_{init})$$

  - If safety condition is not satisfied, iterate the abstraction

- Choice of discrete states

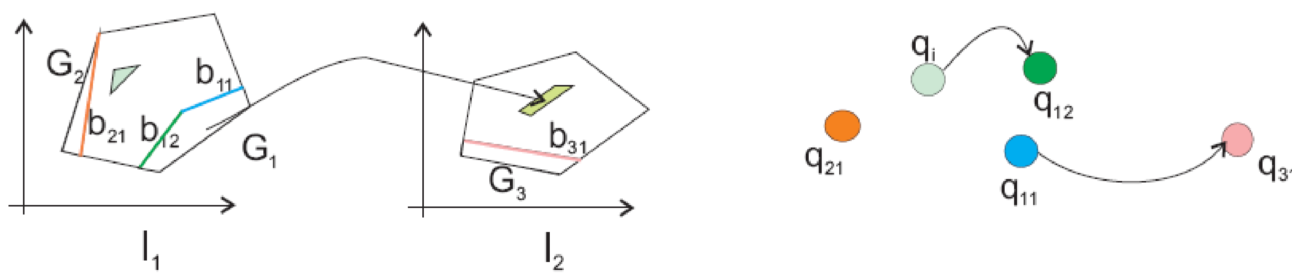
Abstraction on guards
$q_e = \pi(l_k, G_e)$



transition from q_1 to q_3 stems from the continuous reachability of G3 from D2

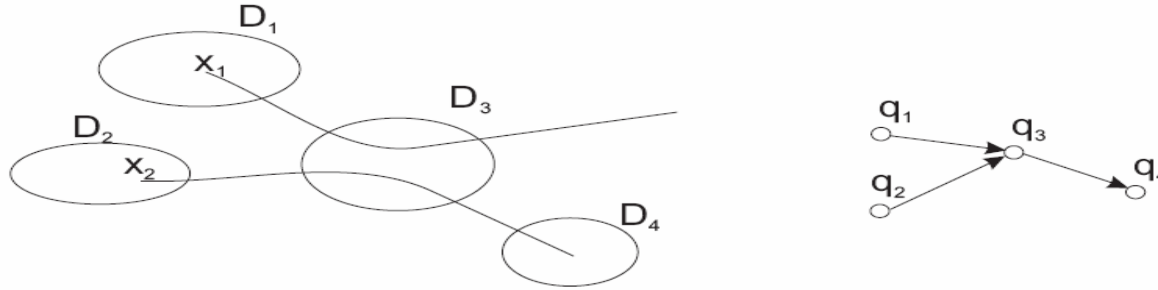
Include a transition from q_a to q_b if $G_b \subset Succ_C(Succ_D(G_a))$

Abstraction on borders:



Include a transition from q_a to q_b if $b_b \subset Succ_C(Succ_D(Succ_C(b_a)))$

Abstraction



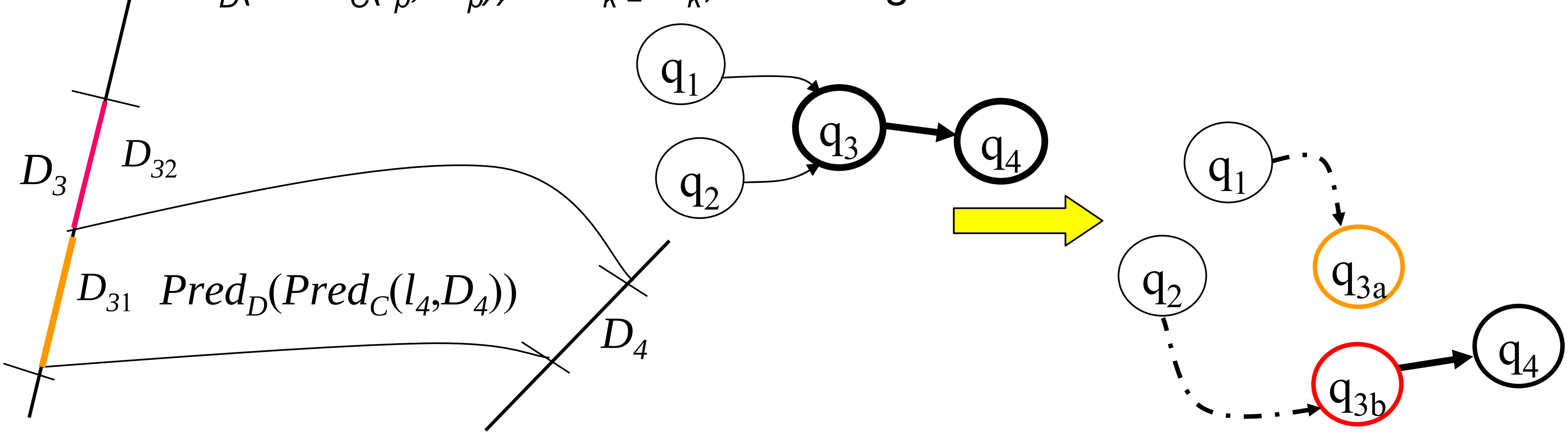
- Spurious transitions due to abstraction
- Iterative algorithme to refine the abstraction (Tabuada et al., 2002)

  Consider a discrete transition & partition the continuous domain of the region mapped to the source location

  If $Pred_D(Pred_C(l_p, D_p)) \cap D_k \neq D_k$, split Dk to:

  $D_{k1} = Pred_D(Pred_C(l_p, D_p)) \cap D_k$; $D_{k2} = D_k - (Pred_D(Pred_C(l_p, D_p)) \cap D_k)$

  If $Pred_D(Pred_C(l_p, D_p)) \cap D_k = D_k$, no change



- Difficulty: choice of transition to refine:
  - transitions leading to regions close to forbidden area
  - Transitions close to counter-example trajectory provided by verification

Outline

- Verification and reachability
- Abstraction

  Building an abstraction requires the determination of reachable regions: 2 types of answers

  - if the problem is to decide whether there is a discrete transition between 2 locations in case of hybredization or 2 discrete states in case of DE abstraction, use methods that gives a yes/no answer

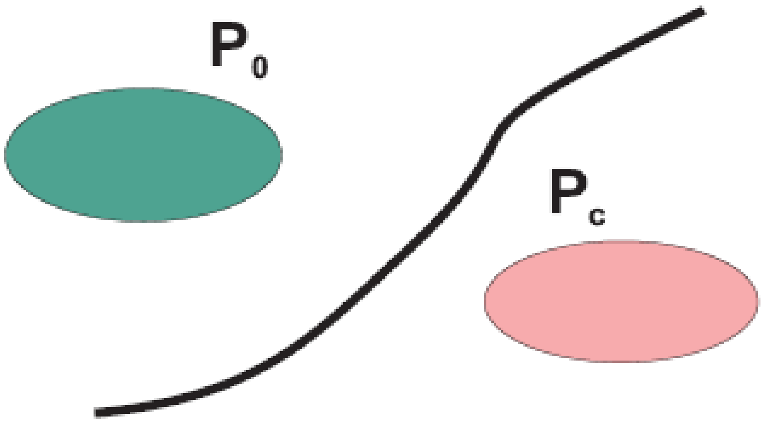
- **Characterizing reachable space**
  - *to refine the DE abstraction*
- Reachable space computation

In both approaches, reachability calculation is only related to 1 location or 2 successive locations

Characterizing reachable space

- Is it possible to reach a region P_c from region P_0 without explicitly computing the reachable space?

- Display borders separating the two domains and uncrossable by continuous trajectories



- Constraints inconsistency: determine partial (easier to compute) characteristics of reachable and goal region and prove their inconsistency

- Existence of Trajectories from P_0 to P_c ??

Uncrossable borders: Use structural properties of continuous dynamics to define borders characterising invariant domains that continuous trajectories never leave & include initial region (Tiwari, 03, Rodriguez & Tiwari 05)

- Example: linear dynamics $\dot{\mathrm{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \mathrm{x}$

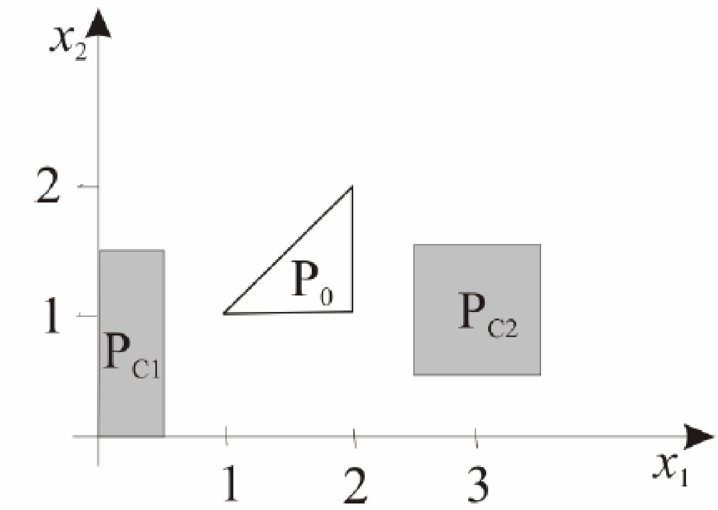
  +ve real eigenvalues λ (2, 4)

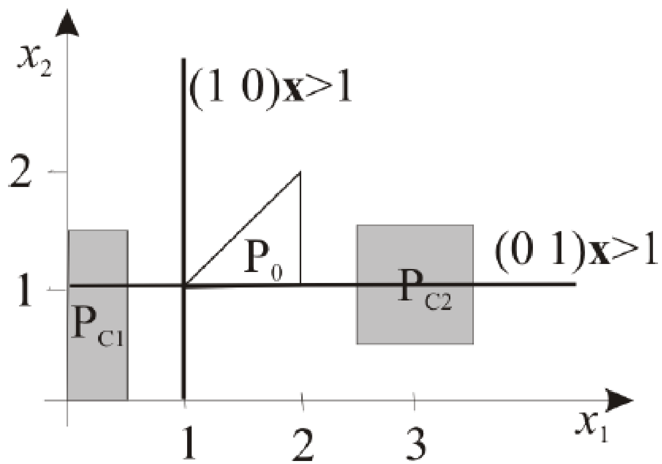
- $c_1=(1\ 0)^T$, $c_2=(0\ 1)^T \rightarrow \boldsymbol{c}^T x \geq min_{P0} (\boldsymbol{c}^T x)$ if $\lambda>0$

  $\rightarrow$ reachable space characterized by: $\boldsymbol{c_1}^T x \geq 1$, $\boldsymbol{c_2}^T x \geq 1$

  $\rightarrow P_{C1}$ unreachable, P_{C2}??

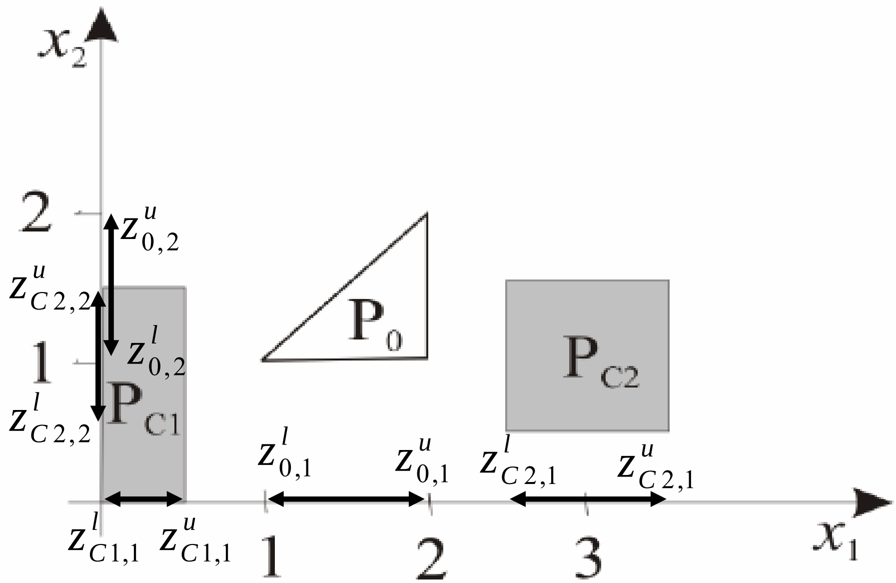
- Extension to complex λ





Inconsistent Temporal constraints on reachability in eigenspaces (Yazarel & Pappas 04)

- $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$: Projection of trajectory from $\mathbf{x}_0$ on eigenspaces (of dimension 1) associated with real eigenvalues
- Compute min & max time necessary (through linear programming) to go from projection of P_0 to projection of P_C for each eigenspace
- Check for –ve value of max time or check emptiness of intersection of time intervals from different eigenvectors

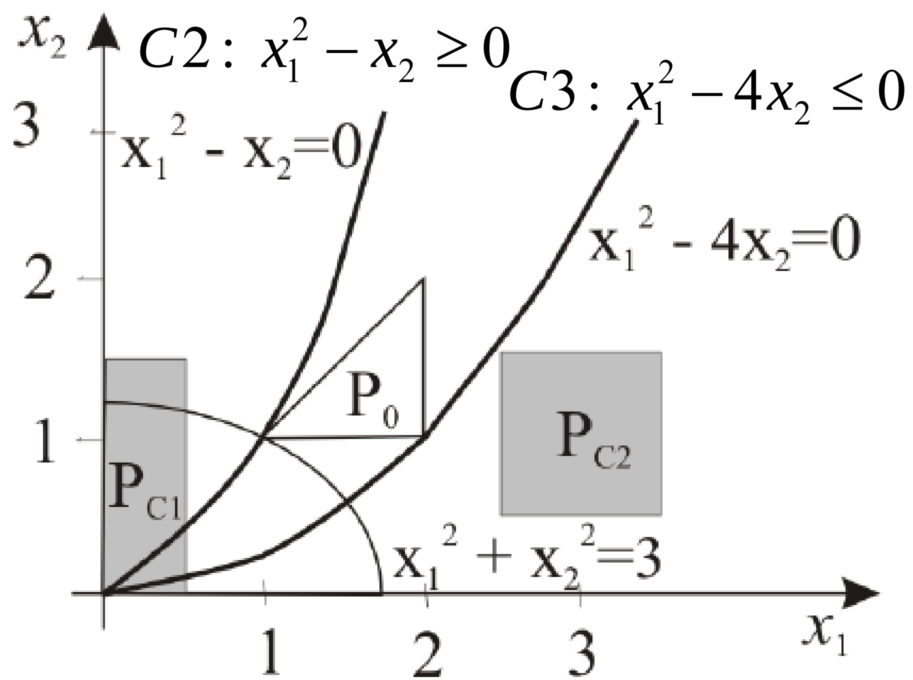


- Projections of P_0 & P_{C1} on subspace defined by eigenvector (1,0):
  bounds: $(-\infty \quad 0.5\ln 0.5)$
  $t_u<0 \rightarrow P_{C1}$ unreachable from P_0
- Projections of P_0 & P_{C2} on (1,0):
  bounds: $(0.5 \ln 1.25 \quad 0.5 \ln 3.5)$
  Projections of P_0 & P_{C2} on (0,1):
  bounds: $(0 \quad 0.25 \ln 1.5)$
  since $0.25 \ln 1.5 < 0.5 \ln 1.25 \rightarrow P_{C2}$ unreachable from P_0
- The more the number of eigenvalues associated with eigen subspace of dimension 1, the more the chances to conclude that P_c is unreachable

Inconsistent Spatial (polynomial) constraints on reachability in eigenspaces (Yazarel et al. 04)

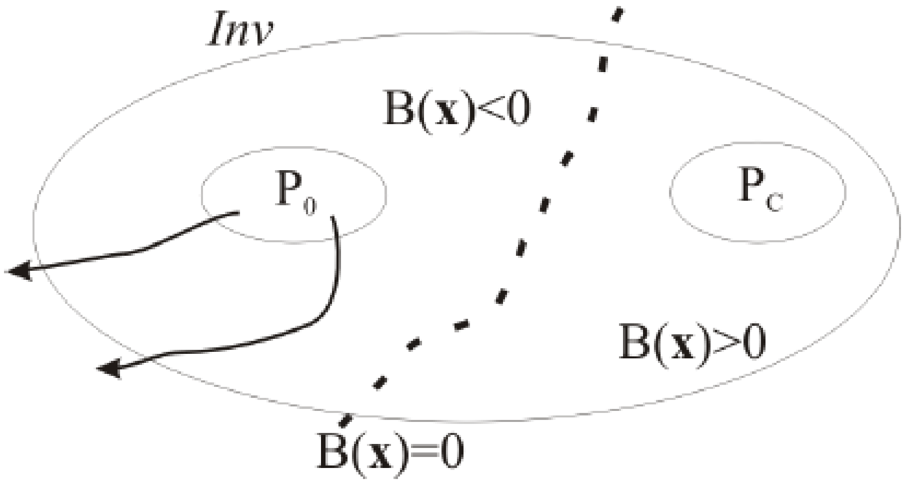
- $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, **A** diagonalizable with rational eigenvalues λ_i or nilpotent with pure imaginary eigenvalues
- reachable points on eigenspace of λ_i can be characterized with a set of polynomial constraints
- Check that no point fulfils all constraints through SOS optimization → goal region unreachable from initial region

- no point in P_{C2} fulfils C2, C3 → P_{C2} unreachable
- Constraint on positivity of time: $x_1^2 + 2x_2^2 \geq 3$
- no point in P_{C1} fulfils C2, C3, C4 → P_{C1} unreachable



Barrier certificates (e.g. Prajna et al. 07, Glavaski et al. 05)

$$\forall \mathbf{x} \in X, \forall u \in U : B(\mathbf{x}) = 0 \Rightarrow \frac{\partial B(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}) \leq 0$$



- Choice of type of $B(x)$
- SOS Optimization if B and dynamics are polynomial

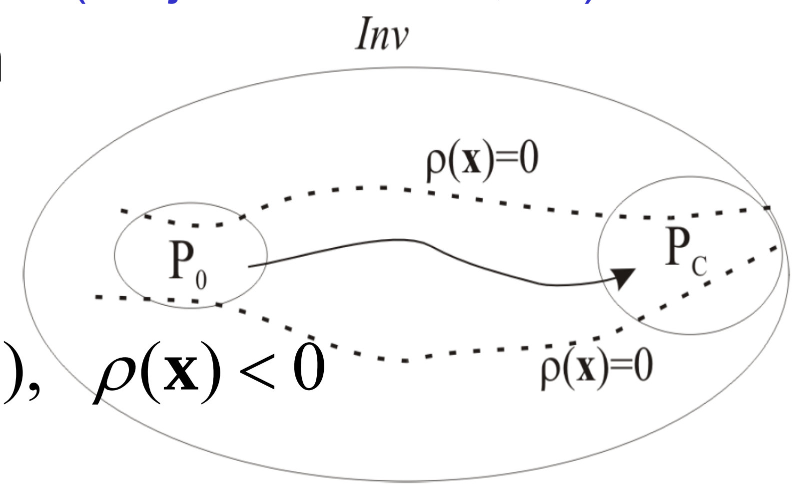
Existence of a trajectory: reachability certificate (Prajna & Rantzer, 05)

- For $\dot{\mathbf{x}} = f(\mathbf{x})$, ∃ a trajectory from P_0 to P_C if ∃ a function p st:

$$\int_{P_0} \rho(\mathbf{x}) dx > 0$$

$$\forall \mathbf{x} \in closure(bound(Inv) - bound(P_C)), \quad \rho(\mathbf{x}) < 0$$

$$\forall \mathbf{x} \in closure(Inv - P_C), \quad div(\rho f)(\mathbf{x}) < 0$$



Outline

- Verification and reachability
- Abstraction
- Characterizing reachable space
- **Reachable space computation**

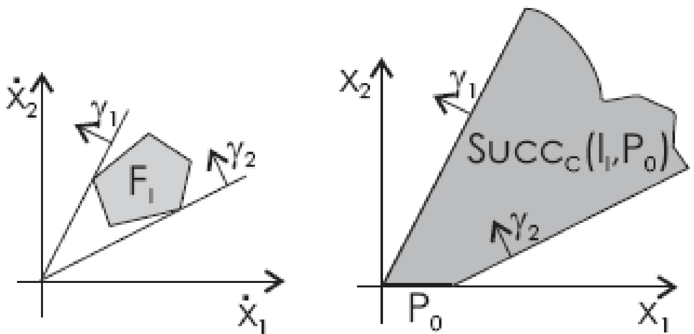
Reachable space calculation

- When refining a DE abstraction

- Difficulty: integration of differential equations (infinite set of trajectories to simulate), time elimination

- Over-approximation to preserve safety property

For continuous systems specified by linear differential inclusions, the overapproximated regions can be determined with geometric considerations and polytopes computations

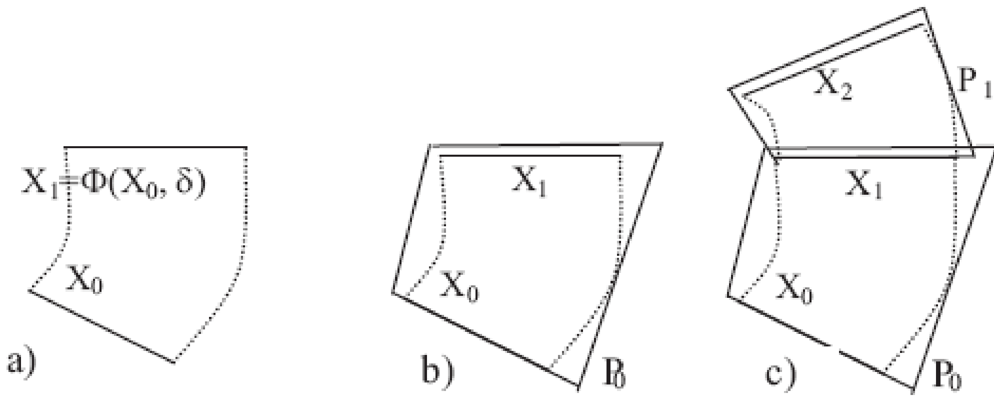


a) b)

Complex and difficult to implement: pay attention to the choice of regions

Finite discrete time integration (Dang, Chutinan & Krog, Asarin et al., Gérard)

- Calculation of series of finite time successor regions, using sample-time computation
  - Guaranteed integration: Time step δ, Finite number of steps



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$P_i = e^{A\delta} P_{i-1}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u}$$

$$P_i = e^{A\delta} P_{i-1} \oplus V$$

Minkowski sum with region *V* that depends on the bounded uncertainty **u**

$$\text{where } A \oplus B = \left\{ a + b \,\middle|\, a \in A \wedge b \in B \right\}$$

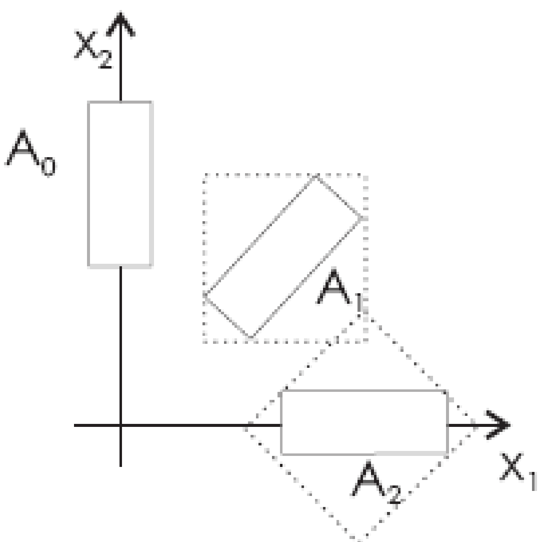
Space regions

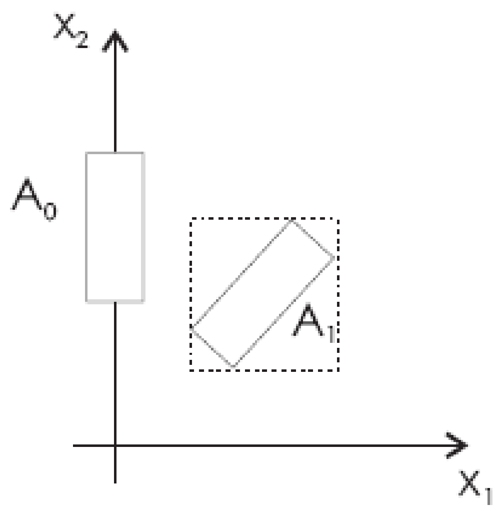
- Choice of a type of sets for continuous space regions:
  - efficiency of their set representation
  - complexity of computation on this type of set (intersection, union, dynamic evolution, Minkowski sum)
  - Closure of this type of set wrt operations needed for reachability calculation to reduce complexity and approximation

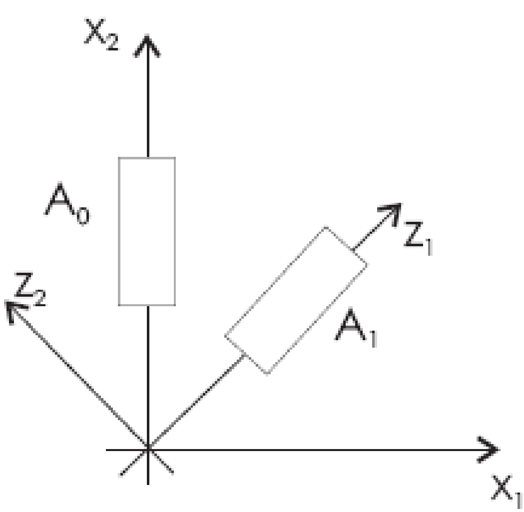
- Polynomial regions (e.g. Dang, 2006)
- Ellipsoids (e.g. Kurzhanski & Variya, 2000)
  - Compact and closed for transformations induced by linear dynamics
  - Not closed for other operations (ex: Minkowski sum), inducing important approximations
- Polyhedral sets
  - hyperrectangles – interval computation (Nedialkov et al., 1999)
  - Polyhedrons (linear constraints, vertex)
  - Zonotopes

Closure

- Hyperrectangles : all borders are normal to one of the basis vectors
- Difficulty: hyperrectangles are not closed for continuous dynamics changes (wrapping effect)
- Express intermediate results in intermediate basis to overcome wrapping effect

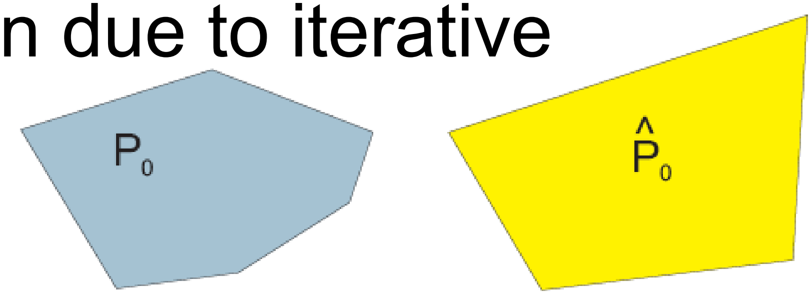




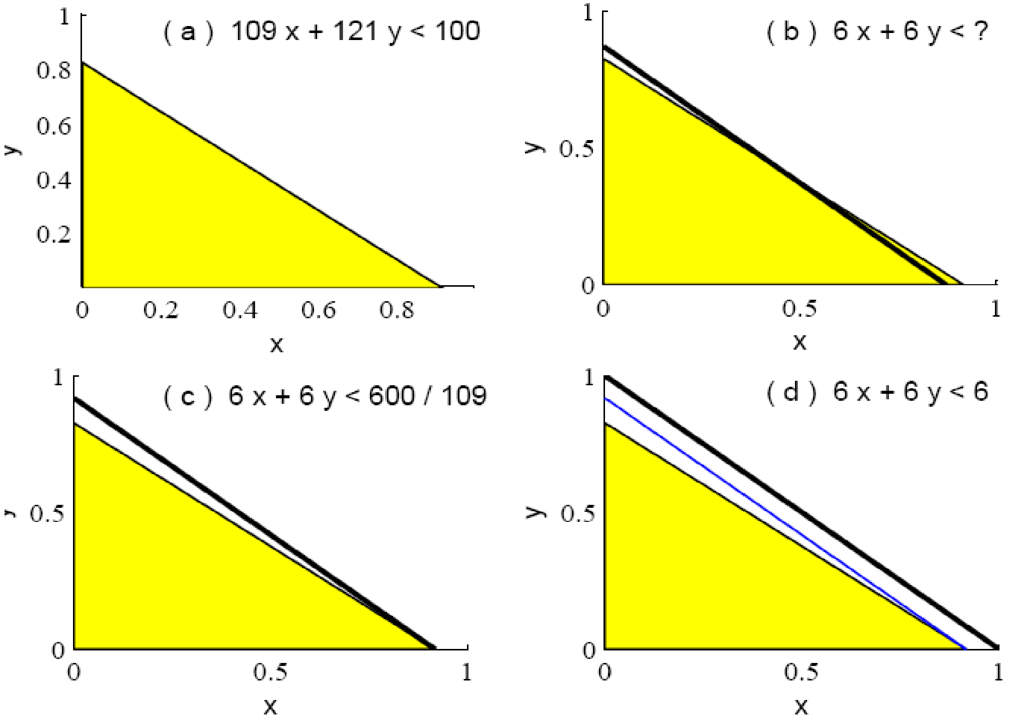


Polyhedral sets: Polyhedrons

- Complexity of representation due to iterative computation
  - Tight overapproximation to reduce number of constraints



  - Efficient coding of constraints (Asarin et al., 06): overapproximation to encode constraints with lower number of bits



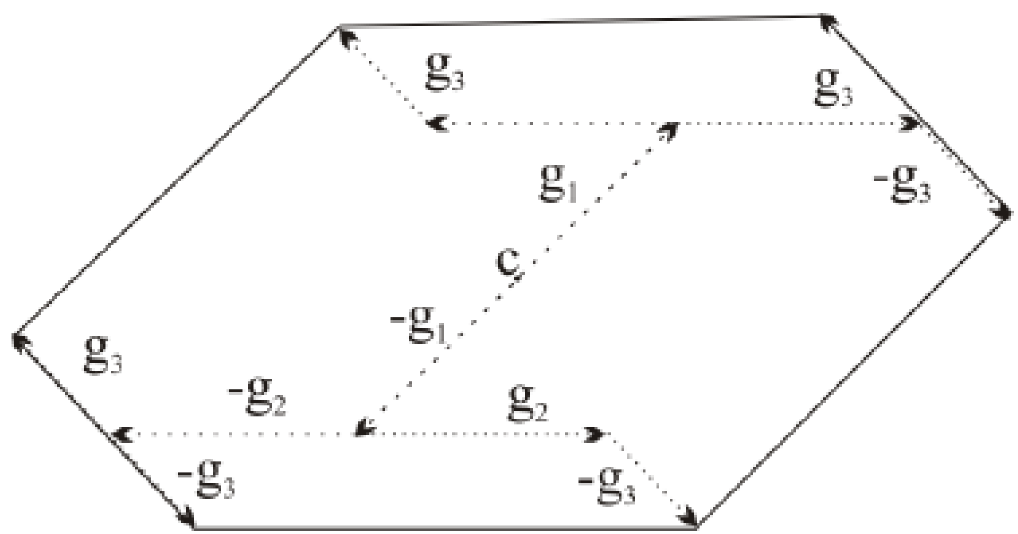
Polyhedral sets: Zonotopes

- Use for high dimension state space due to compact representation
- Closed for most operations involved in reachability computation (linear transformation, Minkowski sum)
- Problems: reduction of number of generators further to iteration of reachability computation, and computation of intersection with guards

Planar zonotope

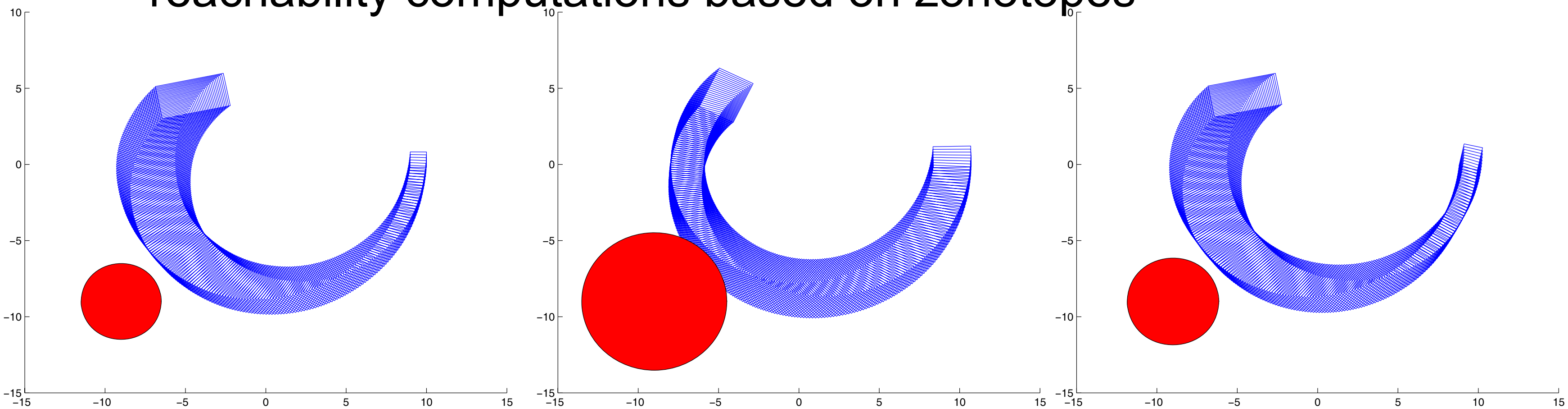
Defined by its center

and 3 generators



Complexity reduction: Continuous space dimension reduction

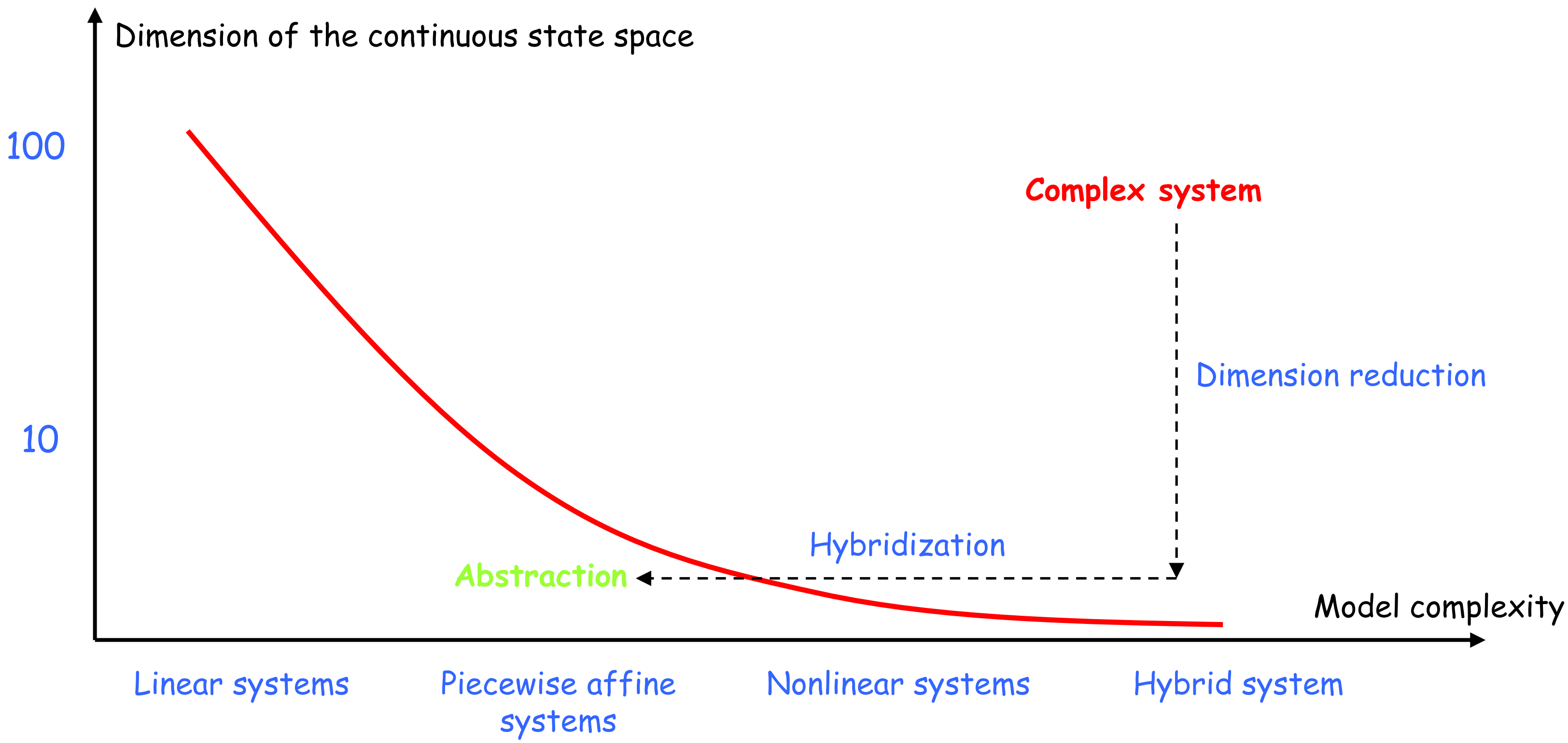
- Projection & uncertainty (e.g. Asarin & Dang, 04; Han & Krogh, 05): identify subspaces of state space st projection of state in one subspace has low influence on the projection of the state of the other
- Trajectories similarities (Girard, Pappas et al., 2006):
  - Approximation as a relaxation of the notion of abstraction
  - distance between trajectories rather than an inclusion relation
  - simulation functions defining approximate simulation relations: Lyapunov-like characterization, Algorithms (LMIs, SOS, Optimization)
  - reachability computations based on zonotopes



10 dimensional original system — 5 dimensional approximation — 7 dimensional approximation

Analysis of complex systems

Abstraction methods for complexity reduction of systems.



Conclusion: Structured presentation of formal verification techniques for Hybrid Systems

- Guaranty correct behavior
  - Methods and tools
- Safety properties: reachability and abstraction
- Non decidability results
- Various propositions
  - General principles
  - Representation of regions
  - Algorithms
- Reference: Annual Reviews in Control, Vol 33, 2009, p. 25-36, H. Guéguen, M.A. Lebfevre, J. Zaytoon

  doi:10.1016/j.arcontrol.2009.03.02

Perspectives

- Safety verification for real-size applications require complementary approaches alternating overapproximation, characterization of reachable space, dimension reduction
- Methodology based on clear criteria to guide the choice of the approaches and their cooperation for a given class of applications and properties
- Integrating such approaches with other control design tools